University of Cambridge

Mathematics Tripos

Part III

Symplectic Topology

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0 Introduction and Motivations

Let M be a manifold. In Riemannian geometry, we put a nondegenerate symmetric bilinear form on T_xM. In symplectic geometry we put a non-degenerate skew symmetric form instead. By basic linear algebra, by a change of basis, such a form is

$$\Omega = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$$

We define the *symplectic group* to be

$$\mathrm{Sp}_{2n}(\mathbb{R}) = \{A \in \mathrm{GL}_{2n}(\mathbb{R}) : A^T \Omega A = \Omega\}.$$

A symplectic manifold is a $2n$-manifold M^{2n} with an atlas of charts such that the derivatives of transition maps are in $\mathrm{Sp}_{2n}(\mathbb{R})$. We will prove that this is equivalent to (M^{2n}, ω), where $\omega \in \Omega^2(M)$ closed ($\mathrm{d}\omega = 0$), everywhere nondegenerate ($\omega^{\wedge n} \neq 0$) at all points.

Exercise. $(\mathbb{R}^{2n}, \sum \mathrm{d}x_i \wedge \mathrm{d}y_i)$ gives Ω with respect to $\frac{\partial}{\partial x_i}, \frac{\partial}{\partial y_i}$. We call this sympletic form ω_{std}.

In fact, this example is the "local model" for all symplectic manifolds. In other words, they have no local invariants.

Motivation 1: mechanices. Given a particle in $\mathbb{R}^n$ and a potential U, define $H = U + \frac{\dot{q}^2}{2}$ to be the energy. Then we can work out the flow of Hamilton's equations

$$\frac{\partial H}{\partial p} = \dot{q}, \frac{\partial H}{\partial q} = -\dot{p}.$$

It is a fact that the flow preserves the symplectic form $\sum \mathrm{d}p_i \wedge \mathrm{d}q_i$.

Motivation 2: symmetry groups. We want to classify groups acting locally on $\mathbb{R}^k$ such that

- act locally transitively (or reduce dim - orbit)
- noo invariant foliations: not of the form $(x, y) \mapsto (f(x), g(x, y))$ (or reduce dimension).

Theorem 0.1 (Lie)**.** *If such a group is finite dimensional, it is one of finitely many families (e.g.* $\mathrm{SO}(n), \mathrm{SU}(n), \mathrm{SO}(p, q)$ *etc).*

Theorem 0.2 (Cartan)**.** *If such a group is infinite dimensional, it is one of*

- $\mathrm{Diff}(\mathbb{R}^k)$*: all diffeomorphisms (preserving orientation),*
- $\mathrm{Vol}(\mathbb{R}^k)$*: all diffeomorphisms preserving volume form,*
- $\mathrm{Symp}(\mathbb{R}^{2\ell})$*: symplectomorphisms, i.e. diffeomorphisms preserving symplectic structure,*

- Cont($\mathbb{R}^{2\ell+1}$)*: contactomorphism (odd dimensional analogue of symplectormorphism)*
- *and their conformal analogues.*

Motivation 3: difference with volume. As $A \in \mathrm{Sp}_{2n}(\mathbb{R})$ implies $\det A = 1$, we have inclusion $\mathrm{Symp}(\mathbb{R}^{2n}) \subseteq \mathrm{Vol}(\mathbb{R}^{2n})$.

Theorem 0.3 (Moser)**.**

1. *Two volume forms on a closed manifold are equivalent if and only if they have the same total volume.*
2. *Suppose U, V are connected open in $\mathbb{R}^k$. There is a volume form-preserving $U \hookrightarrow V$ if and only if* $\mathrm{vol}(U) \leq \mathrm{vol}(V)$.

By contrast

Theorem 0.4 (Gromov non-squeezing)**.** *There is no symplectic embedding $B^{2n}(R) \hookrightarrow B^2(r) \times \mathbb{R}^{2n-2}$ if $R > r$.*

Motivation 4: complex geometry. Any smooth affine variety has a natural symplectic form

Course outline:

- background: extra bit of differential geometry, almost complex structure, first Chern class,
- basic symplecti geometry: distinguished submanifolds, local models, some constants on symplectic manifolds,
- constructions: e.g. new sympletric manifolds from old,
- holomorphic curves: invariants given by generalisation of Cauchy-Riemann equations, proof of non-squeezing theorem

1 (More) Differential geometry

Tensor algebra Let E be a vector space over $\mathbb{F}$. We define the *tensor algebra* of E to be

$$T(E) = \bigoplus_{i \geq 0} E^{\otimes i}$$

where $E^{\otimes 0} \cong \mathbb{F}$. Then we define the *exterior algebra* to be

$$\Lambda^* E = T(E)/\langle v \otimes v \rangle_{\text{as algebra}}$$

which has a natural grading $\Lambda^* E = \bigoplus_{k \geq 0} \Lambda^k E$,

$$\Lambda^k E = E^{\otimes k}/\langle w_1 \otimes \cdots \otimes w_k : w_i = w_j \text{ for some } i \neq j \rangle_{\text{as vector space}}$$

If $\dim_{\mathbb{F}} E = n$ then $\dim \Lambda^* E = 2^n, \dim \Lambda^k E = \binom{n}{k}$. The tensor product on $T(E)$ induces wedge product on $\Lambda^*(E)$ which is bilinear, associative and graded commutative.

If $A : E \to F$ is a linear map then it induces a map $\Lambda^k A : \Lambda^k E \to \Lambda^k F$. In $\dim E = \dim F = n$ then we can identify $\Lambda^n A : \Lambda^n E \to \Lambda^n F$ can be identified with $\det A : \mathbb{F} \to \mathbb{F}$.

Vector fields and differential forms Suppose M^n is a manifold[1]. Then we have the tangent and cotangent bundle TM, T^*M. *Vector fields* and *k-forms* on M are defined to be

$$\begin{aligned} \text{Vect}(M) &= \Gamma(TM) = \Gamma(M, TM) \\ \Omega^k(M) &= \Gamma(M, \Lambda^k T^* M) \end{aligned}$$

The 0-forms are also the smooth functions on M, $C^\infty(M) = \Omega^0(M)$.

In local coordinates $x_1, \dots, x_n$ on M, $X \in \text{Vect}(M)$ can be written locally as

$$X_p = \sum_{i=1}^n X^i \frac{\partial}{\partial x_i}|_p$$

where each X^i is a smooth function.

Given $X \in \text{Vect}(M), f \in C^\infty(M)$, we can differentiate f along X by $(Xf)_p = X_p(f)$. In local coordinates,

$$(Xf)_p = \sum X^i(p) \frac{\partial f}{\partial x_i}|_p.$$

We can check this is well-defined and it is a derivation in the sense that $X(fg) = fX(g) + gX(f)$.

Pullbacks Suppose $f : M \to N$ is smooth. It induces $f^* : C^\infty(M) \to C^\infty(N), g \mapsto g \circ f$ and also induces a function on 1-forms by $(f^*\varphi)_x = (Df_*)^*(\varphi_{f(x)})$. It then induces $f^* : \Omega^*(N) \to \Omega^*(M)$ with the properties that

1. f^* is linear and $f^*(\varphi \wedge \theta) = f^*\varphi \wedge f^*\theta$.
2. $(f \circ g)^* \varphi = g^* f^* \varphi$

[1] In this course all manifolds are assumed to be smooth unless stated otherwise.

Differential on $\Omega^*(M)$ Let $U \subseteq M$ be a chart with coordinates x_i. Then a local basis for $\Lambda^k U$ is $\{\mathrm{d}x_I = \mathrm{d}x_{i_1} \wedge \cdots \wedge \mathrm{d}x_{i_k}\}$ where $I = \{i_1 < \cdots < i_k\}$. If $\varphi : U \to \mathbb{R}$ then we have

$$\begin{aligned} \mathrm{d} : C^\infty(M) &\to \Omega^1(M) \\ \varphi &\mapsto \mathrm{d}\varphi = \sum \frac{\partial \varphi}{\partial x_i} \mathrm{d}x_i \end{aligned}$$

Note that $(\mathrm{d}\varphi)X = X\varphi \in C^\infty(M)$. In general, if $\varphi = \sum \varphi_i \mathrm{d}x_I \in \Omega^k(M)$ where φ_I smooth functions, then $\mathrm{d}\varphi = \sum \mathrm{d}\varphi_I \wedge \mathrm{d}x_I$. Can check this is well-defined and satisfies

1. $\mathrm{d}(\varphi_1 + \varphi_2) = \mathrm{d}\varphi_1 + \mathrm{d}\varphi_2$,
2. $\mathrm{d}(\varphi_1 \wedge \varphi_2) = \mathrm{d}\varphi_1 \wedge \varphi_2 + (-1)^k \varphi_1 \wedge \mathrm{d}\varphi_2$ if $\varphi_1 \in \Omega^k$,
3. $\mathrm{d}^2 = 0$,
4. $\mathrm{d}(f^*\varphi) = f^*(\mathrm{d}\varphi)$.

Moreover we can show these properties uniquely determines d.

It follows that we have the *de Rham complex*

$$0 \longrightarrow \Omega^0(M) \xrightarrow{\mathrm{d}} \Omega^1(M) \xrightarrow{\mathrm{d}} \Omega^2(M) \longrightarrow \cdots \xrightarrow{\mathrm{d}} \Omega^n(M) \longrightarrow 0$$

which gives rise to de Rham cohomology $H^*_{\mathrm{dR}}(M)$. By de Rham theorem this is isomorphic to $H^*(M; \mathbb{R})$, singular cohomology with coefficients in $\mathbb{R}$. $(\Omega^* M, \wedge, \mathrm{d})$ is the de Rham algebra. Morgan 1978 shows that it characterises the rational homotopy type of algebraic varieties.

Isotopies and vector fields

Definition (isotopy)**.** A smooth map $\rho : M \times \mathbb{R} \to M$ is an *isotopy* if $\rho_t = \rho(-, t) : M \to M$ is a diffeomorphism for each t and $\rho_0 = \mathrm{id}_M$.

We could replace $\mathbb{R}$ with open intervals containing 0.

Given an isotopy ρ, we get a time-dependent vector field, say v_t, as follows:

$$v_t|_p = \frac{d}{ds} \rho_s(q)|_{s=t}$$

where $q = \rho_t^{-1}(p)$, i.e.

$$\frac{d\rho_t}{dt} = v_t \circ \rho_t. \qquad (*)$$

Conversely, given a time-dependent vector field v_t, if M is compact or if v_t is compactly supported, by Picard's theorem on existence of solutions to ODEs, there is an isotopy ρ such that $\rho_0 = \mathrm{id}$ and the ODE $(*)$ is satisfied. For compact M we have a one-to-one correspondence

$$\{\text{isotopies of } M\} \longleftrightarrow \{\text{time-dependent vector fields on } M\}.$$

For non-compact M, the flow still exists locally (i.e. at each point p for sufficiently small interval of time) by Picard(-Lindelöf).

> **Definition.** If $v_t = v$ (independent of t), its flow is called the *exponential map* of v, denoted $\exp(tv)$.

Useful formula (III Differential Geometry Example sheet 2 question 3): for $\theta \in \Omega^1(M), X, Y \in \mathrm{Vect}(M)$, have

$$\mathrm{d}\theta(X, Y) = X\theta(Y) - Y\theta(X) - \theta([X, Y]).$$

Interior product Suppose $\alpha \in \Omega^{p+1}(M), X \in \Gamma(TM)$, the we define the *interior product* $X \lrcorner \alpha = \iota_X a \in \Omega^p(X)$ to be

$$\iota_X(\alpha)(u) = \alpha(X \wedge u)$$

for $u \in \Gamma(\Lambda^p TM)$.

Lie derivatives DG ES2 Q 11*

Let M be a manifold, $X \in \Gamma(TM)$ a vector field. Then we have a local flow $\varphi_t : M \to M$ for $t \in (-\delta, \delta)$. Given $\alpha \in \Omega^*(M)$, the *Lie derivative* of α with respect to X is

$$\mathcal{L}_X(\alpha) = \frac{d}{dt}(\varphi_t^* \alpha)|_{t=0} \in \Omega^*(M)$$

and has the same degree as α if α has pure degree. For $V \in \Gamma(\Lambda^k TM)$, we similarly define

$$\mathcal{L}_X V = \frac{d}{dt}((\varphi_{-t})_* V)|_{t=0}$$

where $(\varphi_t)_* = \Lambda^k D\varphi_t$.

Properties:

1. $\mathcal{L}_X f = Xf$ for $f \in C^\infty(M)$.
2. $\mathcal{L}_X(Y) = [X, Y]$ for $X, Y \in \Gamma(TM)$.
3. $\mathcal{L}_X(\mathrm{d}\alpha) = \mathrm{d}\mathcal{L}_X \alpha$ for $\alpha \in \Omega^*(M)$.
4. Cartan's formula: $\mathcal{L}_X = \iota_X \circ \mathrm{d} + \mathrm{d} \circ \iota_X$.
5. For a time-dependent X_t with flow φ_t, $\frac{d}{dt}(\varphi_t^* \alpha) = \varphi_t^* \mathcal{L}_{X_t} \alpha$ for $\alpha \in \Omega^*(M)$.

Sketch proof.

1. $\mathcal{L}_X f = \frac{d}{dt}|_{t=0}(f \circ \varphi_t) = Xf$.
2. Let φ_t be the flow of X. Use the slightly unusual notation $\varphi_t^*(Y)|_p = (D\varphi_t^{-1})(Y_{\varphi_t(p)})$. Check

   $$\varphi_t^*(Y)(f \circ \varphi_t) = Y(f) \circ \varphi_t$$

   so have

   $$\frac{\varphi_t^*(Y)(f \circ \varphi_t) - \varphi_t^*(Y)(f)}{t} + \frac{\varphi_t^*(Y)(f) - Y(f)}{t} = \frac{Y(f) \circ \varphi_t - Y(f)}{t}$$

   take limit as $t \to 0$,

   $$YX(f) + \mathcal{L}_X(Y)(f) = XY(f).$$

3. Omitted.

4. General strategy: check the formula holds for 0-forms, both sides commute with d, both sides are derivations for $(\Omega^*(M), \wedge)$, and use the fact that the equations are local and for a local coordinate patch U, $\Omega^*(U)$ is generated as an algebra by $\Omega^0(U)$ and $\mathrm{d}\Omega^0(U)$.

5. Same as 4.

□

Lemma 1.1. *For a smooth family $\alpha_t \in \Omega^k(M)$,*

$$\frac{d}{dt}(\varphi_t^* \alpha_t) = \varphi_t^*(\mathcal{L}_{X_t}\alpha_t + \frac{d\alpha_t}{dt}).$$

Proof. Treat LHS as the derivative of a function of two variables,

$$\frac{d}{dt}(\varphi_t^*\alpha_t) = \frac{d}{dx}(\varphi_x^*\alpha_t)|_{x=t} + \frac{d}{dy}(\varphi_t^*\alpha_y)|_{y=t} = \varphi_t^*\mathcal{L}_{X_t}\alpha_t + \varphi_t^*\frac{d\alpha_t}{dt}.$$

□

Oreintations Let E be an n-dim $\mathbb{R}$ vector space. Then an orientation on E is an equivalence class of ordered basis $(e_1, \dots, e_n)$ under the equivalence relation $(e_1, \dots, e_n) \sim (f_1, \dots, f_n)$ if and only if the endomorphism $A : e_i \mapsto f_i$ has $\det A > 0$.

