University of
Cambridge

Mathematics Tripos

Part II

Riemann Surfaces

Michaelmas, 2018

Lectures by
H. Krieger

Notes by
Qiangru Kuang

Contents

1 Complex analysis & Branching/Multivalued functions

1.1 Holomorphicity

Definition (holomorphic/analytic function)**.** A smooth function $f : U \to \mathbb{C}$ from a domain (i.e. an open connected subset of $\mathbb{C}$) is *holomorphic* or *analytic* if either of the following holds:

1. f is differentiable in the sense of limits (which is equivalent to satisfying the Cauchy-Riemann equations),
2. for each $a \in U$, f has a power series expansion

$$f(z) = \sum_{n\geq 0} a_n (z-a)^n,$$

valid on some disk $D(a, r)$ with positive radius $r > 0$.

Remark. 1 implies 2 since f being differentiable allows us to construct a_n using Cauchy Integral Formula. 2 implies 1 since f having power series allows term-by-term differentiation.

By 2, if $a \in U$ and f is not identically 0 near a, then there exists some minimal $m \geq 0$ such that $a_m \neq 0$. It follows that $f(z) = a_m(z-a)^m(1+g(z-a))$ where $\lim_{z\to a} g(z-a) = 0$. Therefore for z sufficiently close to a, f is nonzero. This is known as

Theorem 1.1 (principle of isolated zeros)**.** *An analytic function on a domain U which is not identically zero has isolated zeros, i.e. around each $a \in U$, there exists a disk Δ_a on which $f(z) \neq 0$ unless possibly at $z = a$.*

If f is identically 0 near a, then there exists a disk Δ_a on which $f(z) = 0$ for all $z \in \Delta_a$. Consider $V := \bigcup_{a: f|_{\Delta_a}=0} \Delta_a$ and $W := \bigcup_{a: f\neq 0 \text{ near } a} \Delta_a$. V and W are open and disjoint so by connectivity of U, one of them is empty so $f = 0$ on U or has isolated zeros. Thus having isolated zero is a property of a domain, not a local property.

Corollary 1.2. *If f and g are analytic on U then either $f = g$ on U or $f(z) = g(z)$ on a discrete set.*

Definition (isolated singularity)**.** If f is analytic on the punctured disk $D(a, r)^* := D(a, r) \setminus \{a\}$ for some $r > 0$, then f has an *isolated singularity* at a.

In this case, we obtain the analogue of power series, *Laurent series* at a

$$f(z) = \sum_{n=-\infty}^{\infty} c_n (z-a)^n.$$

There are three possibilities:

1. removable singularity: $c_n = 0$ for all $n < 0$.
2. pole: there exists $N < 0$ such that $c_N \neq 0$ and $c_n = 0$ for all $n < N$. We say f has a pole of order $-N$ and can write $f(z) = (z-a)^N g(z)$ where g is analytic and nonzero at a.
3. essential singularity: $c_n \neq 0$ for infinitely many $n < 0$.

However, characterisation in terms of Laurent series is coordinate-dependent. Intrinsically, recall that

Theorem 1.3. *f has a removable singularity at a if and only if f is bounded on $D(a,r)^*$.*

Theorem 1.4 (Casorati-Weierstrass)**.** *f has an essential singularity at a if and only if for every punctured disk $D(a,r)^*$ in the domain of f, the image $f(D(a,r)^*)$ is dense in $\mathbb{C}$.*

For completeness sake, we state that f has a pole at a if and only if neither of the above happens (so $\lim_{z\to a} |f(z)| = \infty$).

This allows us, for example, to extend the definitions to infinity. Consider the Riemann sphere $\mathbb{C}_\infty$, on which a neighbourhood of infinity is the complement of a closed set not including ∞. Mapping it to the complex plane, we define a punctured disk around ∞ to be the complement of a closed disk in $\mathbb{C}$. Then we can talk conveniently about singularity at ∞.

Example. $f(z) = \frac{1}{e^z - 1}$ is meromorphic on $\mathbb{C}$ with poles at $z = 2\pi n i$ where $n \in \mathbb{Z}$. By considering $g(z) = \frac{z}{e^z-1}$ which has a removable singularity at 0, we know f has a pole of order 1 at 0, and therefore at all poles by periodicity.

At ∞, we have an essential singularity : along the imaginary axis, $|f(z)|$ can be arbitrarily big so it cannot be a removable singularity. Along the positive real axis, $|f(z)| \to 0$ so it cannot be a pole.

Definition (meromorphic function)**.** f is *meromorphic* on a domain $U \subseteq \mathbb{C}_\infty$ if it has only isolated singularities, none of which are essential.

1.2 Complex logarithm & Analytic continuation

Given nonzero $z = re^{i\theta}$, if $e^w = z$, we know that $w = \log r + (2\pi n + \theta)i$ for some $n \in \mathbb{Z}$. We can make a continuous choice of $\log z$ on, for example, $U = \mathbb{C} \setminus \mathbb{R}_{\geq 0}$, by choosing $0 < \theta < 2\pi$ and fixing some $n \in \mathbb{Z}$. This makes $f_n(z) := \log r + (2\pi n + \theta)i$ a well-defined continuous analytic function on U.

Note.

1. If $g : U \to V$ is an analytic bijection, then any inverse $h : V \to U$ is analytic.
2. If $g : U \to V$ is analytic, then any *continuous* inverse $h : V \to U$ is analytic.

More naturally,

Proposition 1.5. *Fix $n \in \mathbb{Z}$ and define $h(z) := \int_{-1}^{z} \frac{dw}{w} + (2n+1)\pi i$ for $z \in U$, where the integral is taken over the straight line from -1 to z, then h is analytic on U and inverse to $z \mapsto e^z$.*

Proof. First show h is analytic with $f'(z) = \frac{1}{z}$.

$$\frac{h(z+\tau) - h(z)}{\tau} = \frac{1}{\tau} \int_{z}^{z+\tau} \frac{dw}{w}$$

for τ sufficiently small (such that the triangle formed by -1, z and $z+\tau$ lies in U) by Cauchy's Theorem. Then

$$\left| \frac{1}{\tau} \int_{z}^{z+\tau} \frac{dw}{w} - \frac{1}{z} \right| = \left| \frac{1}{\tau} \int_{z}^{z+\tau} \frac{z-w}{zw} dw \right| \to 0$$

as $\tau \to 0$.

Now define $g(z) = \frac{e^{h(z)}}{z}$ so $g'(z) = \frac{ze^{h(z)}h'(z) - e^{h(z)}}{z}$ and so $g'(z) = 0$ identically. $g(-1) = 1$ so $e^{h(z)} = z$ for all $z \in U$.

□

Definition (direct analytic continuation)**.** A *function element* in a domain U is a pair (f, D) where D is a subdomain of U and f is an analytic function on D. Two function elements (f, D) and (g, E) are equivalent, write $(f, D) \sim (g, E)$ if $D \cap E \neq \emptyset$ and $f = g$ on $D \cap E$.

We say (g, E) is a *direct analytic continuation* of (f, D).

Why do we make such a definition? We know the power series

$$\sum_{r \geq 0} z^k = \frac{1}{1-z}$$

is defined on $D(0,1)$ and cannot be extended to any larger domain due to natural boundary. However, $\frac{1}{1-z}$ is holomorphic on $\mathbb{C} \setminus \{1\}$ so sometimes the domain forced by the definition of a function is not the maximal possible. In other words, sometimes we are looking at the "correct" function with a "wrong" domain.

Definition (analytic continuation along path)**.** We say (g, E) is an *analytic continuation of* (f, D) *along* γ if $\gamma : [0,1] \to U$ and there exist function elements (f_i, D_i), $i \in \{0, \dots, n\}$ and $0 = t_0 < t_2 < \dots < t_n = 1$ such that

$$(f, D) = (f_0, D_0) \sim (f_1, D_1) \sim \dots \sim (f_{n-1}, D_{n-1}) \sim (f_n, D_n) = (g, E)$$

and $\gamma([t_j, t_{j+1}]) \subseteq D_j$ for $j \in \{0, \dots, n-1\}$.

Write $(f, D) \approx_\gamma (g, E)$.

Remark. As $\mathbb{C}$ has a path-connected basis for the topology, domains are path-connected.

> **Definition** (analytic continuation). We say (g, E) is an *analytic continuation* of (f, D) if there exists a path γ such that $(f, D) \approx_\gamma (g, E)$. In this case we write $(f, D) \approx (g, E)$.

Remark.

1. If $(f, D) \approx_\gamma (g, E)$ and $(f, D) \approx_\gamma (h, E)$ then $g = h$ by repeated application of the identity principle. In other words, g is completely determined by f and γ.

2. Analytic continuation is an equivalence relation (exercise), but direct analytic continuation is *not* transitive, even if we require pairwise intersections of the domains to be nonempty. If fact, that is the whole point of analytic continuation along path.

> **Definition** (complete analytic function). An equivalence class of a function element under $\approx$ is a *complete analytic function*.

Example (complex logarithm). Let $U = \mathbb{C}$ be the ambient space. Given $\alpha < \beta$ in $\mathbb{R}$, define

$$E_{(\alpha,\beta)} := \{re^{i\theta} : r > 0, \alpha < \theta < \beta\}.$$

Note $\mathbb{C} \setminus \mathbb{R}_{\geq 0} = E_{(0,2\pi)}$. If $\beta - \alpha \leq 2\pi$, define

$$f_{(\alpha,\beta)}(z) = \log r + i\theta$$

where $z = re^{i\theta}, \alpha < \theta < \beta$. Then $(f_{(\alpha,\beta)}, E_{(\alpha,\beta)})$ is a function element for any such α, β.

Let

$$A = (-\frac{\pi}{2}, \frac{\pi}{2})$$
$$B = (\frac{\pi}{6}, \frac{7\pi}{6})$$
$$C = (\frac{5\pi}{6}, \frac{11\pi}{6})$$

and $\gamma : [0, 1] \to U, t \mapsto e^{2\pi it}$ and choose

$$0 = t_0 < t_1 = \frac{1}{6} < t_2 = \frac{1}{2} < t_3 = \frac{5}{6} < t_4 = 1$$

and $(f_A, E_A), (f_B, E_B), (f_C, E_C)$ the corresponding function elements.

When the *intervals* overlap, the function elements agree so

$$(f_A, E_A) \sim (f_B, E_B) \sim (f_C, E_C),$$

but

$$f_C(z) = f_A(z) + 2\pi i, z \in E_A \cap E_C$$

which shows nontransitivity of $\sim$. In fact, $f_A + 2\pi i \sim f_C$. However we see $(f_A, E_A) \approx_\gamma (f_C, E_C)$ and so $(f_A, E_A) \approx (f_C, E_C)$. By repeating the process with intervals moving to infinity in $\mathbb{R}$, we see that all the $\log r + (2\pi n + \theta)i$ are in the same class for $\approx$. On the other hand, if $(f, D) \approx_\gamma (f_{A'}, E_{A'})$ for some interval A' then applying identity principle along the path to e^{f_i} shows that f is one of the branches of log.

Now we can define a space that contains all branches of logarithm. On $U = \mathbb{C} \setminus \mathbb{R}_{\geq 0}$, define

$$f_n(z) = \log n + (2\pi n + \theta)i$$

where $0 < \theta < 2\pi$. Then (f_n, U) are function elements in the complete analytic function of log, and "almost" all of them. Take $\mathbb{Z}$ copies of U and we can glue them along $\mathbb{R}_{\geq 0}$. More precisely, for any $n \in \mathbb{Z}$ and $\alpha > 0$, there exists a neighbourhood V of α and a function element (g, V) such that

$$(f_{n+1}, E_{(0,\varepsilon)}) \sim (g, V) \sim (f_n, E_{(2\pi-\varepsilon, 2\pi)})$$

for some $\varepsilon > 0$.

This object is the "gluing construction" of the Riemann surface associated to log. Since these (g, V) exist, the resulting surface R will admit a *continuous* function f such that the following diagram commutes:

$$\begin{array}{ccc} R & \xrightarrow{f} & \mathbb{C} \\ \big\downarrow{\scriptstyle \pi} & \swarrow{\scriptstyle \exp} & \\ \mathbb{C}^* & & \end{array}$$

The rigorous construction is as follow. Let $R = \coprod_{k \in \mathbb{Z}} \mathbb{C}^*$ and a basis for the topology on R is

1. disks contained in a single sheet: $D((\eta, k), r)$ disk of radius r about $\eta \in \mathbb{C} \setminus \mathbb{R}_{\geq 0}$ at level k, where r is sufficently small such that the disk does not intersect $\mathbb{R}_{\geq 0}$,
2. disks along $\mathbb{R}_{\geq 0}$: for $\eta > 0, k \in \mathbb{Z}, r < |\eta|$,

   $$A((\eta, k), r) = \{(z, k) : |z - \eta| < r, \operatorname{Im} z \geq 0\} \amalg \{(z, k-1), |z - \eta| < r, \operatorname{Im} z < 0\}.$$

Check that this makes R a Hausdorff, path-connected space. R comes with a natural projection $\pi : R \to \mathbb{C}^*, (\eta, k) \mapsto \eta$. This is a continuous map as the preimage of a small disk $D(\eta, r) \subseteq \mathbb{C}^*$ is the union of countably many copies of that disk, one for each sheet. This is precisely the definition of a covering space.

Definition (covering space). A *covering space* of a topological space X is a continuous map $p : \tilde{X} \to X$ where $\tilde{X}$ and X are Hausdorff and path-connected and p is a local homeomorphism, i.e. for each $\tilde{x} \in \tilde{X}$, there exists a neighbourhood $\tilde{N}$ of $\tilde{x}$ such that $p|_{\tilde{N}}$ is a homeomorphism.

X is the *base space* of p.

The cover is *regular* if for all $x \in X$, there exists a neighbourhood N of x such that $p^{-1}(N)$ is a disjoint union of sets mapped homeomorphically by p to N.

Note. Whether including regularity in the definition of covering space is a matter of taste. It is ususally included in algebraic topology, e.g. in IID Algebraic Topology.

Remark. $\pi : R \to \mathbb{C}^*$ is a regular cover.

Example (a non-regular cover)**.** Consider $p : \tilde{X} \to \mathbb{C}^*, z \mapsto e^z$ where

$$\tilde{X} = \{z \in \mathbb{C} : 0 < \operatorname{Im} z < 4\pi\}.$$

It is a covering space but consider $1 \in \mathbb{C}^*$. Any preimage of a sufficiently small disk centred at 1 will be the disjoint union of one disk at $2\pi i$ and two half disks at 0 and $4\pi i$ each. Thus p fails to be a regular cover as we choose the "wrong" domain.

Define

$$\begin{aligned} f : R &\to \mathbb{C} \\ (\eta, k) &\mapsto \log r + (2\pi k + \theta) i \end{aligned}$$

where $\eta = re^{i\theta}, 0 \le \theta < 2\pi$. Then f is a continuous bijection and the following diagram commutes:

$$\begin{array}{ccc} R & \xrightarrow{f} & \mathbb{C} \\ \downarrow \pi & \swarrow \exp & \\ \mathbb{C}^* & & \end{array}$$

A similar construction can be done for the multivalued function $z^{1/n}$ where $n \in \mathbb{N}$. As a multivalued function,

$$(re^{i\theta})^{1/n} = r^{1/n} e^{i\theta/n} e^{2\pi k i/n}$$

for $k \in \mathbb{Z}/n\mathbb{Z}$. Define $R_n = \coprod_{k \in \mathbb{Z}/n\mathbb{Z}} \mathbb{C}^*$ and glue near modulo n ("top sheet to bottom sheet"). Then we have f_n, π_n such that the following diagram commutes:

$$\begin{array}{ccc} R_n & \xrightarrow{f_n} & \mathbb{C}^* \\ \downarrow \pi_n & \swarrow z \mapsto z^n & \\ \mathbb{C}^* & & \end{array}$$

Definition (regular/singular point)**.** Let $f(z) = \sum_{k \ge 0} a_k z^k$ with radius of convergence 1. A point $z \in \partial D(0,1)$ is *regular* if there exists a neighbourhood N of z and a holomorphic g on N such that $g = f$ on $N \cap D(0,1)$, i.e. g is a direct analytic continuation of f.

If $z \in \partial D(0,1)$ is not regular it is *singular*.

Remark.

1. The regular points of $\partial D(0,1)$ form an open set in the subspace topology on $\partial D(0,1)$.
2. z is regular does *not* mean that the series converges at z. Consider the classical example $f(z) = \sum_{k \ge 0} z^k$, which is regular everywhere except $z = 1$ ($g(z) = \frac{1}{1-z}$).