Let $\pi : E \to B$ be a rank k real vector bundle. An orientation on E is a coherent choice of orientations on each fibre E_b, where "coherent" means that for local trivialisation $\pi^{-1}(U) \cong \mathbb{R}^k \times U$, the choice is constant.

Let M^n be a manifold. An orientation of M is an orientation of TM (if exists). We will denote by $\overline{M}$ the manifold M with opposite orientation. If M^n is a manifold with boundary ∂M, an orientation of M induces an orientation on ∂M: a basis $(e_1, \dots, e_{n-1})$ for $T_x(\partial M)$ is positively oriented if $(n_x, e_1, \dots, e_{n-1})$ is for "$T_x M$", where n_x is the outward pointing normal vector.

Note if M is a compact oriented 1-manifold with boundary the $\sum_{p \in \partial M} \mathrm{or}(p) = 0$.

Integration In vector calculus, we have if $f : (U, x_i) \to (V, y_i)$ is a diffeomorphism of open subsets of $\mathbb{R}^k$, then

$$\int_V a dy_1 \cdots dy_k = \int_U (a \circ f)|\det(Df)| dx_1 \cdots dx_k.$$

In differential geometry we formulate integration in this way: for $\varphi = a\mathrm{d}y_1 \wedge \cdots \wedge \mathrm{d}y_k$, if φ preserves orientation then

$$\int_V \varphi = \int_U f^*\varphi.$$

Lemma 1.2. *If X is an oriented k-manifold, there is a well-defined integration map*

$$\int_X : \Omega_c^k \to \mathbb{R}$$

where Ω_c^k are k-forms with compact support.

A *volume form* on M^k is a nowhere zero section $\mathrm{d}\,\mathrm{Vol} \in \Omega^k(M)$, which is equivalent to a choice of trivialisation $\Lambda^k T^* M \cong \mathbb{R} \times M$. M is orientable if and only if $\Lambda^k T^* M$ is trivial. Note that $\mathrm{d}(\mathrm{d}\,\mathrm{Vol}) = 0$.

Theorem 1.3 (Stokes)**.**

$$\int_M \mathrm{d}\alpha = \int_{\partial M} \alpha.$$

Corollary 1.4. *For X closed oriented, we have a surjection $\int_X : H^k_{dR}(X) \to \mathbb{R}$.*

Sketch proof. Let $U \subseteq \mathbb{R}^k_+$ be an open chart. Use linearity and partition of unity, it suffices to work in U. Then use standard results from multivariate calculus/Fubini's theorem. See example sheet 1. □

2 Symplectic linear algebra

Recall the standard symplectic form $\omega_{\text{std}} = \sum \mathrm{d}x_i \wedge \mathrm{d}y_i$ on $(\mathbb{R}^{2n}, (x_i, y_i))$: with respect to $\frac{\partial}{\partial x_1}, \frac{\partial}{\partial y_1}, \dots, \frac{\partial}{\partial x_n}, \frac{\partial}{\partial y_n}$, it has matrix

$$\Omega_0 = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & & & \\ & & \ddots & & \\ & & & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}$$

Define

$$\begin{aligned} \mathrm{Sp}_{2n}(\mathbb{R}) &= \{A \in \mathrm{GL}_{2n}(\mathbb{R}) : A^*\omega_0 = \omega_0\} \\ &= \{A \in \mathrm{GL}_{2n}(\mathbb{R}) : A^T \Omega_0 A = \Omega_0\} \end{aligned}$$

by identifying A with its matrix representation.

Recall from linear algebra

Lemma 2.1. *Suppose (V, Ω) is a vector space with a non-degenerate alternating (or skew-symmetric) bilinear form Ω (i.e. V is a* symplectic vector space*), then there is a basis $\mathcal{B} = (u_1, v_1, \dots, u_n, v_n)$ of V such that $[\Omega]_{\mathcal{B}} = \Omega_0$.*

Sketch. By non-degeneracy exist u_1, v_1 such that $\Omega(u_1, v_1) = 1$. u_1, v_1 are linearly independent since Ω is alternating. Then $V = \langle u_1, v_1 \rangle \oplus \{w : \Omega(u_1, w) = \Omega(v_1, w) = 0\}$. Proceed by induction. □

Corollary 2.2. *Symplectic vector spaces are even-dimensional and $\Omega \in \Lambda^2 V^*$ is non-degenerate if and only if $\Omega^n \neq 0 \in \Lambda^{2n} V^*$.*

Definition (symplectic complement)**.** Suppose $U \leq (V, \Omega)$. The *symplectic complement* of U in V is

$$U^\Omega = \{w \in V : \Omega(w, u) = 0 \text{ for all } u \in U\}.$$

Definition (symplectic, (co)isotropic, Lagrangian subspace)**.** Let (V, Ω) be a symplectic vector space.

- $U \leq V$ is a *symplectic subspace* if $U \cap U^\Omega = 0$, i.e. $\Omega|_U$ is nondegenerate.
- $U \leq V$ is an *isotropic subspace* if $U \leq U^\Omega$.
- $U \leq V$ is a *coisotropic subspace* if $U^\Omega \leq U$.
- $U \leq V$ is a *Lagrangian subspace* if it is both isotropic and coisotropic.

Proposition 2.3. *An isotropic subspace has dimension at most* $\frac{1}{2}\dim V$ *and a coisotropic subspace has dimension at least* $\frac{1}{2}\dim V$*. If* U *is isotropic of dimension* $\frac{1}{2}\dim V$*, it is also coisotropic (and vice versa), in which case it is Lagrangian.*

Proof. Ω nondegenerate gives a surjection $V \to U^*, v \mapsto \Omega(-, v)$. Thus $\dim U^* + \dim U^0 = \dim V$, so $\dim U + \dim U^\Omega = \dim V$. ☐

3 Symplectic manifolds: first notions

$\varphi \in \Omega^2(M)$ gives

$$\begin{aligned} \mu_\varphi : TM &\to T^*M \\ u &\mapsto (v \mapsto \varphi(u, v)) \end{aligned}$$

i.e. $\mu_\varphi u = \iota_u \varphi$. φ is *non-degenerate* if μ_φ is an isomorphism, which happens if and only if φ^n is nonwhere zero, where $\dim M = 2n$.

Definition (symplectic form)**.** A closed and non-degenerate form $\omega \in \Omega^2(M)$ is called a *symplectic form*.

First condition a linear algebra condition, the second a local condition: to check some integral is zero only have to check locally.

Definition (symplectic manifold)**.** (M^{2n}, ω) is a symplectic manifold.

Definition (symplectic structure)**.** A *symplectic structure* is a symplectic form up to pullback by diffeomorphism.

Proposition 3.1. *A 2-fold is symplectic if and only if it is orientable.*

Proof. A non-degenerate form on M^2 is a volume form. By dimension reason all 2-forms are closed. □

Proposition 3.2. *Suppose a closed manifold M^{2n} is symplectic. Then $H^{2i}_{\mathrm{dR}}(M) \neq 0$ for $0 \leq i \leq n$.*

Proof. $[\omega] \in H^2_{\mathrm{dR}}(M)$ and ω^n is a volume form, say $\mathrm{d}\,\mathrm{Vol}$. Thus

$$[\omega]^n = [\omega^{\wedge n}] = [\mathrm{d}\,\mathrm{Vol}] \neq 0 \in H^{2n}_{\mathrm{dR}}(M).$$

□

Example. S^4 is not symplectic.

3.1 Hamiltonian flows

Suppose (M^{2n}, ω) is symplectic and $f \in C^\infty(M)$. We can construct a vector field as follow: $\mathrm{d}f \in \Omega^1(M)$. Use the bundle isomorphism $\mu_\omega : TM \to T^*M$ we obtain $X_f = (\mu_\omega)^{-1}(\mathrm{d}f)$. It is the unique vector field such that $\iota_{X_f}\omega = \mathrm{d}f$.

Definition (Hamiltonian flow)**.** The flow of X_f is called the *Hamiltonian flow* of f.

Proposition 3.3. *Wherever defined, the Hamiltonian flow of a function acts by symplectomorphism.*

Proof. By Cartan's formula

$$\mathcal{L}_{X_f}\omega = \iota_{X_f}\mathrm{d}\omega + \mathrm{d}\iota_{X_f}\omega = 0 + \mathrm{d}(\mathrm{d}f) = 0.$$

□

Remark. This means that symplectic manifolds have larger spaces of symmetries than Riemannian manifolds. For example compare isometries of S^2 vs. symplectomorphisms.

Example. Hamilton's equations

$$\frac{\partial H}{\partial p} = \dot{q}, \frac{\partial H}{\partial q} = -\dot{p}$$

where q is position and p is momentum. This is same as following the flow of $X_H = (\frac{\partial H}{\partial p}, -\frac{\partial H}{\partial q})$, for $\omega = \sum \mathrm{d}q_i \wedge \mathrm{d}p_i$. Note $\iota_{X_H}\omega = \mathrm{d}H$. This shows that classical Hamiltonian flows are examples of Hamiltonian flows in the sense of symplectic geometry and are through symplectomorphisms of $\mathbb{R}^{2n}$.

3.2 (Almost) complex manifolds

Definition (complex manifolds)**.** A *complex manifold* is a manifold M^{2n} covered by charts $u_\alpha \subseteq \mathbb{C}^n$ such that the transition maps are biholomorphisms. Equivalently, for two charts $(U_\alpha, \varphi_\alpha)$ and (U_β, φ_β), we require $D(\varphi_\alpha \circ \varphi_b^{-1}) \in \mathrm{GL}_n(\mathbb{C}) \subseteq \mathrm{GL}_{2n}(\mathbb{R})$.

The (co)tangent spaces of M are naturally complex vector spaces.

Definition (almost complex structure)**.** An *almost complex structure* (acs) on a smooth manifold M is an endomorphism $J : TM \to TM$ such that $J^2 = -\,\mathrm{id}$.

Definition (integrable)**.** If an almost complex structure J comes from a complex structure then we say J is *integrable*.

Remark.

1. A complex manifold is almost complex: the complex structure is induced by multiplication by i.

2. J extends to a map $TM \otimes_{\mathbb{R}} \mathbb{C} \to TM \otimes_{\mathbb{R}} \mathbb{C}$, which we also denote by J. For complex manifolds we get

$$J(\frac{\partial}{\partial z_j}) = i\frac{\partial}{\partial z_j}, J(\frac{\partial}{\partial \overline{z}_j}) = -i\frac{\partial}{\partial \overline{z}_j}$$

where $TM \otimes \mathbb{C} = \mathbb{C}\langle \frac{\partial}{\partial x_j}, \frac{\partial}{\partial y_j}\rangle$, and

$$\frac{\partial}{\partial z_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}), \frac{\partial}{\partial \overline{z}_j} = \frac{1}{2}(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}),$$

and their dual basis is

$$\mathrm{d}z_j = \mathrm{d}x_j + i\mathrm{d}y_j, \mathrm{d}\overline{z}_j = \mathrm{d}x_j - i\mathrm{d}y_j.$$

3. The complexified cotangent bundle splits as $T^*M \otimes \mathbb{C} = T^*M^{1,0} \oplus T^*M^{0,1}$ where

$$\begin{aligned} T^*M^{1,0} &= \{\alpha : \alpha(Jv) = i\alpha(v)\} = \mathbb{C}\langle \mathrm{d}z_j \rangle \\ T^*M^{0,1} &= \{\alpha : \alpha(Jv) = -i\alpha(v)\} = \mathbb{C}\langle \mathrm{d}\overline{z}_j \rangle \end{aligned}$$

This then induces splitting on sections

$$\Omega^1(M; \mathbb{C}) = \Gamma(T^*M \otimes \mathbb{C}) = \Omega^{1,0} \oplus \Omega^{0,1}.$$

More generally, define

$$\Omega^{p,q}(M) = \Gamma(\Lambda^p T^*M^{1,0} \otimes \Lambda^q T^*M^{0,1}) \leq \Gamma(\Lambda^{p+q}(T^*M \otimes \mathbb{C})) = \Omega^{p+q}(M; \mathbb{C}).$$

For M a complex manifold, a section for $\Omega^{p,q}$ is given in locally coordintes by

$$\sum \alpha_{PQ} \mathrm{d}z_P \wedge \mathrm{d}\overline{Z}_Q$$

where α_{PQ} are smooth functions.

4. $\mathrm{d} : \Omega^k(M) \to \Omega^{k+1}(M)$ induces $\mathrm{d} : \Omega^k(M; \mathbb{C}) \to \Omega^{k+1}(M; \mathbb{C})$. Suppose $\mathrm{d} : \Omega^0(M; \mathbb{C}) \to \Omega^1(M; \mathbb{C})$. By composition with projections to $(1, 0)$-forms and $(0, 1)$-forms, we get $\mathrm{d} = \partial + \overline{\partial}$.

   For a *complex manifold*, we can

   (a) talk about holomorphic functions (functions that are holomorphic on each chart), and $f \in C^\infty(M)$ is holomorphic if and only if $\overline{\partial} f = 0$.
   (b) $\mathrm{d}(\Omega^{p,q}) \subseteq \Omega^{p+1,q} \oplus \Omega^{p,q+1}$ (this needn't be true for almost complex manifolds).
   (c) $\mathrm{d}^2 = 0$ implies $\overline{\partial} = 0$ so we can form the Dolbeault complex

   $$\cdots \longrightarrow \Omega^{\bullet,k} \xrightarrow{\overline{\partial}} \Omega^{\bullet,k+1} \xrightarrow{\overline{\partial}} \cdots$$

   The cohomology of this complex is Dolbeault cohomology $H^{\bullet,k}_{\overline{\partial}}(M)$.

3.3 Kähler manifold

Definition (Kähler manifold)**.** A *Kähler manifold* is a complex manifold with a closed positive-definite real $(1, 1)$-form: $\omega \in \Omega^{1,1}(M)$ such that

- $\partial\omega = \overline{\partial}\omega = 0$,
- $\omega = \frac{i}{2} \sum_{j,k} h_{jk} \mathrm{d}z_j \wedge \mathrm{d}\overline{z}_k$ with (h_{jk}) a hermitian postive definite matrix.

Exercise. Expand the local expression in terms of $\mathrm{d}x_j$ and $\mathrm{d}y_j$ to show that the factor $\frac{i}{2}$ is sensible.

Proposition 3.4. *Kähler forms are symplectic forms.*

Proof. A Kähler form ω is closed, and

$$\omega^n = n!\left(\frac{i}{2}\right)^n \underbrace{\det(h_{jk})}_{>0} \mathrm{d}z_1 \wedge \mathrm{d}\overline{z}_1 \wedge \cdots \wedge \mathrm{d}z_n \wedge \mathrm{d}\overline{z}_n$$

and since $\frac{i}{2}\mathrm{d}z_i \wedge \mathrm{d}\overline{z}_i = \mathrm{d}x_i \wedge \mathrm{d}y_i$, ω^n is nowhere zero. □

Definition (plurisubharmonic)**.** Let M be a complex manifold. A smooth function $\rho : M \to \mathbb{R}$ is strictly *plurisubharmonic* (plush) if

$$\left(\frac{\partial^2 \rho}{\partial z_j \partial \overline{z}_k}\right)$$

is positive definite everywhere.