3. The converse does not hold either. A series converges at z does not imply that it is regular there. For example, $g(z) = \sum_{k\geq 2} \frac{z^k}{(k-1)k}$ converges at all $z \in \partial D(0,1)$. If it was regular at such a point then the second derivative $g''(z) = \sum_{k\geq 0} z^k$ would also be regular at z. But $g''(z) \to \infty$ as $z \to 1$ so f cannot agree on a neighbourhood of 1 with any holomorphic function.

However, regularity does affect radius of convergence:

Proposition 1.6. *Suppose $f(z) = \sum_{k\geq 0} a_k z^k$ with radius of convergence 1. Then there exists a singular point on $\partial D(0,1)$.*

Proof. Suppose not so for each $z \in \partial D(0,1)$ there exists a neighbourhood N_z of z and g_z on N_z holomorphic with $g_z = f$ on $N_z \cap D(0,1)$. These extensions can be glued together by identity principle. As $\partial D(0,1)$ is compact, there exists a finite collection of $z_1, \dots, z_m \in \partial D(0,1)$ such that N_{z_i}'s cover $\partial D(0,1)$. wlog let the neighbourhoods be disks. Then we can choose $\delta > 0$ sufficiently small such that f is holomorphic on $D(0, 1+\delta)$. Contradiction. □

Definition (natural boundary)**.** The disk boundary $\partial D(0,1)$ is the *natural boundary* for f if all points on the boundary are singular.

Example. $f(z) = \sum_{k\geq 0} z^{k!}$ has natural boundary $\partial D(0,1)$. Consider $\omega = e^{2\pi i \frac{p}{q}}$ a root of unity. For $0 < r < 1$,

$$f(r\omega) = \sum_{k\geq 0} r^{k!}\omega^{k!} = \sum_{k\leq q-1} r^{k!}\omega^{k!} + \sum_{k\geq q} r^{k!}$$

so as $r \to 1$ the last term goes to infinity so this cannot agree with a holomorphic function on a neighbourhood of ω. Since the closure of roots of unity is $\partial D(0,1)$, every point is singular.

1.3 Definition of Riemann surface

Definition (Riemann surface)**.** A *Riemann surface* R is a connected, Hausdorff topological space, together with a collection of homeomorphisms $\phi_\alpha : U_\alpha \to D_\alpha \subseteq \mathbb{C}$ with U_α open, so that

1. $\bigcup_\alpha U_\alpha = R$,
2. if $U_\alpha \cap U_\beta \neq \emptyset$ then $\phi_\beta \circ \phi_\alpha^{-1}$ is analytic on $\phi_\alpha(U_\alpha \cap U_\beta)$.

For a given α, (U_α, ϕ_α) is a *chart*, and these compositions $\phi_\beta \circ \phi_\alpha^{-1}$ are *transition functions*. The collection of charts is known as an *atlas* on R.

In other words, a Riemann surface is precisely a connected one-dimensional complex manifold.

Definition (analytic function between Riemann surfaces)**.** Let R, S be Riemann surfaces with atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ respectively. A continuous map $f : R \to S$ is *analytic* or *holomorphic* if whenever $U_\alpha \cap f^{-1}(V_\beta) \neq \emptyset$, then

$$\psi_\beta \circ f \circ \phi_\alpha^{-1}$$

on $\phi_\alpha(U_\alpha \cap f^{-1}(V_\beta))$ is analytic.

Remark. Analyticity is local. An equivalent definition is to say f is analytic at $x \in R$ if whenever $x \in U_\alpha \cap f^{-1}(V_\beta)$ then $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is analytic on a neighbourhood of $\phi_\alpha(x)$.

Example. $(\mathbb{C}, z)$ is a Riemann surface with one chart where we denote by z the map $z \mapsto z$, as is $(\mathbb{C}, z+1)$ and $(\mathbb{C}, \overline{z})$.

Example. The Möbius band cannot be made into a Riemann surface because it is non-orientable. Informally, if we put an atlas on the Möbius band, we could choose it so that the centre circle maps to a space homeomorphic to a circle. And as analytic transition implies conformity, consistent choice of "inside" of the circle leads to a consistent choice on "inside" on the Möbius band, which is a contradiction.

Remark.

1. Each transition function has continuous inverses and so are conformal equivalence on their domains.
2. R is connected with a path-connected basis so R is path-connected.

Definition (equivalent atlas)**.** Two atlases $\{(U_\alpha, \phi_\alpha)\}$ and $\{(V_\beta, \psi_\beta)\}$ are *equivalent* if their union is also an atlas, i.e. whenever $U_\alpha \cap V_\beta \neq \emptyset$ then $\psi_\beta \circ \phi_\alpha^{-1}$ on $\phi(U_\alpha \cap V_\beta)$ is analytic.

Example. $(\mathbb{C}, z)$ and $(\mathbb{C}, z+1)$ are equivalent: $z \mapsto z+1$ (or $z \mapsto z-1$) are analytic. On the other had $(\mathbb{C}, z)$ and $(\mathbb{C}, \overline{z})$ are not equivalent as $z \mapsto \overline{z}$ is not analytic.

We will see later that the notion of equivalence defines an equivalence relation on the collection of atlases on a fixed R.

Definition (conformal structure)**.** An equivalence class of atlases on R is a *conformal structure* on R.

Remark.

1. If R is a Riemann surface and $S \subseteq R$ is open and connected then restrictions of the charts provide a conformal structure on S, for which $i : S \hookrightarrow R$ is analytic.
2. Two atlases are equivalent if and only if the identity map is analytic.

Proposition 1.7. *Let $f : R \to S, g : S \to T$ be analytic maps of Riemann surfaces. Then $g \circ f$ is analytic.*

Proof. Suppose $\{(U_\alpha, \phi_\alpha)\}, \{(V_\beta, \psi_\beta)\}$ and $\{(W_\gamma, \theta_\gamma)\}$ are atlases on R, S and T respectively. Let $h = g \circ f$ which is continuous. Suffices to show that whenever

$$Y := U_\alpha f^{-1}(V_\beta) \cap h^{-1}(W_\gamma)$$

is nonempty then

$$\theta_\gamma \circ g \circ f \circ \phi_\alpha^{-1}$$

is analytic on Y. Since $\psi_\beta \circ f \circ \phi_\alpha^{-1}$ is analytic on $\phi_\alpha(Y)$ and $\theta_\gamma \circ g \circ \psi_\beta^{-1}$ is analytic on $\psi_\beta \circ f(Y)$, we concluded that

$$\theta_\gamma \circ g \circ \psi_\beta^{-1} \circ \psi_\beta \circ f \circ \phi_\alpha^{-1}$$

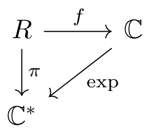
is analytic on $\alpha_\alpha(Y)$. ☐

Corollary 1.8. *Equivalence of atlas is an equivalence relation.*

Proposition 1.9. *Suppose R is a Riemann surface and $\pi : \tilde{R} \to R$ is a covering map. Then there is a unique conformal structure on $\tilde{R}$ which makes π analytic.*

Proof. Given $\tilde{z} \in \tilde{R}$, we can find $\tilde{N}$ of $\tilde{z}$ on which $\pi : \tilde{N} \to N$ is a homeomorphism onto its image. Let (V, φ) be a chart containing the image $\pi(\tilde{z})$. Define $U_{\tilde{z}} = \pi^{-1}(V) \cap \tilde{N}$ and $\varphi_{\tilde{z}} = \varphi \circ \pi$. This defines a chart on some neighbourhood of $\tilde{z}$ and $\{(U_{\tilde{z}}, \varphi_{\tilde{z}})\}_{\tilde{z} \in \tilde{R}}$ defines an atlas: this is clearly a cover and the transition functions $\varphi_{\tilde{z}} \circ \varphi_{\tilde{w}}^{-1}$ are the restrictions of transition functions for R. π is analytic with respect to this conformal structure as the composite maps are transition maps of R. Uniqueness follows from a similar argument. ☐

Example. Let $R = \coprod_{k \in \mathbb{Z}} \mathbb{C}^*$ and $\pi : R \to \mathbb{C}^*, (\eta, k) \mapsto \eta$ be a covering map. Then there exists a unique conformal structure on R for which π is analytic. Note that the following diagram commutes, f is a continuous map and locally f is the composition of inverse of exp and projection so f is analytic.