Check that for such a function, $\frac{i}{2}\partial\overline{\partial}\rho$ defines a Kähler form on M.

Remark. Such an M can't be closed.

Example.

1. $\rho(z) = |z|^2$ on $\mathbb{C}^n \cong \mathbb{R}^{2n}$ gives $\frac{i}{2}\partial\overline{\partial}\rho = \omega_{\mathrm{std}}$.
2. $\rho(z) = \log(1 + |z|^2)$ on $\mathbb{C}^n \cong \mathbb{R}^{2n}$. To check this is plush, look at $(1, 0, \ldots, 0)$ and use $U(n)$-invariance. Check also the induced volume form has finite total volume.
3. A complex submanifold of a Kähler manifold is Kähler with the pullback form.
4. $\mathbb{CP}^n$ is Kähler: $\mathbb{P}^n$ can be covered by charts $U_i = \{z_i \neq 0\}$. The transition functions are of the form

   $$\varphi : (u_1, \ldots, u_n) \mapsto (\frac{1}{u_1}, \frac{u_2}{u_1}, \ldots, \frac{u_n}{u_1})$$

   so

   $$\varphi^*(\frac{i}{2}\partial\overline{\partial}(\log(1 + |z|^2)) = \frac{i}{2}\partial\overline{\partial}(\log(1 + |z|^2) + \log(\frac{1}{|z_1|^2}))).$$

   Thus the local Kähler forms patch to give a global one. For details see III Complex Manifolds.

Theorem 3.5 (Hodge)**.** *If X is compact Kähler then*

- $H^k_{\mathrm{dR}}(X) \otimes \mathbb{C} \cong \bigoplus_{i+j=k} H^{i,j}_{\overline{\partial}}(X)$.
- $H^{i,j}_{\overline{\partial}}(X) \cong \overline{H^{j,i}_{\overline{\partial}}(X)}$. *In particular they have the same dimension.*

Corollary 3.6. *If X is compact Kähler then any Betti number of odd degree is even.*

Theorem 3.7 (Lefschetz)**.** *For X compact Kähler the wedge product*

$$- \wedge [\omega]^k : H_{\mathrm{dR}}^{n-k}(X) \to H_{\mathrm{dR}}^{n+k}(X)$$

is an isomorphism, where n is the complex dimension of X.

Remark.

1. It is easy to write down a compact complex manifold that is not Kähler: consider $(\mathbb{C}^2 \setminus \{0\})/z \sim 2z$. It inherits a complex structure from $\mathbb{C}^2$. It is homeomorphic to $S^1 \times S^3$, so $b_2 = 0$. Thus it can't be symplectic.

2. Most examples of complex Kähler manifold we'll see are projective, but plenty are not (e.g. take deformations of complex surfaces in $\mathbb{P}^3$. c.f. K3 surfaces).

3.4 Almost complex structure on symplectic manifolds

Definition (compatible almost complex structure)**.** An almost complex structure J on a symplectic manfiold (M, ω) is *compatible* with ω if

1. $\omega(Ju, Jv) = \omega(u, v)$,

2. $\omega(v, J(v)) > 0$ unless $v = 0$.

Note. It follows that $\omega(\cdot, J\cdot)$ is a positive-definite symmetric bilinear form, so gives a Riemannian metric $g(w, u) = \omega(u, Jv)$. Such a triple (ω, J, g) is sometimes called a compatible triple. We'll see any two of them determines the third.

Proposition 3.8. *Any symplectic manifold (M, ω) admits a compatible almost complex structure.*

Proof. Let (V, Ω) be a symplectic vector space. Fix any metric g on V. As g and Ω both determine isomorphisms $V \to V^*$, there exists $A \in \operatorname{End} V$ such that $\omega(u, v) = g(Au, v)$. Note that

$$\omega(u, v) = -\omega(v, u) = -g(Av, u) = -g(u, Av)$$

so $A^* = -A$ with respect to g. As $g(AA^*v, v) = g(A^*v, A^*v) > 0$ for all $v \neq 0$ and $(AA^*)^* = AA^*$, AA^* is positive definite symmetric. Thus by choosing an orthnormal basis, $AA^* = BDB^{-1}$, where B is the diagonal matrix with entries $\lambda_1, \cdots \lambda_{2n}$. We can thus take the positive square root and define $J = (\sqrt{AA^*})^{-1}A$.

1. $J^2 = (\sqrt{-A^2})^{-1}A(\sqrt{-A^2})^{-1}A = -\operatorname{id}$.

2. $J^* = -J$ (implies $JJ^* = \operatorname{id}$ so J is orthogonal).

3. For compatibility,
$$\omega(Ju, Jv) = g(AJu, Jv) = g(JAu, Jv) = g(Au, v) = \omega(u, v).$$
4. $\omega(u, Ju) = g(-JAu, u) = g(\sqrt{AA^*}u, u) > 0$.

Since A is uniquely determined and we are taking the positive square root, we can use this construction on each T_xM, for a choice of Riemannian metric on M. Our procedure on (V, Ω) is canonical so this works globally. □

Note. Given (M, Ω), there is a bijection
$$\{\text{compatible complex structure } J\} \longleftrightarrow \{\text{Riemannian metric } M\}$$
Note that RHS is convex so contractible, and it follows that LHS is also contractible. We will see that for many applications, there is "essentially no choice" of J.

Definition (symplectic vector bundle)**.** A *symplectic vector bundle* is vector bundle $\pi : E \to B$ with a section $\Omega \in \Gamma(\Lambda^2 E^*)$ such that $\Omega|_{E_b} = \omega_b$ is symplectic on each fibre and locally trivial, i.e. $(\pi^{-1}(U), \Omega) \cong (U \times \mathbb{R}^{2n}, \omega_0)$.

Corollary 3.9. *Such an E admits the structure of a complex vector bundle, uniquely determined up to a contractible choice.*

Note that this is weaker than a holomorphic bundle.

3.5 First Chern class

What is the (unique) compatible complex structure good for? We can define a topological invariant. The *first Chern class* of a complex vector bundle $E \to B$ is an element of $c_1(E) \in H^2(B; \mathbb{Z})$. We present three definitions here. They are equivalent, but we will not prove so.

1. Algebraic topology: let $\det E = \Lambda^{\operatorname{rank} E} E$, which is a complex line bundle over B, so can be obtained as a pullback of $\mathcal{L}_{\text{taut}} \to BU(1) = \mathbb{CP}^\infty$. Note the tautological bundle can be described as
$$\begin{aligned}\mathcal{L}_{\text{taut}} &= \{(w, z) \in \mathbb{CP}^n \times \mathbb{C}^{n+1} : z \in w\} \\ &= \{(w, z) : z_i w_j = z_j w_i \text{ for all } i, j\} \subseteq \mathbb{CP}^n \times \mathbb{C}^{n+1}\end{aligned}$$
The advantage of the second description is that it is a smooth projective variety so a Kähler manifold.

   Suppose $\det E = \varphi^* \mathcal{L}_{\text{taut}}$. $H^*(\mathbb{CP}^\infty) = \mathbb{Z}[c_1^{\text{univ}}]$ where c_1 has degree 2. We then define $c_1(E) = \varphi^*(c_1^{\text{univ}})$.
2. Algebraic topology, alternative: take $s : B \to \det E$ a generic section. Then $\dim_{\mathbb{R}}(s(B) \cap s_0(B)) = n - 2$ (recall E is complex). Then define $c_1(E)$ to be the Poincaré dual of this class.
3. Differential geometry: suppose d_A is a connection on E, $F_A = \mathrm{d} \cdot \mathrm{d}_A \in \Omega^2(\operatorname{End} E)$ the curvature. Then $c_1(E) = [\frac{1}{2\pi i} \operatorname{tr} F_A]$.

Properties of $c_1(E)$:

1. from the second definition, $c_1(E) = e(\det E) \in H^2(B;\mathbb{Z})$.
2. $c_1(E) \in H^2(B;\mathbb{Z})$ is invariant if J_E change continuously, so we get an invariant of the symplectic bundle.
3. $c_1(f^*E) = f^*c_1(E)$.
4. $c_1(E^*) = -c_1(E)$.
5. For any short exact sequence of complex vector bundles $0 \to E \to \mathcal{E} \to E' \to 0$, then $c_1(\mathcal{E}) = c_1(E) + c_1(E')$.
6. $c_1(E \otimes F) = \mathrm{rk}E c_1(F) + \mathrm{rk}F c_1(E)$.
7. For M compact, complex line bundles over M up to isomorphism is isomorphic to $H^2(M;\mathbb{Z})$ via c_1.

Notation. For a manifold with amost complex structure, define $c_1(M) = c_1(TM)$.

Exercise.

1. $c_1(\Sigma_g) = (2-2g)PD(\mathrm{pt})$. This is Guass-Bonnet: count the signed zeros of a generic vector field. As Σ_g has complex dimension 1, this is just the Euler class of $T\Sigma_g$.
2. $c_1(\mathcal{L}_{\mathrm{taut}} \to \mathbb{CP}^n) = -[H] = PD(\mathbb{CP}^{n-1}) \in H^2(\mathbb{CP}^n) \cong \mathbb{Z}\langle H\rangle$. $[H]$ is the hyperplane class. In algebraic geometry we have $\mathcal{L}_{\mathrm{taut}} = \mathcal{O}(-1)$.
3. $c_1(T\mathbb{CP}^n) = c_1(\mathbb{CP}^n) = (n+1)[H]$.

Constraint for 4-manifolds:

adjunction formula Suppose $C \subseteq (X^4, J)$ is an almost complex curve in an almost complex surface ($JTC = TC \subseteq TX$). Then

$$2g(C) - 2 = -c_1(X) \cdot [C] + [C]^2$$

where $[C]^2$ is the self-intersection number of C. Thus the genus of C is determiend by its homology class.

Proof. We have a short exact sequence

$$0 \longrightarrow TC \longrightarrow TX|_C \longrightarrow \nu_{C/X} \longrightarrow 0$$

where $\nu_{C/X}$ is the normal bundle. So $c_1(TX|_C) = c_1(TC) + c_1(\nu_{C/X})$. Pair this with $[C] \in H_2(C;\mathbb{Z})$,

$$c_1(TX) \cdot [C] = \underbrace{c_1(TC) \cdot [C]}_{\chi(C) = 2-2g(C)} + c_1(\nu_{C/X}) \cdot [C].$$

The last term is $[C]^2$, argued as follow: the self-intersection number is the signed number of intersection points of C and a small (smooth) pushoff of C, say C'. We can identify C with the zero section of the normal bundle and C' a small generic section of $\nu_{C/X}$. □

Recall for oriented X^4, $H^2(X;\mathbb{R})$ has symmetric non-degenerate cup product pairing, so gives rise to *signature* $\sigma(X) = b_+ - b_-$. We state without proof

Theorem 3.10 (Hirzebruch signature theorem). *Let X^4 be an almost complex manifold. Then $c_1(X)^2 = 2\chi(X) + 3\sigma(X)$. In particular this is a topological invariant.*

Fact: $c_1^2 = \sigma \pmod 8$.

Corollary 3.11. *If X^4 admits an almost complex structure then $X\#X, X\#X\#X\#X$ etc do not admit almost complex structures.*

Proof. $1 - b_1 + b_+$ is even if X admits an almost complex structure. Now $b_\pm(X\#X) = 2b_\pm(X)$ etc. □

Local forms for symplectic manifolds

Theorem 3.12 (Moser stability). *If we have a smooth family of symplectic forms $\{\omega_t\}$ on a closed symplectic manifold M such that $[\omega_t] \in H^2_{\mathrm{dR}}(X)$ is constant, then there is a diffeomorphism $f : M \to M$ such that $f^*\omega_1 = \omega_0$.*

Thus if we deform a symplectic form smoothly the symplectic structure remains unchanged.

Remark.

1. Small deformation of symplectic form matters: let $M = \mathbb{CP}^1 \times \mathbb{CP}^1, \Omega_1 = \omega \oplus \omega, \Omega_2 = \omega \oplus t\omega, t > 1$. They are not in the same class.

   Gromov: there is a deformation retract $\mathrm{Symp}(M, \Omega_1) \simeq \mathrm{SO}(3) \times \mathrm{SO}(3)$. On the other hand $\pi_1 \mathrm{Symp}(M, \Omega_2)$ is infinite.

2. A blowup of $S^2 \times T^2 \times S^2 \times S^2$ has two symplectic forms which are cohomologous but there does not exist a diffeomorphism pulling back one to the other.

3. A note on closedness: there exist exotic symplectic structures on $\mathbb{R}^{2n}$ for $n \geq 2$, i.e. no symplectic embedding into $(\mathbb{R}^{2n}, \omega_{\mathrm{std}})$.

Proof. We are going to show there is an isotopy $\varphi_t : M \to M, \varphi_0 = \mathrm{id}, \varphi_1 = f$ and $\varphi_t^*\omega_t = \omega_0$. It's equivalent to looking for a vector field $X_t \in \mathrm{Vect}(M)$ whose flow is φ_t. Recall that the vector field is defined by.

$$\frac{d}{dt}\varphi_t = X_t \circ \varphi_t.$$

Differentiate the relation $\varphi_t^*\omega_t = \omega_0$,

$$0 = \frac{d}{dt}(\varphi_t^*\omega_t) = \varphi_t^*(\frac{d}{dt}\omega_t + \mathcal{L}_{X_t}\omega_t) = \varphi_t^*(\frac{d}{dt}\omega_t + \mathrm{d}\iota_{X_t}\omega_t + \iota_{X_t}\mathrm{d}\omega_t). \qquad (*)$$

Since $[\omega_t]$ is fixed, $\frac{d}{dt}\omega_t = \mathrm{d}\sigma_t$ where $\sigma_t \in \Omega^1(M)$ is smooth. Thus $(*)$ is equivalent to $\mathrm{d}\sigma_t + \mathrm{d}\iota_{X_t}\omega_t = 0$. Thus it is enough to show $\sigma_t + \iota_{X_t}\omega_t = 0$, which unqiuely determines X_t as ω_t is nondegenerate. □

Slogan for local form theorems: check condition infinitesimally, integrate on small neighbourhood using vector field/Moser type argument.

Theorem 3.13 (Darboux). *If $p \in (M^{2n}, \omega)$, there is a local chart $f : D_\varepsilon(0) \to M$ with $f(0) = p$ such that $f^*\omega = \omega_0$. In other words, symplectic manifolds are locally standard.*

Proof. Fix a chart $h : D_r(0) = D \to M$ around p. Both $h^*\omega$ and ω_0 are symplectic forms on D. Postcomposing an element of $\mathrm{Sp}_{2n}(\mathbb{R})$, wlog $h^*\omega$ and ω_0 agree at p. Let $\omega_t = (1-t)\omega_0 + th^*\omega$ be the linear interpolation. Being nondegenerate is an open condition, so there is an open neighbourhood $U \subseteq D$ containing 0 on which ω_t is symplectic for $t \in [0,1]$. $H^2(D) = 0$ so

$$\frac{d}{dt}\omega_t = -\omega_0 + h^*\omega = \mathrm{d}\sigma$$

for some $\sigma \in \Omega^1(D)$. wlog σ vanishes at 0. Let X_t be the vector field defined by $\iota_{X_t}\omega_t = -\sigma$. Note X_t vanishes at 0, so for some $\varepsilon > 0$ the trajectory of every $x \in D_\varepsilon(0) \subseteq U$ under the flow of X_t stays inside U (and is defined) up to at least time 1. Thus on $D_\varepsilon(0)$ we can integrate X_t to $\{\psi_t\}_{t\in[0,1]}$ such that $\psi_1^* h^*\omega = \omega_0$. □

Remark. This shows that closed nondegenerate 2-form is the same as atlas of charts with transition functions whose derivatives are in $\mathrm{Sp}_{2n}(\mathbb{R})$.