As f is a bijection by construction, it has a global analytic inverse.

Definition (conformal equivalence)**.** An analytic map $f : R \to S$ of Riemann surfaces is a *conformal equivalence* if there exists $g : S \to R$ analytic inverse to f.

Example.

1. f as above for the logarithm Riemann surface is a conformal equivalence: the inverse of f is continuous and locally it is given by $\pi^{-1} \circ \exp$ so is analytic. Therefore (R, π) and $(\mathbb{C}, \exp)$ cannot be "told apart".

2. $(\mathbb{C}, z)$ and $(\mathbb{C}, \overline{z})$ are conformally equivalent as $f(z) = \overline{z}$ is a conformal equivalence.

3. $$\begin{array}{ccc} R_n & \xrightarrow{f_n} & \mathbb{C}^* \\ \downarrow{\scriptstyle \pi_n} & \swarrow{\scriptstyle z \mapsto z^n} & \\ \mathbb{C}^* & & \end{array}$$

   Again there exists a unique conformal structure on R_n making π analytic. It follows that f_n is analytic. Note that one could imagine adding two points to R_n and replacing $\mathbb{C}^*$ with $\mathbb{C} \cup \{\infty\} = \mathbb{C}_\infty$. Doing so ruins π as a cover, but sometimes it's worth it (compactness!).

4. $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ equipped with the sphere topology via steoreographic projection. Define two charts: $(\mathbb{C}, z)$ and $(\mathbb{C}_\infty \setminus \{0\}, \frac{1}{z})$. The transition functions are $\frac{1}{z}$ which are anlaytic on $\mathbb{C}^*$. It makes $\mathbb{C}_\infty$ a compact Riemann surface. This is sometimes denoted by $\widehat{\mathbb{C}}$.

Definition (analytic function)**.** If R is a Riemann surface, an analytic map $f : R \to \mathbb{C}$ is an *analytic function*.

Therefore we use "map" to denote maps between Riemann surfaces and reserve "function" for a $\mathbb{C}$-valued map.

Recall from IB Analysis II and IB Complex Analysis

Theorem 1.10 (inverse function theorem)**.** *Given analytic g on a domain $V \subseteq \mathbb{C}$ and $a \in V$ such that $g'(a) \neq 0$, there exists a neighbourhood N of a such that $g|_N : N \to g(N)$ is a conformal equivalence.*

Consider an analytic function $f : R \to \mathbb{C}$. Given $p \in R$, choose a chart (U, φ) with $p \in U$. wlog $f(p) = 0$. and write $a = \varphi(p)$. Locally around a, $f \circ \varphi^{-1}$ is analytic so can be written as $g(z)^r$ where g is a conformal equivalence: we can write any nonconstant analytic function sending $a \mapsto 0$ as $(z-a)^r h(z)$ where h is analytic and nonzero on a neighbourhood of a. Then there is a neighbourhood V of a such that $h(V)$ does not intersect some ray from the origin. This allows us to define a logarithm on $h(V)$ and rth root

$$\ell(z) := \exp(\frac{1}{r} \log h(z)).$$

Then $f \circ \varphi^{-1}$ is of the form $g(z)^r$ where $g(z) = (z-a)\ell(z)$. Then $g'(a) = \ell(a) \neq 0$ so conformal.

Define a chart on the intersection of $\varphi(U)$ with domain of g, together with the chart $\psi = g \circ \phi$. Therefore up to translation, any analytic function on a Riemann surface is locally equivalent to a powering map.

Definition (complex torus)**.** Let

$$\Lambda = \mathbb{Z}\tau_1 \oplus \mathbb{Z}\tau_2 \subseteq \mathbb{C}$$

be a lattice where τ_1, τ_2 are nonzero in $\mathbb{C}$ with $\frac{\tau_1}{\tau_2} \notin \mathbb{R}$, i.e. are linearly independent over $\mathbb{R}$. The quotient group $T = \mathbb{C}/\Lambda$ can be equipped with a complex structure, known as a *complex torus*.

The complex structure is constructed as follow. Equip the quotient group $T = \mathbb{C}/\Lambda$ with quotient topology. $\pi : \mathbb{C} \to T$ is continuous so T is connected. π is also open: if U is an open set in $\mathbb{C}$ then

$$\pi^{-1}(\pi(U)) = \bigcup_{\omega\in\Lambda} \omega + U$$

a union of open sets so open. Note that any closed parallelogram

$$P_z = \{z + r\tau_1 + s\tau_2 : r, s \in [0, 1]\}$$

maps onto T by π. So T is the continuous image of a compact set so compact. T is also Hausdorff: note first that Λ is a discrete set: if Λ contained an accummultaion point then 0 would also be a limit point, i.e. for all $k \in \mathbb{N}$ there exists $m_k, n_k \in \mathbb{Z}$ (and wlog $n_k \neq 0$) such that

$$|m_k\tau_1 - n_k\tau_2| < \frac{1}{k}$$

but then

$$\left|\frac{m_k}{n_k} - \frac{\tau_2}{\tau_1}\right| < \frac{1}{k|n_k|\tau_1} \leq \frac{1}{k|\tau_1|} \to 0$$

as $k \to \infty$ so $\frac{\tau_2}{\tau_1} \in \mathbb{R}$, contradiction. Thus given two points $w_1, w_2 \in T$ we can choose preimages $x_i \in p^{-1}(w_i)$ and neighbourhoods N_i of x_i such that

$$\left(\bigcup_{\omega\in\Lambda} N_1 + \omega\right) \cap \left(\bigcup_{\omega\in\Lambda} N_2 + \omega\right) = \emptyset,$$

i.e. $\pi(N_1)$ and $\pi(N_2)$ are open disjoint with $w_i \in \pi(N_i)$.

Now show π is a covering map: by the above π is a covering map, in fact regular: given $w \in T$, choose $z \in \mathbb{C}$ such that $\pi^{-1}(w)$ lies in the interior of Λ-translates of P_z, then choose a neighbourhood N of the unique preimage of w in P_z which is contained in the interior of P_z. Then $\pi(N)$ satisfies

$$\pi^{-1}(\pi(N)) = \bigcup_{\omega\in\Lambda} \omega + N$$

is a disjoint union of $\pi(N)$.

Finally for the complex structure of T, given $a \in T$, choose $z \in \mathbb{C}$ such that $\pi(z) = a$ and a neighbourhood N_a of a on which the regularity is realised. In particular, the component N_z of $\pi^{-1}(N_a)$ containing z has $\pi|_{N_z} : N_z \to N_a$ a homeomorphism. Define a chart to be the image of a disk D_z about z contained in N_z. Write $U_a = \pi(D_z)$ and define a chart map $\phi_a = (\pi|_{N_z})^{-1}$ on U_a. Claim this defines an atlas on T: clearly this is a cover and claim the trasition maps are translations: suppose $U_a \cap U_b \neq \emptyset$, then for each $w \in U_a \cap U_b$ there exists $\omega_w \in \Lambda$ such that $\phi_b^{-1} \circ \phi_a(w) = w + \omega_w$. But $w \mapsto \omega_w$ is a continuous function on a connected set and it takes values in a discrete set so is constant. Thus the transition functions are translations so analytic.

In example sheet 1 we'll show that different lattices can yield conformally equivalent tori. In example sheet 2 we give characterisation of conformal equivalence classes of tori in terms of Λ. Complex tori are an important class of Riemann surfaces.

Theorem 1.11 (open mapping theorem)**.** *Let $f : R \to S$ be a nonconstant analytic map of Rieman surfaces. Then f is an open map.*

Proof. Suppose $W \subseteq R$ is open. Choose $z \in W$ and charts (U, ϕ) of z, (V, ψ) of $f(z)$. Choose a disk D about $\phi(z)$ sufficiently small such that

$$\phi^{-1}(D) \subseteq W \cap f^{-1}(V) \cap U.$$

Then

$$(\psi \circ f \circ \phi^{-1})(D)$$

is open so $(f \circ \phi^{-1})(D) = f(\phi^{-1}(D))$ is open. Thus

$$f(z) \in (f \circ \phi^{-1})(D)) \subseteq f(W)$$

so $f(W)$ is open. □

Corollary 1.12. *Let $f : R \to S$ be a nonconstant analytic map. If R is compact then $f(R) = S$ and S is compact.*

Proof. $f(R)$ is open because f is open. It is also closed as it is compact in S, a Hausdorff space. As S is connected, the nonempty clopen set $f(R)$ is precisely S. The second claim follows. □

Corollary 1.13. *Complex tori and $\mathbb{C}_\infty$ admit no analytic function which are nonconstant.*

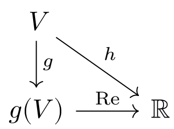
We have seen a special case of this in IB Complex Analysis: if $f : \mathbb{C}_\infty \to \mathbb{C}$ is analytic then $f(\infty) \in \mathbb{C}$ so f is bounded on a neighbourhood of ∞. By Liouville's theorem f is constant.

Definition (harmonic)**.** Let $h : R \to \mathbb{R}$ be a continuous function on a Riemann surface R. h is *harmonic* if for all charts (U, ϕ) of R, $h \circ \phi^{-1}$ is harmonic on $\phi(U)$.

Recall that a harmonic function on a domain in $\mathbb{C}$ is the real part of some analytic funciton locally, same is true for harmonic functions on Riemann surfaces. Thus harmonicity is well-defined independent of charts.

Proposition 1.14. *Suppose $h : R \to \mathbb{R}$ is harmonic on a Riemann surface R. Then if h is nonconstant, h is open. In particular if R is compact, R admits no nonconstant harmonic function.*

Proof. Given such a nonconstant $h : R \to \mathbb{R}$ and open set $U \subseteq R$ and $z \in U \subseteq R$, choose $z \in V \subseteq U$ open such that $h = \operatorname{Re} g$ for some analytic function g on V.



By open mapping theorem if g is nonconstant then it is open. Since Re is open, their composition h is as well. For a proof that g is nonconstant, see example sheet 1 Q13.

The second claim follows. □

Here we digress a little bit on non-examibable content before heading to the next chapter. A fundamental result about harmonic functions on Riemann surfaces is that they "almost" exist. We cannot find nonconstant harmonic function from a compact Riemann surface. But as the next best alternative we have

Theorem 1.15. *Let R be a Riemann surface, $P \neq Q \in R$. Then there exists a harmonic function $h : R \setminus \{P, Q\} \to \mathbb{R}$ such that for any chart $\phi : U \to \mathbb{C}$ about P with $\phi(P) = 0$, $h \circ \phi^{-1}$ is $\log|z|$ plus a bounded function near 0, and for any chart $\psi : V \to \mathbb{C}$ about Q with $\psi(Q) = 0$, $h \circ \psi^{-1}$ is $-\log|z|$ plus a bounded function near 0.*

Theorem 1.16 (Riemann existence theorem, classical version)**.** *Let R be a compact Riemann surface and $P \neq Q$ in R. Then there exists a meromorphic function f on R with $f(P) \neq f(Q)$.*

2 Meromorphic functions

Definition (meromorphic)**.** A *meromorphic* function on a Riemann surface R is an analytic map to $\mathbb{C}_\infty$.

Proposition 2.1. *Let $U \subseteq \mathbb{C}$ is a domain. A function $f : U \to \mathbb{C}_\infty$ is meromorphic if and only if it is meromorphic as a map from a Riemann surface.*

Proof. Assume $f : U \to \mathbb{C}_\infty$ is analytic. Given $a \in U$, if $f(a) \in \mathbb{C}$ then f is an analytic function near a so meromorphic. If $f(a) = \infty$ then by considering the chart $(\mathbb{C} \setminus \{0\}, \frac{1}{z})$ of $\mathbb{C}_\infty$ near ∞, we see that $g(z) = \frac{1}{f(z)}$ is analytic on a neighbourhood of a with $g(a) = 0$. Thus $g(z) = (z-a)^r h(z)$ where h is analytic nonzero on a neighbourhood of a so $f(z) = (z-a)^{-r}\frac{1}{h(z)}$, which is meromorphic as a complex function.

All the implications above are equivalences so the reverse also holds. □

Example. In example sheet 1 Q15 we show that $\{(z, w) : w^2 = z^3 - z\} \subseteq \mathbb{C}^2$ admits a conformal structure via the coordinate projection maps. We may alternatively do this geometrically by gluing. Define $f(z) = z^3 - z$ and define $U = \mathbb{C} \setminus ([-1, 0] \cup [1, \infty))$. Claim that we can define a square root of f on U (in other words, direct analytic continuation is transitive): this can be done locally at any point of U. To show it's well-defined, consider a closed path $\gamma \subseteq U$. By a result about winding number in example sheet 1 Q1,

$$I(f \circ \gamma, 0) = I(\gamma, -1) + I(\gamma, 0) + I(\gamma, 1).$$

We can check that $I(\gamma, 1) = 0$ and $I(\gamma, -1) = I(\gamma, 0)$ so $I(f \circ \gamma, 0) \in 2\mathbb{Z}$. Therefore if we define locally some $\exp(\frac{1}{2} \log f(z))$, as we travel along γ, the change in log is

$$\int_\gamma \frac{f'(z)}{f(z) - 0} dz = 2\pi i I(f \circ \gamma, 0) = 2n\pi i$$

for some $n \in 2\mathbb{Z}$ by argument principle. Thus $\frac{1}{2} \log f(z)$ change by $n\pi i$.

If we let U_+, U_- be two copies of U and denote by $g_+ : U_+ \to \mathbb{C}$ the map we just constructed and let $g_- = -g_+$, glue according to the identifying segments (see image) to obtain a single surface R and an analytic function g on R which agrees with g_+ on U_+ and g_- on U_-. Topologically, this is a torus minus four points.

It might be instructive to compare algebraic and gemeotric/topological construction and advantage of each. Later we'll learn to extract topological information *directly* from the algebraic definition.

2.1 Space of germs and monodromy

Definition (lift)**.** Suppose $\pi : \tilde{X} \to X$ is a (topological) covering map, and $\gamma : [0, 1] \to X$ is a path. Then a *lift* of γ is a path $\tilde{\gamma} : [0, 1] \to \tilde{X}$ such that $\pi \circ \tilde{\gamma} = \gamma$.

Proposition 2.2. *If $\tilde{\gamma}_1, \tilde{\gamma}_2$ are lifts of γ with $\gamma_1(0) = \gamma_2(0)$ then $\gamma_1 = \gamma_2$.*

Proof. Define

$$I_1 = \{t \in [0,1] : \tilde{\gamma}_1(t) = \tilde{\gamma}_2(t)\}$$
$$I_2 = \{t \in [0,1] : \tilde{\gamma}_1(t) \neq \tilde{\gamma}_2(t)\}$$

Claim that both are open in $[0,1]$. First suppose $\tau \in I_2$. As $\tilde{X}$ is Hausdorff, there exist open disjoint U_1, U_2 with $\tilde{\gamma}_1(\tau) \in U_1, \tilde{\gamma}_2(\tau) \in U_2$. Paths are continuous so $\tilde{\gamma}_1^{-1}(U_1)$ and $\tilde{\gamma}_2^{-1}(U_2)$ are open neighbourhoods of τ in $[0,1]$, their intersection is thus open and contained in I_2, so I_2 is open.

Suppose now that $\tau \in I_1$. Choose an open neighbourhood $\tilde{N}$ of $\tilde{\gamma}_1(\tau) = \tilde{\gamma}_2(\tau)$ in $\tilde{X}$ such that $\pi|_{\tilde{N}}$ is a homeomorphism onto its image. We have $\pi(\tilde{\gamma}_1(t)) = \pi(\tilde{\gamma}_2(t))$ for all t as they are both lifts for γ, so on $\tilde{N}$ this implies that $\tilde{\gamma}_1(t) = \tilde{\gamma}_2(t)$. By continuity of paths, there exists $\delta > 0$ such that $t \in (\tau-\delta, \tau+\delta) \subseteq [0,1]$ implies $\tilde{\gamma}_1(t), \tilde{\gamma}_2(t) \in \tilde{N}$. So the interval $(\tau-\delta, \tau+\delta) \subseteq [0,1] \subseteq I_1$ so I_1 is open. Thus $I_1 = [0,1]$ by connectivity. □

In summary, lifts are unique up to choice of basepoints.

As for existence, lifts may not exist if the cover is not regular. c.f. nonregular cover exmaple. However, it is the *only* obstruction to the construction of a lift.