Theorem 3.14 (Poincaré lemma). *Suppose M is a C^∞ manifold, $Q \subseteq M$ closed smooth submanifold, $\alpha_1, \alpha_2 \in \Omega^k(M)$ agreeing on $TM|_Q$. Then exists $\beta \in \Omega^{k-1}(U_Q)$, where U_Q is an open neighbourhood of Q, such that $\mathrm{d}\beta = \alpha_1 - \alpha_2$ and $\beta = 0$ on $TM|_Q$.*

Proof. Example sheet 2. Use partition of unity. □

Note. If $Q \subseteq (M, \omega)$ is a symplectic submanifold. Then

$$\nu_{Q/M} \cong (TQ)^\omega \subseteq TM|_Q$$

as a subbundle. On each fibre $V \cong W \oplus W^\omega$ for $W \leq V$. Moreover $\nu_{Q/M} = (TQ)^\omega$ is a symplectic vector bundle.

Theorem 3.15. *If $Q \subseteq (M, \omega)$ is a compact symplectic submanifold, the symplectic structure of M near Q is determined by $\omega|_Q$ and $\nu_{Q/M}$ as symplectic vector bundle, i.e. if two submanifolds have the same data then they have symplectomorphic tubular neighbourhoods.*

Proof. Choice of a metric gives $\exp : \nu_{Q/M} \to U_Q$ defined for $|t| < \varepsilon$. If $Q_i \subseteq M_i$ symplectic, $\varphi : Q_1 \to Q_2$ a symplectomorphism then it lifts to $\Phi : \nu_{Q_1/M_1} \to \nu_{Q_2/M_2}$, an isomorphism of symplectic bundles. Using metric to get $\tilde{\varphi} : \exp_{M_2} \circ \Phi \circ \exp_{M_1}^{-1} : U_{Q_1} \to U_{Q_2}$. ω_1 and $\tilde{\varphi}^*\omega_2$ are symplectic forms on U_{Q_1} which agree on $TM|_{Q_1}$ so by Poincaré lemma exists $\sigma \in \Omega^1(U'_{Q_1})$ which vanish on TM_{Q_1} and such that $\mathrm{d}\sigma = \omega_1 - \tilde{\varphi}^*\omega_2$ on U'_{Q_1}. Now apply Moser's method to the path $\omega_t = t\omega_1 + (1-t)\tilde{\varphi}^*\omega_2$. Similar to the proof of Darboux, we can integrate the vector field to time 1 on a sufficiently small neighbourhood of Q_1. □

Corollary 3.16. *A neighbourhood of a closed symplectic surface (i.e. real 2 dimensional) $C \subseteq (M^4, \omega)$ is determined by*

- *topological type of C (i.e. genus), $\int_C \omega$ (these determine C as symplectic manifold).*
- *$[C]^2$ (determines $\nu_{C/M}$).*

Definition. If (M^{2n}, ω) is symplectic, $L \subseteq M$ is *Lagrangian* if $\dim_{\mathbb{R}} L = n$ and $\omega|_L = 0$, i.e. $i^*\omega = 0$ where $i : L \hookrightarrow M$.

Example.

1. $S^1 \subseteq \Sigma$ a surface is Lagrangian as $\Omega^2(S^1) = 0$.
2. $(S^1)^n \subseteq (\mathbb{C}^n, \omega_0)$, the Clifford torus. Note that the radius of S^1's can be arbitrarily small so by Darboux, these exist in any symplectic manifold.

Cotangent bundle $\mathbb{R}^{2n} = T^*\mathbb{R}^n$. Let q_j be coordinates on $\mathbb{R}^n$, $p_j = \mathrm{d}q_j$. Together they give coordinates on $\mathbb{R}^{2n}$. Together they give a 1-form

$$\lambda = \sum p_j \mathrm{d}q_j \in \Omega^1(T^*\mathbb{R}^n)$$

and $\mathrm{d}\lambda = \omega_0$. A diffeomorphism $q \mapsto q'$ of $\mathbb{R}^n$ induces a diffeomorphism of $T^*\mathbb{R}^n$ via pullback which preserves λ: $\sum p_j \mathrm{d}q_j \mapsto \sum p'_j \mathrm{d}q'_j$, where p'_j are dual to q'_j. (It doesn't work in general). For a manifold, patch λ's together to get $\lambda_{\mathrm{can}} \in \Omega^1(T^*X)$ with $\mathrm{d}\lambda_{\mathrm{can}}$ symplectic.

Coordinate-free description: for $v \in T_{(q,p)}T^*X$,

$$\lambda_{\mathrm{can}(q,p)}(v) = \langle p, D\pi(v) \rangle$$

where $\pi : T^*X \to X$.

Exercise. If $\sigma : M \to T^*M$ is a 1-form then

$$\sigma^*\lambda_{\mathrm{can}} = \sigma$$

where on LHS we interpret σ as a morphism and on RHS we treat it as an element of $\Omega^1(M)$.

Note $M \subseteq (T^*M, \mathrm{d}\lambda_{\mathrm{can}})$, the zero section, is Lagrangian. It turns out to be the prototype for Lagrangian submanifolds.

Theorem 3.17 (Weinstein tubular neighbourhood theorem)**.** *If $L \subseteq (M, \omega)$ is compact Lagrangian, then there is a tubular neighbourhood $U(L)$ of L in M which is symplectomorphic to a neighbourhood $U'(L) \subseteq T^*L$ of the zero section.*

Proof. Choose an ω-compatible almost complex structure J and let $g = \omega(\cdot, J\cdot)$. Note the g-orthogonal complement to $T_qL \subseteq T_qM$ is JT_qL. Let $\Phi : T^*M \to TM$ be induced by g: $g(\Phi(f), v) = f(v)$. Define

$$\begin{aligned} \varphi : T^*L &\to M \\ (q, f) &\mapsto \exp_q(J\Phi_q(f)) \end{aligned}$$

Check

$$D\varphi_{(q,0)}(v,f) = v + J\Phi_q(f)$$

for $(v,f) \in T_qL \oplus (T_qL)^* \cong T_{(q,0)}(T^*L)$.

$$\begin{aligned}(\varphi^*\omega_M)(_{(q,0)}((v,f),(v',f')) &= \omega_M|_q(v + J\Phi f, v' + J\Phi f') \\ &= \omega_M|_q(v, J\Phi f') - \omega_M(v', J\Phi f) \\ &= g|_q(v, \Phi f') - g|_q(v', \Phi f) \\ &= f'(v) - f(v') \\ &= (\mathrm{d}\lambda_{\mathrm{can}})_{(q,0)}((v,f),(v',f'))\end{aligned}$$

so $\varphi : T^*L \to M$ is such that $\varphi^*\omega_M$ and $\mathrm{d}\lambda_{\mathrm{can}}$ agree on the zero section, i.e. $T(T^*L)|_L$. The Poincaré lemma now gives $\sigma \in \Omega^1(T^*L)$ for which $\mathrm{d}\lambda_{\mathrm{can}} - \varphi^*\omega_M = \mathrm{d}\sigma$ on $U(L)$ an open neighbourhood of the 0-section, $\sigma = 0$ on 0-section. Apply Moser's trick to the family of forms $\omega_t = (1-t)\varphi^*\omega_M + t\mathrm{d}\lambda_{\mathrm{can}}$. □

Corollary 3.18. *A neighbourhood of a Lagrangian only depends on the smooth topology of L as a symplectic manifold.*

Exercise. Let $L \subseteq M^4$ be Lagrangian.

1. $\chi(L) = -[L]^2$ (because $T^*L \cong \nu_{L/M}$).
2. If $L \subseteq M$ is homologically trivial, i.e. $[L] = 0$, and L is connected closed then L is a torus or a Klein bottle.

Proposition 3.19. *If $f : M \to M$ is a diffeomorphism then $f^*\omega = \omega$ if and only if $\Gamma_f \subseteq (M \times M, \omega \oplus -\omega)$ is Lagrangian.*

Proof. $f^*\omega = \omega$ if and only if $f^*\omega - \omega = 0$ if and only if $i^*(\omega \oplus -\omega) = 0$. □

Example. The antidiagonal Γ_{id} is Lagrangian.

Proposition 3.20. *If $f : M \to M$ is antisymplectic, i.e. $f^*\omega = -\omega$ then* $\mathrm{Fix}(f)$*, the fixed points of f, is Lagrangian where smooth.*

Example. Complex conjugation on $\mathbb{C}^n$ or $\mathbb{CP}^n$ is antisymplectic. The fixed points are $\mathbb{R}^n$ and $\mathbb{RP}^n$ respectively.

More generally if a quasi-affine or projective smooth variety $X(\mathbb{C})$ (smooth submanifold of $\mathbb{C}^n$ or $\mathbb{CP}^n$ cut out by polynomial equations) is defined over $\mathbb{R}$, then $X(\mathbb{R}) \subseteq X(\mathbb{C})$ is a Lagrangian submanifold where smooth.

Corollary 3.21. *A neighbourhood of* $\mathrm{id} \in \mathrm{Symp}(M)$ *where M compact is homeomorphic to a neighbourhood of* 0 *in the space of closed 1-forms on M. In particular* $\mathrm{Symp}(M)$ *is locally path connected.*

Proof. If f is close to id then $\Gamma_f \subseteq M \times M$ is close to the Lagrangian antidiagonal. By Weinstein $\Gamma_f \subseteq U$ for $U \subseteq T^*M$ a neighbourhood of the zero section of M. Moreover the proof of Weinstein shows Γ_f gives a section of T^*M (near the 0-section). This means that $\Gamma_f \subseteq T^*M$ can be thought of as the graph of a 1-form, say σ.

$$\sigma^* \mathrm{d}\lambda_{\mathrm{can}} = \mathrm{d}\sigma^* \lambda_{\mathrm{can}} = \mathrm{d}\sigma$$

so Γ_f is Lagrangian if and only if σ is closed. □

Corollary 3.22. *Suppose (M, ω) is compact and $H^1_{\mathrm{dR}}(M) = 0$. Then any $f \in \mathrm{Symp}(M)$ which is C^1 close to id has at least 2 fixed points.*

Proof. If f is C^1 close to id then $\Gamma_f \subseteq T^*M$ can be written as the graph of a closed 1-form σ (C^1 is enough. See proof of Weinstein). But $\sigma = \mathrm{d}h$ as $H^1_{\mathrm{dR}}(M) = 0$. $p \in \mathrm{Fix}(f)$ if and only if $\mathrm{d}h_p = 0$, if and only if $p \in \mathrm{Crit}(h)$. M is compact so h has at least 2 critical points, a minimum and a maximum. □

Proposition 3.23. *Suppose (M, ω) is connected. Then $\mathrm{Symp}(M, \omega)$ acts transitively on points.*

Proof. M is path-connected so enough to work in a single Darboux chart. Let $x \in \mathbb{R}^{2n}$. Translation from 0 to x is symplectic and is the time 1 flow of the constant vector field $X = 0x$. Now need to cut out a chart, which we do by passing to forms. Let $\sigma = \iota_X \omega_0$. Then

$$\mathrm{d}\sigma = \mathrm{d}\iota_X \omega_0 = \mathcal{L}_X \omega_0 = 0$$

so σ is exact as $H^1(\mathbb{R}^{2n}) = 0$. Say $\sigma = \mathrm{d}f$ for $f \in C^\infty(\mathbb{R}^{2n})$. Now pick a suitable cut-off function ψ and replace f with ψf, X with Y such that $\iota_Y \omega_0 = \mathrm{d}(\psi f)$. □

Stengthening of Darboux chart.

Theorem 3.24 (Gromov-Lees)**.** *There is a Lagrangian immersion of L into $\mathbb{C}^n \cong \mathbb{R}^{2n}$ if and only if $TL \otimes_{\mathbb{R}} \mathbb{C} \cong L \times \mathbb{C}^n$ as complex vector bundles.*

Proof. We prove only if and the other direction is beyond the scope of the course so omitted. Note $V \leq (\mathbb{C}^n, \omega_0)$ is a Lagrangian subspace if and only if $V \perp iV$ (with respect to standard metric) and $\dim_{\mathbb{R}} V = n$ (recall this is also the observation we used for Weinstein). An immersion $\iota : L \to \mathbb{C}^n$ is Lagrangian if and only if $\mathrm{Im}(D\iota_x) \perp i\,\mathrm{Im}(D\iota_x)$ for all $x \in L$. This gives a map

$$\begin{aligned} T_x L \otimes \mathbb{C} &\to \mathbb{C}^n \\ v \otimes (a + ib) &\mapsto aD\iota_x(v) + ibD\iota_x(v) \end{aligned}$$

on each fibre and varying over x induces the trivialisation.

The other direction is hard. Uses h-principle. □

Proposition 3.25. *If $W \subseteq (M, \omega)$ is an isotropic submanifold, i.e. $\omega|_W = 0$, then a neighbourhood of W is determined symplectically by the smooth topology of W and the bundle $TW^\perp/TW$ (which is trivial in Lagrangian* *case).*

Proof. Example sheet 2. □

Lemma 3.26. *Suppose $W \looparrowright X$ is an isotropic immersion and* $\dim W < \frac{1}{2}\dim M$ *then isotropic perturbation of W will be embedded generically.*

Proof. General position argument: flow one branch of W by a compactly supported vector field near individual self-intersection points to remove them locally.

Slogan: stengthen a smooth perturbation to be isotropic. □

Corollary 3.27. *If L compact has a Lagrangian immersion in $\mathbb{C}^n$ then $L \times S^1$ embeds in $\mathbb{C}^{n+1}$.*

Note by Gromov-Less this is in fact if and only if.

Proof. Have isotropic immersion $L \looparrowright \mathbb{C}^n \times \mathbb{C} = \mathbb{C}^{n+1}$. This can be perturbed to get an isotropic embedding $L \hookrightarrow \mathbb{C}^{n+1}$. Note TL^ω/TL is trivial and hence we get a symplectic embedding of an open neighbourhood of $L \subseteq T^*L \times \mathbb{C}$. This containes a Lagrangian $L \times S^1$ by taking the radius of S^1 to be sufficiently small. □

Fact: every compact orientable 3-manifold Y is parallelisable. (proof uses characteristic class)

Corollary 3.28. *If Y^3 is compact orientable then we have a Lagrangian immersion $Y \looparrowright \mathbb{C}^3$ and a Lagrangian embedding $Y \times S^1 \hookrightarrow \mathbb{C}^4$.*

Remark. Any compact three manifold can be Lagrangian immersed in $\mathbb{C}^3$. On the other hand if L^4 compact is Lagrangian immersible into $\mathbb{C}^4$ then by Gromov-Rees $\chi(L) = 0$ so $b_1 > 0$. Thus $\pi_1(L)$ is infinite.