Proposition 2.3. *Suppose $\pi : \tilde{X} \to X$ is a regular covering map. Given γ in X and $z \in \tilde{X}$ such that $\pi(z) = \gamma(0)$, there is a (unique) lift $\tilde{\gamma}$ of γ with $\tilde{\gamma}(0) = z$.*

Proof. Define

$$I = \{t \in [0,1] : \text{ exists lift } \tilde{\gamma} : [0,1] \to \tilde{X} \text{ of } \gamma \text{ with } \tilde{\gamma}(0) = z\}$$

and let $\tau = \sup I$. Suppose for contradiction $\tau \neq 1$. Choose an open neighbourhood U of $\gamma(\tau)$ such that $\pi^{-1}(U) = \coprod_j \tilde{U}_j$ and $\pi|_{\tilde{U}_j}$ is a homeomorphism onto U. By continuity of γ, there exists $\delta > 0$ such that $\gamma([\tau-\delta, \tau+\delta]) \subseteq U$. Since τ is the supremum, exists $\tau_1 \in [\tau-\delta, \tau]$ such that γ lifts to $\tilde{\gamma}$ on $[0, \tau_1]$ with $\tilde{\gamma}(0) = z$. Choose j such that $\tilde{\gamma}(\tau_1) \in \tilde{U}$. Define an extension of $\tilde{\gamma}$ on $[\tau, \tau+\delta]$ by $(\pi|_{\tilde{U}_j})^{-1} \circ \gamma$. This gives a lift of γ to $[0, \tau+\delta]$, contradicting $\tau = \sup I$. Thus $\tau = 1$. □

Definition (homotopy)**.** We say paths α, β in X are *homotopic* in X if there exists a family γ_s of paths where $s \in [0,1]$ such that

1. $\gamma_0 = \alpha, \gamma_1 = \beta$,
2. $\gamma_s(0) = \alpha(0) = \beta(0)$ and $\gamma_s(1) = \alpha(1) = \beta(1)$ for all $s \in [0,1]$,
3. $[0,1] \times [0,1] \to X, (s,t) \mapsto \gamma_s(t)$ is continuous.

Definition (simply connected)**.** We say X is *simply connected* if any closed path in X is homotopic to a constant path.

Theorem 2.4 (monodromy theorem)**.** *Let $\pi : \tilde{X} \to X$ be a covering map and α, β be paths in X. Assume that*

1. *α and β are homotopic in X,*
2. *α and β have lifts $\tilde{\alpha}$ and $\tilde{\beta}$ respectively with $\tilde{\alpha}(0) = \tilde{\beta}(0)$,*
3. *every path in X with $\gamma(0) = \alpha(0) = \beta(0)$ has a lift $\tilde{\gamma}$ with $\tilde{\gamma}(0) = \tilde{\alpha}(0) = \tilde{\beta}(0)$.*

Then the lifts $\tilde{\alpha}$ and $\tilde{\beta}$ are homotopic. In particular, $\tilde{\alpha}(1) = \tilde{\beta}(1)$.

Proof. Non-examinable and omitted. See, for example, IID Algebraic Topology. □

Example. Consider $z \mapsto z^n$ on $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$. This is a regular covering map. Consider a loop γ based at 1. The preimages of 1 are the nth roots of unity ζ_n^k, $1 \leq k \leq n$. Any lift of γ will start at some ζ_n^k and end at ζ_n^{k+1}. As this is a regular cover, monodromy theorem tells that any path based at 1 has a lift whose endpoints are the same as if we lifted γ^{0n} for some $n \in \mathbb{Z}$. Note that to any path γ we have an associated permutation of the set $\{\zeta_n^k\}_{1 \leq k \leq n}$ by considering where the lift starting at ζ_n^k ends, i.e. an element of S_n. The subgroup of S_n arising in this way is generated by $(123 \dots n)$, which is the cyclic subgroup C_n.

(It is an exercise to show that any closed path in the punctured plane is homotopic to an integer multiple of γ.)

2.2 Space of germs

Suppose $G \subseteq \mathbb{C}$ is a domain throughout this section.

Definition (germ)**.** Given $z \in G$ and (f, D) and (g, E) function elements. We say $(f, D) \equiv_z (g, E)$ if $z \in D \cap E$ and $f = g$ on a neighbourhood of z. The equivalence class under $\equiv_z$ of (f, D) is called the *germ* of f at z, denoted by $[f]_z$.

Compare this with direct analytic continuation, which is *not* an equivalence relation.

Note that two germs $[f]_z, [g]_w$ are equal if and only if $z = w$ and $f = g$ on a neighbourhood of $z = w$.

Definition. The *space of germs on G* is the set

$$\mathcal{G} = \{[f]_z : z \in G \text{ and } (f, D) \text{ is a function element with } z \in D\}.$$

Notation. Given a function element (f, D), write

$$[f]_D = \{[f]_z : z \in D\} \subseteq \mathcal{G}.$$

The goal is to show that $\mathcal{G}$ is a Riemann surface. First we define the topology on $\mathcal{G}$ to be the one generated by basis element of the form $[f]_D$. Given $[f]_D$ and $[g]_E$, if $[h]_z \in [f]_D \cap [g]_E$ then $z \in D \cap E$ and $h = f = g$ on a neighbourhood of z. Thus there exists domain D' with $z \in D'$ and $[h]_{D'} \subseteq [f]_D \cap [g]_E$.

The topology is Hausdorff: suppose $[f]_z \neq [g]_w$ in $\mathcal{G}$, represented by (f, D) and (g, E) repsectively. If $z \neq w$ choose $D \cap E = \emptyset$ so $[f]_z \in [f]_D$ and $[g]_w \in [g]_E$ and these open sets are disjoint. If $z = w$ choose $D = E$. Claim that $[f]_D \cap [g]_E = \emptyset$: for suppose $[h]_s \in [f]_D \cap [g]_E$ then by definition exists neighbourhood N of s such that $h = f = g$ on N so that $f = g$ on $D = E$. In particular $[f]_z = [g]_z = [g]_w$, contradiction.

The connected components of $\mathcal{G}$ cover G via the forgetful map $\pi([f]_z) = z$. To show this is a cover, let $V \subseteq G$ be an open set, then

$$\pi^{-1}(V) = \{[f]_z : z \in V\} = \bigcup_{D \subseteq V} \{[f]_D : (f, D) \text{ is a function element}\}$$

which is open. Locally on $[f]_D$, π is a bijection. On such a set $[f]_D$, $U \subseteq [f]_D$ is open if and only if $U = \bigcup_\alpha [f]_{D_\alpha}$, if and only if $\pi(U) = \bigcup_\alpha D_\alpha$, if and only if $\pi(U)$ is open.

For conformal structure on $\mathcal{G}$, we know by a previous proposition that on each connected component of $\mathcal{G}$, there exists a unique conformal structure making π analytic. These charts can be taken to be (U, φ) with $U = [f]_D$ and $\varphi = \pi|_U$.

Moreover $\mathcal{G}$ comes with an evaluation map

$$\begin{aligned} E : \mathcal{G} &\to \mathbb{C} \\ [f]_z &\mapsto f(z) \end{aligned}$$

which is analytic: given a chart $([f]_D, \pi|_{[f]_D})$ of $\mathcal{G}$,

$$E \circ (\pi|_{[f]_D})^{-1}(z) = E([f]_z) = f(z)$$

which is analytic in z. So E is analytic.

The stalk space $\mathcal{G}$ incorporates all information about analytic functions on G. The following theorem translates topological information of $\mathcal{G}$ to analytic information of complete analytic functions:

> **Theorem 2.5.** *Let (f, D) and (g, E) be function elements on G and $\gamma : [0, 1] \to G$ a path with $\gamma(0) \in D, \gamma(1) \in E$. Then (g, E) is analytic continuation of (f, D) along γ if and only if there exists a lift $\tilde{\gamma} : [0, 1] \to \mathcal{G}$ of γ such that $\tilde{\gamma}(0) = [f]_{\gamma(0)}, \tilde{\gamma}(1) = [g]_{\gamma(1)}$.*

Proof. Suppose there exists $(f_j, D_j)_{j=1}^n$ and $0 = t_0 < t_1 < \cdots < t_n = 1$ with

$$(f, D) = (f_1, D_1) \sim (f_2, D_2) \sim \cdots \sim (f_n, D_n) = (g, E)$$

and $f_{j-1} = f_j$ on $D_{j-1} \cap D_j$ and $\gamma([t_{j-1}, t_j]) \subseteq D_j$ for all j. We can define a lift

$$\tilde{\gamma}(t) = [f_j]_{\gamma(t)}, t \in [t_{j-1}, t_j]$$

which is well-defined. Claim it is continuous: suppose $[h]_U \subseteq \mathcal{G}$ and $\tilde{\gamma}(\tau) \in [h]_U$. Then

$$\tilde{\gamma}(\tau) = [f_j]_{\gamma(\tau)}$$

for some j so $f_j = h$ on an open neighbourhood N of $\gamma(\tau)$. As γ is continuous, there exists $\delta > 0$ such that if $|t - \tau| < \delta$ then $\gamma(t) \in N$. Then for such t,

$$\tilde{\gamma}(t) = [f_j]_{\gamma(t)} = [h]_{\gamma(t)} \in [h]_U$$

so $\tilde{\gamma}$ is continuous. $\tilde{\gamma}$ satisfies the lifting properties.