How to remove double point?

Proposition 3.29. *Suppose $M \supseteq L_1, L_2$ contains 2 Lagrangian submanifolds which meet transversally at a point p. Then there is a Darboux chart $\varphi : B(\varepsilon) \to M$ such that $\varphi(0) = p, \varphi^{-1}(L_1) = \mathbb{R}^n \cap B(\varepsilon), \varphi^{-1}(L_2) = i\mathbb{R}^n \cap B(\varepsilon)$.*

Proof. Exercise. □

Polterovich surgery Polterovich surgery replaces $L_1 \cup L_2$ with L_γ, local on a neighbourhood of p.

Let $\gamma : \mathbb{R} \to \mathbb{C}$ smooth, coincide with $\mathbb{R}_+\{0\} \cup \{0\}\{0\} \times \mathbb{R}_-$ away from the origin.

$$L_\gamma = \{z_j = \gamma a_j : (a_j) \in S^{n-1}\}$$

where z_j is the complex coordinate, $a_j \in S^{n-1} \cap \mathbb{R}^n \subseteq \mathbb{C}^n$. This is a Lagrangian handle.

Example. For $n = 1$, $L_\gamma = \{z = \gamma a : a \in S^0 = \{\pm 1\}\}$. Away from a neighbourhood of 0, L_γ agrees with $L_1 \cup L_2$.

For $n = 2$, suppose $\gamma = (\gamma_1(t), \gamma_2(t)) \in \mathbb{R}^2 \cong \mathbb{C}, S^1 = (\cos\theta, \sin\theta)$. Then

$$L_\gamma = \{(\gamma_1 \cos\theta, \gamma_2 \cos\theta, \gamma_1 \sin\theta, \gamma_2 \sin\theta)\} \subseteq \mathbb{R}^4 \cong \mathbb{C}^2.$$

If $\gamma_1 = 0$ then we get $i\mathbb{R}^n$ minus a neighbourhood of 0. If $\gamma_2 = 0$ then we get $\mathbb{R}^n$ minus a neighbourhood of 0.

This depends on the ordering of L^1 and L^2 (corresponding to the canonical order $\mathbb{R}^n, i\mathbb{R}^n$), doesn't depend on the choice of γ, as long as γ agrees with half-axes outside the ball and $\gamma \cap (-\gamma) = \emptyset$.

Corollary 3.30. *If L_1, L_2 are Lagrangian submanifolds which meet transversally at a point, there is an embedding Lagrangian submanifold diffeomorphic to the connected sum $L_1 \# L_2$.*

General case: can perform surgery separately at any finite number of transverse intersections of Lagrangians. Topologically, each surgery replaces $B^n \amalg B^n$ with $\mathbb{R} \times S^{n-1}$.

Example. If $L^n \looparrowright M^{2n}$ Lagrangian with a single double point, get Lagrangian embedding of $L \# (S^1 \times S^{n-1}) \hookrightarrow M$.

Corollary 3.31. *If Y is a compact orientable 3-manifold then for some $k \geq 0$ there is a Lagrangian embedding of $Y \# k(S^1 \times S^2)$ into $\mathbb{C}^3$.*

Highly restrictive

Definition (prime manifold)**.** A closed manifold M^n is *prime* if it can't be written as $M = M_1 \# M_2$ unless M_1 or M_2 is S^n.

Theorem 3.32 (Fukaya)**.** *A compact prime orientable 3-manifold has a Lagrangian embedding in $\mathbb{C}^3$ if and only if and only if it is diffeomorphic to $S^1 \times \Sigma_g$.*

Fix a primitive θ of ω_0 in $\mathbb{C}^n$, i.e. $\mathrm{d}\theta = \omega_0$. If L is Lagrangian then $\omega_0|_L = 0$ so $[\theta|_L] \in H^1(L)$. (By Stoke's this measures the symplectic area of disc with boundary on L). Define L to be exact if $[\theta|_L] = 0$.

Theorem 3.33 (Gromov)**.** *There is no compact exact Lagrangian in $\mathbb{C}^n$.*

3.6 Symplectic submanifolds

Theorem 3.34 (Gromov)**.** *Fix symplectic manifold (V, ω_V) compact and (X, ω_X) with $\dim V \leq \dim X - 4$. Suppose we are given a smooth embedding $f : V \hookrightarrow X$ such*

- *$f^*[\omega_X] = [\omega_V] \in H^2(V)$,*
- *Df is symplectic through bundle maps $TV \to TX$ to a fibrewise symplectic embedding.*

The f is smoothly isotopic to an embedding $\tilde{f} : V \to X$ such that $\tilde{f}^*\omega_X = \omega_V$.

Proof. Omitted. This is an instance of h-principle. □

This fails in general for codimension 2 but

Theorem 3.35 (Donaldson)**.** *If $[\omega] \in H^2(X;\mathbb{Z})$ then for $k \gg 0$ there are symplectic submanifolds representing $PD(k[\omega])$.*

Idea: $[w] \in H^2(X;\mathbb{Z})$ represents a complex line bundle $\pi : L \to X$ with $c_1(L) = [\omega]$. If $s : X \to L$ is a section of π, cleanly intersecting 0-section, then

$$s^{-1}(0) = PD[c_1(L)] = PD[\omega].$$

(By Sard's theorem, $s^{-1}(0)$ is a $(2n-2)$ dimensional manifold). If we use instead $L^{\otimes k}$ then $s^{-1}(0) = PD(k[\omega])$. Donaldson's idea is to construct, for sufficiently large k, sections which are "approximately holomorphic" (they satisfy Cauchy-Riemann equations up to an error term). The error term is small enough that $s^{-1}(0)$ is symplectic.

c.f. ample/very ample line bundles in algebraic geometry.

$$\begin{aligned} \sigma : X &\to \mathbb{P}(H^0(X, L^{\otimes k})^*) = \mathbb{P}^N \\ x &\mapsto [\varphi_x : x \mapsto s(x)] \end{aligned}$$

σ is injective for $k \gg 0$. If that case, $\mathbb{P}^{N-1} \cap \sigma(X)$ represents $PD(kc_1(L))$.

3.7 Blow-ups

Definition (blow-up)**.** The blow-up of 0 in $\mathbb{C}^n$ is

$$Z = \{(z, \ell) \in \mathbb{C}^n \times \mathbb{P}^{n-1} : z \in \ell\}$$

together with two projections $\pi : Z \to \mathbb{C}^n, p : Z \to \mathbb{P}^{n-1}$.

π is one-to-one away from 0 and over 0 the fibre is $\mathbb{P}^{n-1}$. $p : Z \to \mathbb{P}^{n-1}$ is the tautological bundle. $c_1 = -PD[\mathbb{P}^{n-2}]$.

Note. Our choice of Kähler form on $\mathbb{P}^n$ is normalised so that $\omega(\mathbb{P}^1) = \pi$.

Define

$$\omega_\lambda = \pi^*\omega_{\mathbb{C}^n} + \lambda^2 p^*\omega_{\mathbb{P}^{n-1}}.$$

Lemma 3.36. *For $\lambda > 0$, ω_λ is Kähler and for $\delta > 0$, let*

$$Z(\delta) = \{(z, \ell) \in Z : |z| \leq \delta\}.$$

Then $(Z(\delta) \setminus Z(0), \omega_\lambda)$ is symplectomorphic to $(B(\sqrt{\lambda^2 + \delta^2}) \setminus B(\lambda), \omega_0)$.

Proof. Recall our definition of Fubin-Study metric: for the natural projection $\Phi : \mathbb{C}^n \setminus \{0\} \to \mathbb{P}^{n-1}$, have

$$\Phi^*\omega_{\mathbb{P}^{n-1}} = \frac{i}{2}\partial\overline{\partial}\log|z|^2.$$

Let

$$\mu_\lambda = \frac{i}{2}\partial\overline{\partial}(|z|^2 + \lambda^2 \log |z|^2).$$

On $Z(\delta) \setminus Z(0)$, $\pi^*\mu_\lambda = \omega_\lambda$. Define a bijection

$$\begin{aligned} F : \mathbb{C}^n \setminus \{0\} &\to \mathbb{C}^n \setminus B(\lambda) \\ z &\mapsto \frac{z}{|z|}\sqrt{|z|^2 + \lambda^2} \end{aligned}$$

Note $F^*\omega_0 = \mu_\lambda$. Using the fact that ω_0 is Kähler, easy to get plush for μ_λ so Kähler. □

Definition (blowup)**.** The *weight λ blowup* of (M, ω) at p is a symplectic embedding $\varphi : B(\sqrt{\lambda^2 + \delta^2}) \hookrightarrow M, \varphi(0) = p$, and

$$\widetilde{M} = (M \setminus \operatorname{im} \varphi) \cup Z(\delta)$$

with $\tilde{\omega}_M = \omega_{M \setminus \operatorname{im} \varphi} \cup \omega_\lambda$.

Note.

1. $\mathrm{Vol}(\tilde{M}) = \mathrm{Vol}(M, \omega) - \mathrm{Vol}(B(\lambda))$ so the volume decreases under blowup.
2. $[\tilde{\omega}_\lambda] = \pi^*(\omega) - \pi\lambda^2 PD(E) \in H^2_{\mathrm{dR}}(M)$ where $E = Z(0) \cong \mathbb{P}^{n-1}$ is the exceptional divisor.

3.8 Fibre sums

Lemma 3.37. *If $(Q^{2n-2}, \omega_Q) \hookrightarrow (M^{2n}, \omega_M)$ is a closed symplectic submanifold of M then $(Q^{2n-2} \times B(\varepsilon), \omega_q \oplus \omega_0) \hookrightarrow (M^{2n}, \omega_M)$ symplectically if and only if $\nu_{Q/M}$ is symplectically trivial (i.e. $c_1(\nu_{Q/M}) = 0$).*

Already proved!
Consider

$$\begin{aligned} \psi : B(\varepsilon) \setminus \{0\} &\to B(\varepsilon) \setminus \{0\} \\ (r, \theta) &\mapsto (\sqrt{\varepsilon^2 - r^2}, -\theta) \end{aligned}$$

(turn inside out and then flip orientation). Check

$$\psi^*\omega_0 = \psi^*(r\mathrm{d}r \wedge \mathrm{d}\theta) = r\mathrm{d}r \wedge \mathrm{d}\theta.$$

Thus ψ is area preservaing and “turns the annulus inside out”.

Suppose $Q^{2n-2} \hookrightarrow M_i^{2n}$ symplectically, $i = 1, 2$, Q closed symplectic submanifold such that ν_{Q/M_i} is symplectically trivial. We can define

Definition (fibre sum)**.** The *fibre sum* of M_1 and M_2 along Q is

$$M_1 \#_Q M_2 = (M_1 \setminus Q) \cup_{\mathrm{id} \times \psi} (M_2 \setminus Q)$$

where $\mathrm{id} \times \psi : M_1 \setminus Q \supseteq Q \times B(\varepsilon)^* \to Q \times B^*(\varepsilon) \subseteq M_2 \setminus Q$.

Note. As ν_{Q/M_i} is trivial, $Q \hookrightarrow M_1 \# M_2$ symplectically.

More generally, suppose $Q \hookrightarrow M_i$ symplectically and $c_1(\nu_{Q/M_1}) = -c_1(\nu_{Q/M_2})$, then we can define $M_1 \#_Q M_2$ completely analogously.

Note that on each local chart of Q we use $\mathrm{id} \times \psi$. These patch together to give a symplectomorphism

$$\begin{array}{ccc} \nu_{Q/M_1} \setminus \{0\} & \longrightarrow & \mathcal{L} \setminus \{0\} \\ \downarrow & & \downarrow \\ Q & \longrightarrow & Q \end{array}$$

for some complex line bundle to be determined. This flips the signs of intersections of section and zero section, so $c_1(\mathcal{L}) = -c_1(\nu_{Q/M_1})$.

Remark. There is a hidden choice: the *framing* of Q needn't be unique. Fo example if $\nu_{Q/M}$ is trivial, the choices of symplectic trivialisations of $\nu_{Q/M}$ are in one-to-one correspondence with $H^1(Q;\mathbb{Z})$ ($\nu_{Q/M}$ is trivial and choices of trivialisation up to homotopy is $[Q, U(1)] = [Q, S^1] = H^1(Q;\mathbb{Z})$). More generally, maps $\nu_{Q/M} \hookrightarrow U(Q)$ can differ (up to homotopy) be an element of $H^1(Q;\mathbb{Z})$.

Example.

1. Suppose $X^4 \supseteq E$ where E a symplectic sphere of self-intersection number -1. Pick $Y^4 \supseteq S^2$ symplectic sphere of self-intersection number $+1$, for example $H \subseteq \mathbb{P}^2$ a hyperplane. Then we can form $X^4 \#_E \mathbb{CP}^2$. This is called the *blowdown* of X along E.

   Note if $\widetilde{W}$ is the blowup of W^4 at p then $\widetilde{W} \#_E \mathbb{CP}^2 \cong W$.

2. Recall if $C \subseteq \mathbb{P}^2$ smooth degree d curve then $[C] = d[\mathbb{P}^1]$, $[C] \cdot [C] = d^2, g(C) = \frac{(d-1)(d-2)}{2}$. Suppose $X^4 \supseteq Q$ symplectic sphere of self-intersection -4, $\mathbb{CP}^2 \supseteq C$ curve of degree 2. Then can form $X^4 \#_{Q/C} \mathbb{CP}^2$.

   Claim $\mathbb{CP}^2 \setminus C \cong D^*_\varepsilon(\mathbb{RP}^2)$, (truncated) cotangent bundle of Lagrangian $\mathbb{RP}^2$ (becuas given $C \subseteq \mathbb{CP}^2$ of genus 0, exists Lagrangian $\mathbb{RP}^2$ which doesn't intersect it. Check "no other topology")

   Replcaed a symplectic sphere with a Lagrangian $\mathbb{RP}^2$. This is a purely symplectic operation — we can't usually do this in the algebraic geometry world.

3.9 Lefschetz principles

Definition (Lefschetz pencil)**.** A *Lefschetz pencil* on a closed oriented 4-manifold X is a smooth map $f : X \setminus \{b_1, \ldots, b_k\} \to \mathbb{P}^1$ such that

- Df is onto except at a finite collection of points $\{p_1, \ldots, p_m\} \subseteq X \setminus \{b_1, \ldots, b_k\}$,
- Near the b_i (p_j respectively) there are central local complex coordinates $z, w \in B_\varepsilon(0) \subseteq \mathbb{C}$ such that $f(z, w) = \frac{z}{w}$ ($f(z, w) = z^2 + w^2$ respectively)

The b_i's are called *base points*, p_i's *critical points*.

To get such a form, we use the complex Morse lemma

Lemma 3.38. *If $U \subseteq \mathbb{C}^n$ is an open subset, $f : U \to \mathbb{C}$ homomorphic maps 0 to 0 with a non-degenerate critical point at 0. Then exists local coordiantes $z_1, \ldots, z_n$ such that $f(z_1, \ldots, z_n) = \sum_{i=1}^n z_i^2$.*

Compare with the real Morse lemma which says that we can find coordinates such that $f(x_1, \ldots, x_n) = \sum_{i=1}^p x_i^2 - \sum_{i=p+1}^n x_i^2$. In the complex case we can run the same proof and absorb -1 into the coordinates.