Conversely, suppose there is a lift $\tilde{\gamma}$ of γ in $\mathcal{G}$ with $\tilde{\gamma}(0) = [f]_{\gamma(0)}$ and $\tilde{\gamma}(1) = [g]_{\gamma(1)}$. For each $t \in [0,1]$, there exists a function element (f_t, D_t) with $\tilde{\gamma}(t) = [f_t]_{\gamma(t)}$. Note that $[f_t]_{D_t}$ contains $\tilde{\gamma}(t)$. We have for each t an open interval I_t with $\tilde{\gamma}(I_t) \subseteq [f_t]_{D_t}$. By compactness there exists a finite subcover, say intervals $[a_k, b_k]$, ordered so that $a_{k+1} < b_k$ for $k = 1, \dots, n-1$. Choose for each k some $t_k \in (a_{k+1}, b_k)$ and rename the corresponding open sets in $\mathcal{G}$ $[f_k]_{D_k}$. wlog assume all D_k's are disks. Since $\tilde{\gamma}(0) = [f]_{\gamma(0)}$ and $\tilde{\gamma}(1) = [g]_{\gamma(1)}$, we can also assume $D_1 \subseteq D, D_n \subseteq E$ so $f = f_1$ on D_1 and $g = f_n$ on D_n. for each $1 \leq k \leq n-1$, we have

$$\tilde{\gamma}(t_k) \in [f_k]_{D_k} \subseteq [f_{k+1}]_{D_{k+1}},$$

so $f_k = f_{k+1}$ on $D_k \cap D_{k+1}$ by the identity principle, as $f_k = f_{k+1}$ on a neighbourhood of $\gamma(t_k)$. So

$$(f, D) \sim (f_1, D_1) \sim \cdots \sim (f_n, D_n) \sim (g, E).$$

Finally, on $[t_{k-1}, t_k]$, we have

$$\gamma([t_{k-1}, t_k]) = \pi(\tilde{\gamma}([t_{k-1}, t_k])) \subseteq \pi([f_k]_{D_k}) = D_k,$$

thus completing the proof. □

Once we have established the correspondence between analytic continuation in the base space and lift of paths in stalk space, we can use monodromy thoerem (which we stated as a result purely in topology) to deduce uniqueness of analytic continuations:

> **Proposition 2.6.** *If (g, E) and (h, E) are analytic continuations of (f, D) along $\gamma \subseteq G$ then $g = h$ on E.*

Proof. Let (g, E) and (h, E) correspond to lifts $\tilde{\gamma}$ and $\tilde{\gamma}'$ respectively based at $[f]_{\gamma(0)}$. Uniqueness of lifts implies that $\tilde{\gamma}(1) = \tilde{\gamma}'(1)$, i.e. $[g]_{\gamma(1)} = [h]_{\gamma(1)}$, so $g = h$ on a neighbourhood of $\gamma(1)$ so on E by identity principle. □

We can also derive the so-called classical monodromy theorem

> **Theorem 2.7** (classical monodromy theorem)**.** *Suppose (f, D) can be continued analytically along all paths in G starting in D. Then if (g, E) and (h, E) are analytic continuations of f along paths α and β respectively, and α is homotopic to β then $g = h$ on E.*

Proof. Find lifts $\tilde{\alpha}$ and $\tilde{\beta}$ corresponding to (g, E) and (h, E) respectively. Note $\tilde{\alpha}(0) = [f]_{\alpha(0)} = [f]_{\beta(0)} = \tilde{\beta}(0)$. By monodromy theorem we have $\tilde{\alpha}(1) = \tilde{\beta}(1)$ so $g = h$ on E again by identity principle. □

Corollary 2.8. *Suppose G is a simply connected domain and (f, D) is a function element on G which can be analytically continued along all $\gamma \subseteq G$ paths with $\gamma(0) \in D$. Then f extends to G.*

Proof. Define for $z \in G$, $f(z)$ as follows: we fix $z_0 \in D$ and find a path γ on G with $\gamma(0) = z_0$ and $\gamma(1) = z$. By assumption f can be analytically continued along the path so by classical monodromy theorem and simply connectedness this is well-defined for all $z \in G$. □

Corollary 2.9. *Let $\mathcal{F}$ be a complete analytic function on G and define*

$$\mathcal{G}_{\mathcal{F}} = \bigcup_{(f,D)\in\mathcal{F}} [f]_D.$$

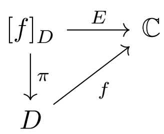
Then $\mathcal{G}_{\mathcal{F}}$ is a connected component of $\mathcal{G}$.

Proof. Each $\mathcal{G}$ is locally path-connected, so path-connected component is the same as connected component. The corollary follows from the theorem. □

Definition (Riemann surface associated to complete analytic function)**.** $\mathcal{G}_{\mathcal{F}}$ is the *Riemman surface associated to the complete analytic function $\mathcal{F}$.*

Remark.

1. For each $(f, D) \in \mathcal{F}$, the evaluation map E provides a single valued extension $f \circ \pi$ on $[f]_D$ to all of $\mathcal{G}_{\mathcal{F}}$.



2. In example sheet 2 Q7 we will show that in general $\pi : \mathcal{G}_{\mathcal{F}} \to G$ is not a regular cover.

Example. Let $R' = \{(z, w) \in \mathbb{C}^2 : w^2 = z^3 - z, w \neq 0\}$ and let $\mathcal{G}_{\mathcal{F}}$ be the Riemann surface associated to $\sqrt{z^3 - z}$ over the domain $G = \mathbb{C} \setminus \{-1, 0, 1\}$. Recall that the Riemann surface structure on R' can be obtained via π_z.

Define

$$\begin{aligned} g : \mathcal{G}_{\mathcal{F}} &\to R' \\ [f]_z &\mapsto (\pi([f]_z), E([f]_z)) \end{aligned}$$

g is continuous as a product of continuous map. g is also analytic: if $([f]_D, \pi)$ is a chart of $\mathcal{G}_{\mathcal{F}}$ then

$$(\pi_z \circ g \circ \pi^{-1})(s) = (\pi_z \circ g)([f]_s) = \pi_z(\pi([f]_s), E([f]_s)) = \pi([f]_s) = s$$

so analytic and open.

Define an inverse h of g: given $(z, w) \in R'$, choose a neighbourhood N on which π_z is a local homeomorphism. Define $h((z, w)) = [\pi_w \circ \pi_z^{-1}]_z$, then this is inverse to g so g is a conformal equivalence.

We have so far seen three constructions of this Riemann surface:

1. embedded curve construction,
2. space of germs $\mathcal{G}_{\mathcal{F}}$ of $\sqrt{z^3 - z}$,
3. gluing construction.

The above shows 1 and 2 are equivalent. 1 and 3 are shown to be equivalent in example sheet 1, and 2 and 3 in example sheet 2. The advantage of each is

1. inherits properties of $\mathbb{C}^2$,
2. always exists, although quite abstract. Moreover it is a covering space and is equipped with analytic maps π and E,
3. can get our hands on topology. Compactification is easy to describe and visualise.

2.3 Compactifying Riemann surfaces

Recall the construction of Riemann sphere. We one-point compactify $\mathbb{C}$ by adding a point ∞. Then we define charts $(\mathbb{C}, z)$ and $((\mathbb{C} \setminus \{0\}) \cup \{\infty\}, \frac{1}{z})$. The result is a map $\mathbb{C} \hookrightarrow \mathbb{C}_\infty$ that is not only a (dense) topological embedding into a compact space, but also an analytic map.