Suppose $t \in \mathbb{P}^1$ is not a critical value of f. Then $\overline{f^{-1}(t)} \subseteq X$ (gluing back b_i), the fibre of t is smooth. Around each critical point p_j of the pencil, the equation for a fibre looks like $\{z^2 + w^2 = 0\}$, i.e. two complex lines intersecting transversally. p_j is also called *ordinary double point* or *node*.

Note all smooth fibres are closed, orientable and have fixed genus (pick a path avoiding the critical points, then the fibre varies smoothly).

Suppose X^4 compact Kähler, $L \to X$ a very ample holomorphic line bundle. For the purpose of this course, this means a holomorphic embedding $i : X \hookrightarrow \mathbb{P}^N$ uch that $i^*\mathcal{O}(1) = L$, where $O(1)$ is the line bundle such that $c_1(\mathcal{O}(1)) = PD[\mathbb{P}^{N-1}]$. Key fact: the restriction of a generic hyperplane $\mathbb{P}^{N-1}$ gives $s^{-1}(0) \cap X$ for some holomorphic sections s of L.

Let s_1, s_2 be generic holomorphic sections. Then we can define a rational map $\pi : X \to \mathbb{P}^1, p \mapsto [s_1(p) : s_2(p)]$. $\{s_1 = s_2 = 0\}$ is a set of finitely many points. These are the base points of p. Among critical points of a holomorphic functions, non-degenerate ones are generic, so by complex Morse lemma we have correct local forms near critical points.

What's the local model near $b \in \{s_1 = s_2 = 0\}$? (s_1, s_2) give local holomorphic coordintes

Example (pencil of cubics in $\mathbb{P}^2$)**.** Define

$$\begin{aligned} \mathbb{P}^2 &\to \mathbb{P}^1 \\ [x : y : z] &\mapsto [x^3 + y^3 + z^3 : x^3 + y^3 + z^3 + xyz] = [f : g] \end{aligned}$$

f and g are sections of $\mathcal{O}(3)$. The fibre

$$\pi^{-1}([\lambda : \mu]) = \{\mu f - \lambda g = 0\} \subseteq \mathbb{P}^2$$

is given by a cubic equation (so torus if smooth). The base points are given by $\{f = g = 0\}$, i.e. $x^3 + y^3 + z^3 = 0, xyz = 0$, which is a set of 9 points (this can also be seen by $[C] \cdot [C'] = 9$ for C, C' cubic). Explicitly, they are given by $[0 : 1 : \xi^i]$ and there cyclic permutations, where $\xi^3 = -1$.

When is the set $\{\mu f - \lambda g = 0\} \subseteq \mathbb{P}^2$ singular? either $\mu = \lambda$, so $xyz = 0$ three coordinate lines, 3 critical points. If $\mu \neq \lambda$, this can be rephrased as $\{x^3 + y^3 + z^3 + axyz = 0\}$. The critical points are given by the zeroes of all three partial derivatives. We get $xyz = 0$ (already seen) or $a^3 + 27 = 0$. Thus we get 3 more cricial fibres $a = 3\xi^i$. We can factorise $x^3 + y^3 + z^3 - 3xyz$, which is again three lines.

In either case, there are three base points on each line.

(pic)

Near a base point, maps $(z, w) \mapsto \frac{z}{w}$ (this is the direction of the line). Then we get an honest map to $\mathbb{P}^1$ by blowing up each of the base points:

$$\pi : E(1) = \mathbb{CP}^2 \# q\overline{\mathbb{CP}^2} \to \mathbb{P}^1$$

$E(1)$ is the *rational elliptic surface*.

- Smooth fibre C with $C \cdot C = 0$ inside $E(1)$ (fibration is locally trivial so can push C of itself).
- $\pi_1(E(1)) = 0$.
- Example sheet 3: $\pi_1(E(1) \setminus C) = 0$.

Theorem 3.39. *Suppose X^4 connected closed oriented is the total space of a Lefschetz pencil with at least one basepoint. Then X admits a symplectic structure.*

Proof. We first outline the general strategy. It is enough to construct a symplectic form on $\tilde{X}$ obtained by blowing up basepoints. More precisely, near basepoint b, we have local holomorphic coordinates (z, w) such that $\pi : X \to \mathbb{P}^1$ is given by $(z, w) \mapsto \frac{z}{w}$. Replace $D_\varepsilon(0)$ with $Z(\varepsilon)$ (local blowup model) at every basepoint to get $\tilde{X}$. Now π induces a well-defined map $\pi : \tilde{X} \to \mathbb{P}^1$. Each basepoint is replaced with exceptional divisor E ($\cong \mathbb{P}^1$), which gives a section of π.

We will construct a symplectic form on $\tilde{X}$ such that each smooth fibre is symplectic (ditto symplectic fibre away from critical points). We'll also see that for any $N \geq 0$, $\omega + N\pi^*\omega_{\mathbb{P}^1}$ is also symplectic. Taking N sufficiently large ensures that the sections coming from the basepoints are all symplectic. Now blow each of these down using fibre connected sum to get a symplectic form on X.

Working on $\tilde{X}$. Claim we can find $\eta \in \Omega^2(\tilde{X})$ closed such that

- $\eta|_F$ is symplectic on neighbourhood of any smooth point points of a fibre.
- near a critical point p_i, $\eta|_{U_i(p_i)} = \frac{i}{2}(\mathrm{d}z\mathrm{d}\overline{z} + \mathrm{d}w\mathrm{d}\overline{w})$, where $U_i(p_i)$ open in $\tilde{X}$ (not the fibre), in local coordinates.

We postpone the proof. Let $\omega = \eta + k\pi^*\pi_{\mathbb{P}^1}$. Claim for $k \gg 0$, ω is symplectic. This automatically gives us the sought after form.

Proof of step 3. Near a smooth point $T_x\tilde{X} = \ker D\pi \oplus (\ker D\pi)^\eta$. $\ker D\pi = T_xF$ is called the vertical subspace, and the symplectic complement is called a choice of horizontal space. The matrix form for $\eta + k\pi^*\omega_{\mathbb{P}^1}$ is

$$\begin{pmatrix} \eta|_F & 0 \\ 0 & \eta|_{\text{hor}} + k\pi^*\omega_{\mathbb{P}^1} \end{pmatrix}$$

Since $\tilde{X}$ is compact, we can choose k sufficiently large such that the this is symplectic.

Near singular points, the local model is $\pi : \mathbb{C}^2 \to \mathbb{C}, (z, w) \mapsto z^2 + w^2$. Check

$$(\omega_{\text{std}} + k\pi^*\omega_{\mathbb{P}^1})^2 = (1 + k|(z, w)|^2)\,\text{Vol} > 0$$

so symplectic. □

Proof of step 2. Model fibre F. Pick a symplectic 2-form $\sigma \in \Omega^2(F)$ with area 1. Claim exists $\xi \in \Omega^2(\tilde{X})$ closed such that $\int_{F'} \xi = 1$ for all fibre F'. This is Poincaré duality: the expression says $[\xi] \cap [F'] = 1$. This requires $[F] \neq 0$, which is OK if $\#\{b_j\} > 0$ as $E \pitchfork F = \{pt\}$, E section.

Pick open sets $\tilde{X}$, $\{U_\alpha\}_{\alpha \in A}, \{V_i\}_{i \in I}$ such that

- $U_\alpha \cong F \times B_\alpha \to B_\alpha$ for some balls $B_\alpha \subseteq \mathbb{P}^1$ away from the critical points.
- $V_i \cong B_i^4 \to \mathbb{C}(z^2 + w^2)$ near cricical points.
- On U_α, $f : F \times B_\alpha \to F$, $f^*\sigma - \xi = \mathrm{d}\lambda_\alpha$ for some $\lambda_\alpha \in \Omega^1(U_\alpha)$.
- On V_i, $\pi : B_i^4 \to \mathbb{C}$ smooth fibre, locally $z^2 + w^2 = a$, $a \neq 0$. Standard symplectic form restricts to a symplectic form. $\mathrm{d}\theta_i = \omega_{\mathrm{std}}$.

Pick partitions of unity subordinate to $\pi(\{U_\alpha\} \cup \{V_i\})$, say $\{\varphi_\alpha, \psi_i\}$. Set (?)

$$\eta = \xi - \mathrm{d}(\varphi_\alpha \circ \lambda_\alpha) + d((\psi_i \circ \pi)\theta_i) \cdot N$$

□

□

As a corollary

Theorem 3.40. *If (X^4, ω) is symplectic and $[\omega] \in H^2(X, \mathbb{Z})$ then for all $k \gg 0$ there exists a Lefschetz pencil X with fibre Poincaré dual to $k[\omega]$.*

Proof. Omitted. □

Thus every integral symplectic 4 manifold is a Lefschetz pencil.

3.10 Grompf's theorem

Theorem 3.41 (Grompf). *If $G = \langle g_1, \dots, g_n | r_1, \dots, r_\ell \rangle$ is a finitely presented group then there exists (Y_G, ω) a symplectic 4 manifold with $\pi_1 Y_G = G$.*

This is really saying the symplectic category is big enough: the topological version is a basic result in algebraic topology. Symplectic 2 manifolds are surfaces so symplectic 4 manifolds are the simplest objects that the theorem could possibly hold.

Proof. The proof strategy is as follow. From example sheet 3 $\pi_1(E(1) \setminus T) = 0$ and $c_1(\nu_{T(E(1))}) = 0$. We'll use this to kill off generators in a bigger space. Consider $(T^2 \times \Sigma_g, \omega_{T_2} \oplus \omega_{\Sigma_g})$. Find in here disjoint symplectic tori T_i's and construct the fibre sum

$$M = (T^2 \times \Sigma_g) \#_{T_1} E(1) \#_{T_2} E(1) \#_{T_3} \dots \#_{T_k} E(1)$$

and then $\pi_1 M = \pi_1(T^2 \times \Sigma_g) / \langle \pi_1(T_i) \rangle$.

We first find disjoint tori. Let $M = T^2 \times \Sigma_g$, regarded as a fibration with base Σ_g (pic). Then

$$\pi_1(M) = \langle u \rangle \times \langle v \rangle \times \langle a_1, b_1, \dots, a_g, b_g | \prod [a_i, b_i] \rangle.$$

Candidate tori: given loops $\alpha_1, \dots, \alpha_k \in \Sigma_g$ and loops $u_1, \dots, u_k$. Then $\alpha_i \times u_i \subseteq M$ are (Lagrangian) tori.

Step 1: can we find enough tori so that $\pi_1(M)/\langle \pi_1 T_i \rangle$ is the free group?

First let $T_0 = T$ a torus fibre. It is an honest symplectic torus and will kill off $\langle u \rangle \times \langle v \rangle$. Next consider tori given by $u_i \times b_i$. This will kill off generators b_i in $\pi_1 M$. Claim the resulting space has free fundamental group: killing off b_i is the same as adding handlebodies to the "holes" on Σ_g and the resulting space is homotopic to g loops attached to S^2, whose fundamental group is indeed F_g.

For a finitely generated group $G = F_g/\langle r_1, \dots, r_\ell \rangle$, for each r_i we find a loop $\alpha_i \in \Sigma_g$ of a_i which spell out the relation. We run into a problem: α_i's may not be embedded (as a simple example, a_1^2 in Σ_1). We can resolve the problem by adding more generators: for example $F_1/\langle a_1^2 \rangle = F_2/\langle a_1 a_2, a_1 a_2^{-1} \rangle$ (pic). Note that it does not matter if these two loops intersect, as we can move them off to different fibres.

So now we should be convinced that there are many Lagrangian tori $T_i \subseteq T^2 \times \Sigma_g$ such that $\pi_1(T^2 \times \Sigma_g)/\pi_1(T_i) = G$.

Remains to show exists $\omega \neq \omega_{T^1} \oplus \omega_{\Sigma_g}$ making T_i symplectic. Oftentimes being Lagrangian is a "closed" condition on the space of symplectic forms. For example $S^2 \subseteq T^* S^2$ is certainly Lagrangian. Consider $T\mathbb{P}^1$ which is homeomorphic to $T^* S^2$. $T\mathbb{P}^1$ has symplectic form compatible with standard complex structure, and $\mathbb{P}^1$ is complex so symplectic.

Claim exists ω making the T_i's symplectic.

Exists a closed form $\beta \in \Omega^2(M, \mathbb{R})$ such that the pullback $i^*[\beta] \in \Omega^2(T_i^2, \mathbb{R})$ is positive. Note that β is in the same class as the usual symplectic form but may not itself by symplectic.

Idea of proof: take forms s_1, s_2 Poincaré dual to loops U_1, α_1 with $T^2 = u_1 \times a_i$. Then $\beta = s_1 \wedge s_2$.

Suppose β is such a form on M, WTS we can modify this form so its pullback is symplectic. Since T_i's are Lagrangian, we know that their neighbourhood is trivial. Let $j : B_\varepsilon(0) \times T^2 \hookrightarrow M$ be a neighbourhood of T_i. The pullback $j^*(\beta)$ on $B_\varepsilon(0) \times T^2$ is in the same cohomology class of $p_{T^2}^* \omega_{T^2}$ (?). By abuse of notation, $\beta, \omega_{T^2} \in \Omega^2(B_\varepsilon(0) \times T^2)$ with $[\beta] = [\omega_{T^2}]$.

Let $\eta = \Omega^1(B_\varepsilon \times T^2)$ with $\mathrm{d}\eta = \omega_{T^2} - \beta$. Pick $\rho : B_\varepsilon(0) \to \mathbb{R}$ which is 1 in neighbourhood of 0 and 0 in a neighbourhood of the boundary. Look at the forms

$$\tilde{\beta} = \rho \omega_{T^2} + (1 - \rho)\beta - \mathrm{d}\rho \wedge \eta$$

where the last term is presente to make sure $\tilde{\beta}$ is closed:

$$\mathrm{d}\tilde{\beta} = \mathrm{d}\rho \omega_{T^2} - \mathrm{d}\rho \wedge \beta - \mathrm{d}\rho \wedge \mathrm{d}\eta = 0.$$

Substitute β on M with $\tilde{\beta}$. Take ω for $T^2 \times \Sigma_g$ to be $\omega_{T^2} \oplus \omega_{\Sigma_g} + \varepsilon\beta$ for ε small. This is nondegenerate since it is an open condition. □

Rcall that a Lefschetz pencil is a map $\pi : (\tilde{X}, \omega, J) \setminus \{b_i\} \to \mathbb{P}^n$ that is

1. J-holomorphic,

2. at the crtical points of the map exists honestly holomorphic coordinates such that $\pi(z_1, z_2) = z_1^2 + z_2^2$.

Goal: understand Lagrangian submanifolds in Lefschetz pencilss. For today let $F_p = \pi^{-1}(p)$.

Idea: suppose $\ell \subseteq F_p$ is a Lagrangian and $\gamma \subseteq \mathbb{P}^1$ is a Lagrangian, can we create L in $\tilde{X}$ based on this data? For example if $\tilde{X} = \Sigma_g \times \mathbb{P}^1$ then any product $\ell \times \gamma$ is Lagrangian. We expect this to work locally in the general case.

First claim $(F_p, \omega|_{F_p})$ is symplectic. Let $i : F_p \hookrightarrow \tilde{X}$. To check closedness

$$\mathrm{d}i^*\omega = i^*\mathrm{d}\omega = 0.$$

To check nondegeneracy, note that F_p is a complex submanifold. Take $v \in TF_p$. Then

$$\omega(v, Jv) = g_J(v, v) > 0$$

so ω is nondegerate. Thus the Lagrangian in the fibre component always makes sense. What about the other?

Observe that at $x \in \tilde{X}$, the tangent spaces splits as

$$T_x\tilde{X} = \ker D\pi \oplus (\ker D\pi)^\omega.$$

As a result $\tilde{X} \to \mathbb{P}^1$ carries a connection. Notation: given $\gamma : [0,1] \to \mathbb{P}^1 \setminus \{\text{critical values of } \pi\}$, let $f_\gamma : F_{\gamma_0} \to F_{\gamma_1}$ be the induced parallel transport map.

Theorem 3.42. *f_γ is a symplectomorphism.*

Proof. Need to show that if $v \in T_x\tilde{X}$ is a horizontal vector then $\mathcal{L}_V\omega$ vanishes on the fibre.

$$\mathcal{L}_V\omega = \iota_V\mathrm{d}\omega + \mathrm{d}\iota_V\omega = \mathrm{d}\eta$$

with $\eta = \iota_V\omega$. η vanishes on the vertical tangent space $\ker D\pi$ as

$$\eta(w) = \omega(v, \omega) = 0.$$

Write $\eta = \pi^*\alpha$ for α a 1-form on $\mathbb{P}^1$. Then $\mathrm{d}\eta(v_1, v_2) = 0$ if $v_1, v_2 \in \ker D\pi$. □

Corollary 3.43. *If $\ell \subseteq F_{\gamma(0)}$ is a Lagrangian submanifold for $\omega|_{F_{\gamma(0)}}$ then $\ell \times I \subseteq \tilde{X}$, given by the parallel transport along a curve γ, is a Lagrangian submanifold.*

Observation: if $\gamma : [0,1] \in \mathbb{P}^1$ with $\gamma(1) \in \mathrm{CritVal}(\pi)$, there is still a map from $F_{\gamma(0)} \to F_{\gamma(1)}$.

Definition (vanishing path)**.** A *vanishing path* is a path $\gamma : [0,1] \to \mathbb{P}^1$ with $\gamma(t) \in \mathrm{CritVal}(\pi)$ if and only if $t = 1$.

The *vanishing cycle* is $V_{\gamma(0)} = f_\gamma^{-1}(p)$ for $p \in \mathrm{Crti}(\pi)$, where f_γ is the parallel transport map.

Proposition 3.44. *V_p is a sphere.*

Proof. Look at local model near the critical point: $\mathbb{C}^2 \to \mathbb{C}, (z_1, z_2) \mapsto z_1^1 + z_2^2$. The regular fibre is $\{z_1^2 + z^2 = a\}$, $a \neq 0$ and the singular fibre is the union of two lines. Observe that $\pi(\mathbb{R} \times \mathbb{R}) = \mathbb{R}$. $\mathbb{R} \times \mathbb{R}$ is Langrangian and therefore the image is under parallel transport. Thus $V_p = \{z_1^2 + z_2^2 = 1\}, z_1, z_2 \in \mathbb{R}$. Note that we can increase the dimension of fibration $z_1, z_2, z_3, \ldots$. □

Since parallel tarnsport along curves gives symplectomorphisms of fibres, a loop γ passing through a gives $f_\gamma : F_a \to F_a$. Claim if γ is contractible in $\mathbb{P}^1 \setminus \mathrm{CritVal}(\pi)$ then f_γ is Hamiltonian isotopy.

Question: what happens to F_a if γ travels around a critical point? We study an easier picture first: consider a Lefschetz fibration $\mathbb{C} \to \mathbb{C}, z \mapsto z^2$. Then over the regular value 1 sits two fibres, and the parallel transport induces the (only) nontrivial symplectomorphism that transposes them. (pic)

Consider next $\mathbb{C}^2 \to \mathbb{C}$. The symplectomorphism twists the hyperboloid.

Now consider the projection z_2

In general

Theorem 3.45. *There exists a symplectomorphism of $T^*S^n \to T^*S^n$ fixing the boundary, τ_{S^n} the Dehn twist.*

Idea: geodesic flow on T^*S^n is a symplectomorphism which fixes S^n. Time π flow is antipodal.

For $n \geq 2$, $\tau_{S^n}^2$ is isotopic to identity but not symplectic isotopic.

4 J-holomorphic curves

Let (X, ω) be a symplective manifold and fix J a compatible almost complex structure.

Definition (J-holomorphic curve)**.** A *J-holomorphic curve* $f : \Sigma \to X$ is a triple (Σ, j, f) where (σ, j) is a smooth real 2-manifold equipped with a complex structure j, and $f \in C^\infty(\Sigma, X)$ such that $Df \circ j = J \circ Df$ or equivalently, $Df + J \circ Df \circ j = 0$.

Remark. In real dimension 2, all almost complex structure are complex.

Remark.

1. $Df(T_x\Sigma) \subseteq F_{f(x)}X$ is J-invariant.
2. f needn't be an embedding.
3. J-holomorphic curves are parameterised (we care about f rather than merely about $\operatorname{im} f$.

Lemma 4.1. *If $f : \Sigma \to X$ is J-holomorphic then its image is a symplectic submanifold at the smooth points and*

$$\int_{f(\Sigma)} \omega = \int_\Sigma f^*\omega \geq 0.$$

Proof. At a smooth point $f(x) \in f(\Sigma)$, $J(T_{f(x)}f(\Sigma)) = T_{f(x)}f(\Sigma)$. As ω is J-compatible, this is nondegenerate

Basis for $T_{f(x)}f(\Sigma)$ is v, J_v for some $v \neq 0$ and $\omega(v, Jv) \geq 0$. □

In local coordinates, suppose $x + iy$ is a local complex coordiante on Σ, the J-holomorphic condition says

$$(\frac{\partial f}{\partial x} + J\frac{\partial f}{\partial y})\mathrm{d}x + (\frac{\partial f}{\partial y} - J\frac{\partial f}{\partial x})\mathrm{d}y = 0$$

or equivalently,

$$\partial_x f + J\partial_y f = 0.$$

This is called the *generalised Cauchy-Riemann equation*, or $\overline{\partial}$-equation.

Definition (energy)**.** Suppose (X, ω, J) is a symplectic manifold with a compatible almost complex structure, with associated metric $g_J = |\cdot|_J$. Let $u : \Sigma \to X$ be smooth where Σ is a surface. $\mathrm{d}u \in \Omega^1(\Sigma, u^*(T^*X))$. The *energy* of u is

$$E(u) = \frac{1}{2}\int_\Sigma |\mathrm{d}u|_J^2 \mathrm{d}\,\mathrm{Vol}_\Sigma$$

where for $L : T_z\Sigma \to T_{u(z)}X$,

$$|L|_J = |\xi|^{-1}\sqrt{|L(\xi)|_J^2 + |L(j\xi)|_J^2}$$

where $0 \neq \xi \in T_z\Sigma$.

Note. $E(u)$ depends on

- metric g_J on X,
- the complex structure on Σ (but not the volume form itself). More compactly,$|\alpha|^2 \mathrm{d}\,\mathrm{Vol}_\Sigma = -\alpha \wedge (\alpha \circ j)$.

Lemma 4.2.
$$E(u) = \int_\Sigma \frac{1}{2} |\overline{\partial}_J u|_J^2 \mathrm{d}\,\mathrm{Vol}_\Sigma + \int_\Sigma u^*\omega$$
where $\overline{\partial}_J u = Du + J \circ Du \circ j$.

Proof. Use conformal coordinates $x + iy$ on Σ,
$$\begin{aligned} |\mathrm{d}u|^2 \mathrm{d}\,\mathrm{Vol} &= (|\partial_x u|^2 + |\partial_y u|^2)\mathrm{d}x \wedge \mathrm{d}y \\ &= (|\partial_x u + J\partial_y u|^2 + 2\langle \partial_x u, J\partial_y u\rangle)\mathrm{d}x \wedge \mathrm{d}y \\ &= |\partial_J u|^2 \mathrm{d}\,\mathrm{Vol} + 2u^*\omega \end{aligned}$$
as J is ω-compatible. □

First properties of J-holomorphic curves:

1. local existence: there are always locally defined J-holomorphic curves through any point p of an almost complex structure. (In fact morally there is a one-to-one correspondence with curves through $0 \in \mathbb{C}^n$).
2. unique continuation: if $f, g : \Sigma \to X$ are J-holomorphic curves and agree to infinite order (as smooth maps) at a point $p \in \Sigma$ then $f = g$.
3. corollary of energy identity: J-holomorphic curves are energy minimising within their cohomology class.
4. Positivity of intersection for almost complex submanifolds.

   If C is a J-holomorphic curve and $Y \subseteq X$ is a codimension 2 submanifold which is also J-holomorphic. If $C \cap Y$ is discrete and nonempty, each intersection point has a positive sign, i.e. if the intersection is nonempty then $[C] \cdot [Y] > 0$ (could be infinite).

   Proof. Pick local almost-complex coordinates near an intersection point so that $Y = \{(z_1, \ldots, z_{n-1}, 0)\} \subseteq X$. Local analysis analogous to the proof of 5 (stated below) tells us that C is locally given by $z \mapsto (f(z), az^k + O(|z|^{k+1}))$ for some $a \neq 0$. Perturb this to $z \mapsto (f(z), az^k + \varepsilon + O(|z|^{k+1}))$. This gives transverse intersection points, each of which is positive as they are locally modelled on complex intersection points. □

   Corollary: (X^4, ω) compat with J a compatible almost complex structure. Suppose $u : C \to X$ is J-holomorphic, $[u(C)] \cdot [u(C)] < 0$. Any other holomorphic curve $v : C' \to X$ representing the same cohomology class must have the same image as u (it's given by a holomorphic reparameterisation of u).

5. Let Σ be closed. If $u : \Sigma \to X$ is J-holomorphic nonconstant then both $\{z \in \Sigma | du_z = 0\}$ and $u^{-1}(\mathrm{Crit}(u))$ are finite.

   Proof. Work in a coordinate patch. Let $u : D \to \mathbb{C}^n$ is J-holomorphic for $J : D \to \mathrm{End}_{\mathbb{R}}(\mathbb{C}^n)$. wlog $u(0) = 0, Du_0 = 0$, i.e. 0 a critical point and $J(0)$ is the action of i on $\mathbb{C}^n$. We've assumed that u is non-constant so exists $\ell \geq 2$ such that $u(z) = O(|z|^\ell)$ near 0, $u(z) \neq O(|z|^{\ell+1})$. In this step we're using unique continuation property to say that all derivatives can't be zero. $J(u(z)) = i + O(|z|^\ell)$ near 0. Let's look at Taylor expansion, say $T_\ell u$, the ℓth order expansion. Taking the Taylor expansion of the Cauchy-Riemann equation

   $$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = 0,$$

   we obtain

   $$\frac{\partial(T_\ell u)}{\partial s} + i\frac{\partial(T_\ell u)}{\partial t} = 0$$

   so $T_\ell u$ is holomorphic in the standard sense. By Taylor's theorem $T_\ell u(z) = cz^\ell$ for some $c \neq 0$, so $u(z) = cz^\ell + O(|z|^{\ell+1})$. Thus 0 is an isolated critical point. Finiteness follows from closedness of Σ. □

6. Two versions

   - Monotonicity theorem: suppose $u : (D, 0) \to (B^{2n}(r), 0)$ is J-holomorphic for $(B^{2n}(r), \omega_0, J)$ for some J compatible, and $u(\partial D) \subseteq \partial B^{2n}(r)$. Then $\mathrm{area}(u(D)) \geq \pi r^2$, where the area is $\int_{u(D)} \omega$.

     We will not prove this theorem. This is a special case of the monotonicity theorem for minimal surfaces (which are energy minimising surfaces). We say last time that J-holomorphic curves are minimal.
   - Weaker version: same statement with $\mathrm{area}(u(D)) \geq \frac{\pi}{4} r^2$.

     Proof. Assume we have such a u. Let $S_t = u^{-1}(B_0^{2n}(t))$. Let $a(t) = \mathrm{area}(S_t)$, $\ell(t) = \mathrm{length}(f|_{\partial S_t})$. We have
     - $a'(t) = \ell(t)$ almost everywhere. This part does not use any property of J-holomorphicity.
     - $a(t) \leq \frac{\ell(t)^2}{\pi}$. Proof later.

     Putting these together

     $$\frac{d}{dt}\sqrt{a(t)} = \frac{a'(t)}{2\sqrt{a(t)}} = \frac{\ell(t)}{2\sqrt{a(t)}} \geq \frac{\sqrt{\pi}}{2}.$$

     Integrating gives $\sqrt{a(t)} \geq \frac{\sqrt{\pi}}{2} t$.

     Proof of second claim. $u(\partial D)$ bounds a 2-complex C which can be made arbitrarily close to a union of "flat 2-simplicies" (pieces of linear subplanes). These can be approximated by union of small flat discs. For a disc of radius ε, $\pi\varepsilon^2 = \frac{(2\pi\varepsilon)^2}{4\pi}$ so area$C \leq \frac{\ell(\partial D)^2}{\pi}$. By Stokes'

     $$\mathrm{area}(u(D)) = \int_D f^*\omega = \int_C \omega \leq \int_C 1 = \mathrm{area}(C)$$

     because $\|\omega\|_p \leq 1$ for any $p \in B^{2n}(0)$. □

□

7. Reomoval of singularity: given J-holomorphic map $f : D^* \to X$ of finite energy, f extends J-holomorphically to D.

   Sketch: can use monotonicity to get continuous extension (argue that $f(D^*)$ must have a unique limit as $x \to 0$, as else the energy would be infinite).

   > **Definition.** A J-holomorphic curve $f : \Sigma \to X$ is *simple* if it isn't multiplely covered, i.e. it doesn't factor as $\Sigma \to \Sigma' \to X$ where $\Sigma \to \Sigma'$ is a nontrivial branched cover.

   Corollary of 5: if $f : \Sigma \to X$ is J-holomorphic, Σ closed and f simple then the set of injective points, i.e. $\{x \in \Sigma : Df_x \neq 0, f^{-1}(f(x)) = \{x\}\}$, is open and dense. In fact, non-injective points are at most countable, and can only accummulate at critical points.

8. Suppose Σ_0, Σ_1 closed, $u_j : \Sigma_j \to M$ J-holomorphic simple such that $u_0(\Sigma_0) = u_1(\Sigma_1)$, then exists a biholomorphic $u : \Sigma_0 \to \Sigma_1$ such that $u_0 = u_1 \circ \varphi$. "J-holomorphic curves with same energy as the same up to reparameterisation".

Note for 6B: area is calculated using the Riemannian metric g_J, and for J-holomorphic curves it is the same as $\int_C \omega$.

For a general (X, g) Riemannian, say g has injectivity radius $r \geq 0$ if $s < r$, $\gamma \subseteq B_x(r)$ closed C^∞ loop, then γ bounds a disc of area $A_\gamma \leq \frac{\ell(\gamma)^2}{\pi}$.

Another remark: we used $u : (D, \partial D) \to (B^{2n}(r), \partial B^{2n}(r))$ where J on the codomain is only required to be compatible. If instead we use the standard complex structure then we can do better using standard isoperimetric inequality: $u(\partial D) = (u_1(t), \dots, u_n t) \in \mathbb{C}^n, t \in S^1$, can fill them with $C_i \subseteq \mathbb{R}^2$ such that $\text{area}(c_i) \leq \frac{\ell_i^2}{4\pi}$, and we can actually get the stronger inequality.

4.1 Moduli space of J-holomorphic curves

Let (X, ω, J) be a symplectic manifold with a fixed compatible almost complex structure J. Let (Σ, j) closed.

> **Definition.** For $A \in H_2(X, \mathbb{Z})$, define the moduli space of J-holomorphic curves to be
>
> $$\mathcal{M}_J(A, X) = \{J\text{-holomorphic curves } \Sigma \to X \text{ such that } [\Sigma] = A\}.$$

Fact: in favourable circumstances (J is regular), $\mathcal{M}_J(A, X)$ is a manifold of real dimension

$$n(2 - 2g) + 2\langle c_1, A\rangle$$

where $g = g(\Sigma), 2n = \dim_{\mathbb{R}} X, c_1 = c_1(X)$.

> **Theorem 4.3.** *If X is compact, A is simple in homology (i.e. primitive) then regular J's are dense in the space of all almost complex structures.*

Let S_0, S_1 be Riemann surfaces diffeomorphic to S^2. They are biholomorphic, in constrast to elliptic curves. $\mathbb{CP}^1$ admits a $\mathrm{PSL}_2(\mathbb{C})$-action by reparameterisation. If $A = [\Sigma]$ where $\Sigma \cong S^2$, $\mathrm{PSL}_2(\mathbb{C})$ acts on $\mathcal{M}_J(X, A)$ so we can form the quotient $\overline{\mathcal{M}}_J(X, A)$. For regular J, $\overline{\mathcal{M}}_J(A, X)$ is a manifold of dimension $2(n - 3 + \langle c_1, A\rangle)$.

- If $X = \mathbb{P}^2, A = [\mathbb{CP}^1]$ the the dimension is $2(2 - 3 + 3) = 4$. This agrees with the fact that the Grassmannian of lines in $\mathbb{P}^2$ is isomorphic to $\mathbb{P}^2$, which has real dimension 4.
- $X = \mathbb{P}^2, A = 2[\mathbb{CP}^1]$ (conic, still isomorphic to $\mathbb{CP}^1$): dimension $2(2 - 3 + 6) = 10$. c.f. conic through 5 points in generic position.

Example of non-regular behaviour: X^4 Kähler, $C \cap X^4$ smooth holomorphic curves. Take local coordinate z on C. Blow up at points $0, t$ on C (in X). If $t \neq 0$ get two exceptional divisors E_1, E_2. If $t = 0$ then we blow up twice. Symplectically they are the same: $\mathbb{P}^2 \# 2\overline{\mathbb{P}}^2$, wlog with the same symplectic form. The construction for varying t gives a 1-(complex) parameter family of J's on $\mathbb{P}^2 \# 2\overline{\mathbb{P}}^2$, say J_2. There are two curves for $t = 0$, lies in class $E_1 = E_2$. However for $t \neq 0$, no connected smooth holomorphic curve C in that class: suppose not, $(E_1 + E_2) \cdot E_1 = -1$, so would get a contradiction to positivity of intersection unless $E_1 \subseteq C$ or $C \cap E_1$ (impossible), similar for E_2. Thus J_0 is not regular for the class $E_1 + E_2$.

Note: blowing up $p \in C^2 \subseteq X^4$ gives $\pi : \tilde{X} \to X$, $\pi^{-1}(C) = \tilde{C} + E$ where $\tilde{C}$ is the strict transform of C, $E \cong \mathbb{CP}^1$ is the exceptional divisor. Then $[\tilde{C}]^2 = [C]^2 - 1$.

Question: suppose we have a sequence of points in $\mathcal{M}_J(X, A)$. When do we have convergence?

Example: consider the family

$$\begin{aligned} \mathbb{P}^1 &\to \mathbb{P}^2 \\ [x : y] &\mapsto [x^2 : y^2 : txy] \end{aligned}$$

The iamge is $\{t^2 XY = Z^2\}$. If $t \neq 0, \infty$ then have smooth conic. As $t \to \infty$, it becomes $\{XY = 0\}$. This is an example of *bubbling*.

Example. More bubbling: consider

$$\begin{aligned} f_n : \mathbb{P}^1 &\to \mathbb{P}^1 \times \mathbb{P}^1 \\ z &\mapsto (z, \frac{1}{zn^2}) \end{aligned}$$

Away from 0, $|Df_n|$ is bounded and f_n's coverge uniformly on compact sets. As $n \to \infty$, $\{|z| \leq \frac{1}{n}\}$ collapses to a point. However, $\mathrm{im}\{f_n\}$ converges pointwise to $\mathbb{P}^1 \times \{0\} \cup \{0\} \times \mathbb{P}^1$.

> **Definition** (bubble)**.** A J-holomorphic curve $g : \mathbb{P}^1 \to X$ occurs as a *bubble* in a seuqnece $f_n : \Sigma \to X$ of J-holomorphic curves if for some $p \in \Sigma$, exist holomorphic charts $\{\varphi_n : B(R_n) \to \Sigma\}_{R_n \to \infty}$ such that for all $z \in \mathbb{C}$, $\varphi_n(z) \to p$ (for $n \gg 0$) on Σ and $f_n \circ \varphi_n \to g$ uniformly on compact sets of

$\mathbb{C}$.

Theorem 4.4. *If $f_n : \Sigma \to X$ is a sequence of J_n-holomorphic curves (some sequence $J_n \to J$ compatible almost complex structures) with compact image, then either some subsequence coverges to a J-holomorphic map, or there is bubbling.*

Note. Each bubble splits off its symplectic area's worth of energy (recall that the energy is zero if and only if the J-holomorphic curve is constant.

Definition. We say that a class $\alpha \in H^2(X, \mathbb{Z})$ has *least area* if $\omega(\alpha) > 0$ and there does not exist $\beta \in H_2(X, \mathbb{Z})$ such that $0 < \omega(\beta) < \omega(\alpha)$.

If α has least area then any bubble "eats" at least $\omega(\alpha)$ energy, there are $\leq \omega(\Sigma)/\omega(\alpha)$ bubbles, so

Corollary 4.5. *If X has a class of least area, any sequence in $\mathcal{M}_J(X, \beta)$, for any β, can only have finitely many bubbles.*

Theorem 4.6. *Let X^{2n} be a closed symplectic manifold. Suppose $\alpha \in H_2(X, \mathbb{Z})$ has least symplctic area. Then exists a dense collection of compatible almost complex structures J which are regular for α. In this case $\mathcal{M}_J(X, \alpha)$ is a compact manifold (every sequence of nets converges).*

Fact: the dnese collection is path-connected.
Idea for bubbling theorem:

- If there is a uniform bound on $|Df_n|$ then there is a convergent subsequence of f_n's by Arzela-Ascoli.
- Could fail to converge if exists points $p_n \in \Sigma$ with $|Df_n(p_n)| \to \infty$. Since Σ is compact, $p_n \to p$, say. Slogan: energy concentrates near p, i.e. bubble at p.

Sketch application: conisder $\mathrm{ev} : \mathcal{M}_J(\mathbb{P}^2, [H]) \to \mathrm{Sym}^2(\mathbb{P}^2)$ given by evaluation at two fixed points on $\mathrm{im}\, \mathbb{P}^1$. Note by adjunction the only J-holomorphic curves in $[H]$ are isomorphic to $\mathbb{P}^1$. For the standard J (which is regular), $[\mathrm{im}\, \mathrm{ev}] = [\mathrm{Sym}^2 \mathbb{P}^2]$. For another regular J, $[\mathrm{im}\, \mathrm{ev} : \mathcal{M}_J(\mathbb{P}^2, [H])]$ is unchanged as a cycle. In particular *mathrmev* must hit every point in $\mathrm{Sym}^2(\mathbb{P}^2)$ (if not, $\mathrm{im}\, \mathrm{ev}$ would not be fundamental class). Thus for a regular J (thought as the standard Kähler form deformed a bit), there is a J-holomorphic $\mathbb{P}^1$ between any two points.

4.2 Gromov non-squeezing

Recall Gormov non-squeezing theorem says if there is a symplectic embedding $B^{2n}(R) \hookrightarrow B^2(r) \times \mathbb{R}^{2n-2}$ then $r \leq R$.

First proof

Assume $B^{2n}(r) \hookrightarrow B^2(R) \times \mathbb{R}^{2n-2}$ is a symplectic embedding. We get $\overline{B}^{2n}(r_0) \hookrightarrow B^2(R) \times \mathbb{R}^{2n-2}$ for any $r_0 < r$. $\overline{B}^{2n}(r_0)$ has compact images so exists symplectic embedding $\overline{B}^{2n}(r_0) \hookrightarrow (\mathbb{P}^1, \Sigma) \times V$ where $\int_{\mathbb{P}^1} \omega = \pi R^2 + \varepsilon$ for some ε small and $V = \mathbb{R}^{2n-2}/L\mathbb{Z}^{2n-2}$ for L sufficiently large. We have $\varphi : B^{2n}(r_0) \hookrightarrow M = \mathbb{P}^1 \times V, 0 \mapsto (a,b)$. Blow up $\mathbb{P}^1 \times V$ by cutting out this ball and replacing it by a ball of any weight $\rho < r_0$ to get $(\tilde{M}, \omega_\rho)$. $u : \mathbb{P}^1 \to \mathbb{P}^1 \times V, x \mapsto (x,b)$ is a J-holomorphic curve through (a,b) on M. We can lift it to $\tilde{u} : \mathbb{P}^1 \setminus u^{-1}(a,b) \to \tilde{M}$ which is $\tilde{J}$-holomorphic automatically, where $\tilde{J}$ is picked so that it agrees with J away from the exceptional divisor E. Finiteness of energy/area means that we can apply removal of singularities to extend to a $\tilde{J}$-holomorphic map $\tilde{u} : \mathbb{P}^1 \to \tilde{M}$. By positivity of intersection, $[\tilde{u}(\mathbb{P}^1)] \cdot E > 0$. But

$$\pi R^2 + \varepsilon = \int_{\mathbb{P}^1} u^*\omega = \int_{\mathbb{P}^1} (\pi \circ \tilde{u})^*\omega = \underbrace{\int_{\mathbb{P}^1} \tilde{u}^*\omega_\rho}_{>0} + \pi\rho^2 \underbrace{[\tilde{u}(\mathbb{P}^1)] \cdot E}_{\geq 1}$$

so $\pi R^2 + \varepsilon > \pi \rho^2$ for ε arbitrarily small. Thus $r \leq R$.

Could use monotonicity to get bounds on areas of bubbule

***

Moduli space for non-primitive classes: define $M_J^*(X,A) \subseteq M_J(X,A)$ to be the subspace of curves which are simple (for example generic degree d in $\mathbb{CP}^2$ represents $d[H]$ and is simple. We have a version of the theorem before

> **Theorem 4.7.** *for $J \in \mathcal{J}$ where $\mathcal{J}$ is a subspace of the space of all compatible almost complex structures which is of the second category (i.e. contains an intersection of countably many dense open subsets), $M_J^*(X,A)$ is a smooth manifold of dimension $n(2-2g) + 2c_1(A)$, where $g = g(\Sigma)$ and $u : \Sigma \to X$. Such a J is "regular".*

4.3 Application of Gromov non-squeezing

> **Theorem 4.8** (Eliashberg)**.** *Let M be a symplectic manifold. Then* $\mathrm{Symp}(M)$ *is C^0 closed in* $\mathrm{Diff}(M)$*, i.e. the topology of uniform convergence of compact sets).*

> **Lemma 4.9.** *Volume (Lebesgue measure) is preserved under C^0 limit. This means that if $\{f_n\} \subseteq \mathrm{Symp}(M)$ and $f_n \to f$ in C^0 where $f \in \mathrm{Diff}(M)$ then f is volume preserving.*

Proof. Work near a point, wlog $0 \in \mathbb{R}^{2n}, Df|_0 = A$. Let $B(\delta) = B_\delta(0)$. $\frac{\mathrm{Vol}(f(B(\delta)))}{\mathrm{Vol}(B(\delta))} \to \det A$ as $\delta \to 0$. Now note f_n's are volume preserving and converge uniformly on compact sets to f, so $\det A = 1$. □

Proof of Theorem 4.8. This is a local result. Suppose we have a sequence of symplectic embeddings $B^{2n}(r) \hookrightarrow (\mathbb{R}^{2n}, \omega)$ such that $f_n \to f$ in C^0, $f \in C^\infty$. Need to show $A = Df|0 \in \mathrm{Sp}_{2n}(\mathbb{R})$. We will show $A^*\omega = \pm\omega$, and this is enought to show $A^*\omega = \omega$ by considering $f_n \times \mathrm{id} : B^{2n+2}(r) \to \mathbb{R}^{2n+2}$ which C^0 converges to $f \times \mathrm{id}$.

Assume $A^*\omega \neq \pm\omega$. Work with $A^\tau = -J_0 A^T J_0$ where J_0 is the standard complex structure. This is such that $\omega(Ax, y) = \omega(x, A^\tau y)$. Exist $u, v \in \mathbb{R}^{2n}$ such that $\omega(A^\tau u, A^\tau v) \neq \pm\omega(u, v)$. $|\det A| = |\det A^\tau| = 1$. We can assume

$$0 < \lambda^2 = |\omega(A^\tau u, A^\tau)| < \omega(u, v) = 1.$$

Fix signs, wlog $\omega(A^\tau u, A^\tau v) = \lambda^2$. Extend u, v to a basis $\mathcal{B}_1$ for $\mathbb{R}^{2n}$ with respect to which ω is given by the standard matrix. Similarly extend $\frac{A^\tau u}{\lambda}, \frac{A^\tau v}{\lambda}$ to a basis $\mathcal{B}_2$ for $\mathbb{R}^{2n}$ such that ω has the same matrix representation. Let $S \in \mathrm{Sp}_{2n}(\mathbb{R})$ be the linear map taking $\mathcal{B}_1$ to $\mathcal{B}_2$. Now with respect to $\mathcal{B}_1$, $A \circ S$ has matrix

$$\begin{pmatrix} \lambda & 0 & 0 \\ 0 & \lambda & \\ * & & * \end{pmatrix}$$

$A \circ S = D(f \circ S)_0$ and this takes $B^{2n}(\varepsilon) \hookrightarrow B^2(\lambda\varepsilon) \times \mathbb{R}^{2n-2}$. For sufficiently small ε, $f \circ S : B^{2n}(\varepsilon) \hookrightarrow B^2(\lambda\varepsilon) \times \mathbb{R}^{2n-2}$. Thus exists n such that $f_n \circ S : B^{2n}(\varepsilon) \hookrightarrow B^2(\lambda\varepsilon) \times \mathbb{R}^{2n-2}$, contradicting Gromov non-squeezing. □

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