In general, suppose X and Y are topological spacs, $U \subseteq X, V \subseteq Y$ open and $\phi : U \to V$ a homeomorphism. Let $Z = X \amalg Y / \sim_\phi$ where $a \sim_\phi b$ if and only if $a = b, a = \phi(b)$ or $a = \phi^{-1}(b)$. Z is known as the *gluing of X and Y along ϕ*.

Proposition 2.10. *Suppose X and Y are Riemann surfaces and $U \subseteq X$ and $V \subseteq Y$ are nonempty open sets with $\phi : U \to V$ an isomorphism of Riemann surfaces. If $Z = X \amalg Y / \sim_\phi$ is Hausdorff then there exists a unique conformal structure on Z for which $i_X : X \hookrightarrow Z, i_Y : Y \hookrightarrow Z$ are analytic.*

Proof. Note i_X, i_Y are homeomorphisms. For each chart (W, ψ) of X we define a chart $(i_X(W), \psi \circ i_X^{-1})$ on Z, similarly for charts of Y. Transition maps come from those of X or Y or those composed with ϕ so are analytic. Z is connected for if we could disconnect Z we could disconnect X or Y. So Z admits a conformal structure which makes inclusions analytic. Uniqueness is immediate. □

Example. $R = \{(z, w) \in \mathbb{C}^2 : w^2 = z^3 - z\}$. We have seen via gluing that R minus points where $w \neq 0$ is a topological torus minus 4 points. Now we compactify it.

Consider $t = \frac{1}{z}, u = \frac{1}{w}$. Then the defining equation becomes

$$\frac{1}{u^2} = \frac{1}{t^3} - \frac{1}{t},$$

i.e.

$$t^3 = u^2 - u^2 t^2 = u^2(1 - t^2).$$

Unfortunately it is not a Riemann surface via either π_t or π_u at $(0, 0)$. But not all hope is lost. Write

$$t = \left(\frac{u}{t}\right)^2 (1 - t^2)$$

and let $v = \frac{u}{t} = \frac{z}{w}$. Then the surface becomes $Y = \{(t, v) \in \mathbb{C}^2 : t = v^2(1 - t^2)\}$. Y does have one or both projections π_t, π_v a local homeomorphism around each point, including $(0, 0)$, so Y admits a conformal structure. Consider the isomorphism

$$U \to V$$
$$(z, w) \mapsto (t, v) = (\frac{1}{z}, \frac{z}{w})$$

where $U \subseteq R$ are points where neither z nor w is 0 and V its isomorphic image in Y. Consider the gluing of R and Y along this isomorphism, call it X, with inclusions $i_R : R \hookrightarrow X, i_Y : Y \hookrightarrow X$. The image of R in X is $X \setminus \{1 \text{ points}\}$ and all points in $i_R(R)$ can be separated, similarly in $i_Y(Y)$. If $P \in X \setminus i_Y(Y)$ and $Q \in X \setminus i_R(R)$ so P is $(0, 0)$ and Q is $(0, 0)$ in local coordinates then

$$\{(z, w) \in R : |z| < 1\}$$
$$\{(t, v) \in Y : |t| < 1\}$$

separate P and Q.

X admits a conformal structure for which i_R, i_Y are analytic. Consider

$$D_R = \{(z, w) \in R : |z| \leq 2\}$$
$$D_Y = \{(t, v) \in Y : |t| \leq 2\}$$

these are compact in $R \amalg Y$ so map to compact sets in X via the continuous quotient map. Thus as a finite union of compact sets X is compact. Note this agrees with our topological intuition that R can be compactified by the addition of a single point.

2.4 Branching

Note these projection maps are *not* coverings on R (or X) but they still have controlled behaviour.

Definition (multiplicity/valency)**.** Let $f : R \to S$ be an nonconstant analytic map of Riemann surfaces and $z_0 \in R$. Locally we can write

$$\hat{f}(z) = \hat{f}(z_0) + (z - z_0)^{m_f(z_0)} g(z)$$

where $g(z)$ nonzero analytic. $m_f(z_0)$ is the *multiplicity* or *valency* of f at z_0.

Lemma 2.11. *Suppose g, h are nonconstant analytic on domains in $\mathbb{C}$ and the image of h is contained in the domain of g. Then*

$$m_{g \circ h}(z) = m_h(z) m_g(h(z)).$$

Proof. Exercise. □

As a corollary, multiplicity is well-defined. Indeed if $z \in R, f(z) \in S$ and $(U, \phi), (\tilde{U}, \tilde{\phi})$ are charts for z, $(V, \psi), (\tilde{V}, \tilde{\psi})$ are charts for $f(z)$ then $m_f(z)$ is given by the multiplicity of its local expression, which is

$$\begin{aligned}\tilde{\psi} \circ f \circ \tilde{\phi}^{-1} &= \tilde{\psi} \circ (\psi^{-1} \circ \psi \circ f \circ \phi^{-1} \circ \phi) \circ \tilde{\phi}^{-1} \\ &= (\tilde{\psi} \circ \psi^{-1}) \circ (\psi \circ f \circ \phi^{-1}) \circ (\phi \circ \tilde{\phi}^{-1})\end{aligned}$$

the transition maps have multiplicity 1 everywhere so by the lemma the multiplicity of the local expressions agree.

Note that the points at which $m_f(z) > 1$ are isolated, by the (local) principle of isolated zeros. In particular if R is compact then $\{z \in R : m_f(z) > 1\}$ is finite.

Definition (ramification point, ramification index, branch point)**.** Let $f : R \to S$ be nonconstant analytic. If $z \in R$ has $m_f(z) > 1$, we call z a *ramification point* of f and $m_f(z)$ in this case is called the *ramification index* at z, and $f(z)$ is a *branch point* of f.

Example. Let $p(z) = \sum_{k=0}^d a_k z^k$ be an analytic map $\mathbb{C} \to \mathbb{C}$ with $d \geq 1, a_d \neq 0$. p extends to an analytic map of the Riemann sphere via $p(\infty) = \infty$. At ∞ the local expression is

$$\frac{1}{p(\frac{1}{z})} = \frac{1}{\sum_{k=0}^d a_k z^{-k}} = \frac{z^d}{\sum_{k=0}^d a_k z^{d-k}} = z^d g(z)$$

for some g analytic and nonzero near 0. Thus $m_p(\infty) = d$.

Theorem 2.12 (valency theorem)**.** *Let $f : R \to S$ be a nonconstant analytic map of Riemann surfaces. If R is compact then there exists $n \geq 1$ such that f is an n-to-1 map counting multiplicity, i.e. for all $w \in S$,*

$$\sum_{z \in f^{-1}(w)} m_f(z) = n.$$

See how false this can be for noncompact Riemann surfaces!

Proof. By the principle of isolated zeros $f^{-1}(w)$ is a finite set for all $w \in S$. Define then

$$n(w) = \sum_{z \in f^{-1}(w)} m_f(z).$$

We want to show $n : S \to \mathbb{Z}$ is constant. But S is connected so suffice to show n is locally constant. Fix $w_0 \in S$ and let $f^{-1}(w_0) = \{z_1, \dots, z_q\}$. For each z_k, By choosing appropriate charts centred at z_k and w_0, f is locally $z \mapsto z^{m_f(z_k)}$. Moreover we can wlog choose a chart (N_k, ϕ) around z_k such that $\phi(N_k)$ is a disk around $\phi(z_k)$, on which $f|_{N_k}$ is an $m_f(z_k)$-to-1 map to its image. wlog choose the N_k disjoint. Note that $R \setminus \bigcup N_k$ is compact so $f(R \setminus \bigcup N_k)$ is compact, and there exists open neighbourhood M of w_0 such that $f(R \setminus \bigcup N_k) \cap M = \emptyset$. Let $N = f(N_1) \cap \dots \cap f(N_q) \cap M$, an open neighbourhood of w_0. For $w \in N$, $f^{-1}(w) \subseteq \bigcup_{k=1}^q N_k$ so

$$n(w) = \sum_{z \in f^{-1}(w)} m_f(z) = \sum_{z \in f^{-1}(w_0)} m_f(z) = n(w_0).$$

□

Definition (degree/valency)**.** Let $f : R \to S$ be a nonconstant analytic map with R compact. Then we call the number n the *degree* or *valency* of f.

Corollary 2.13 (fundamental theorem of algebra)**.** *Let p be nonconstant polynomial of degree d. Then p has d roots in $\mathbb{C}$.*

Proof. p extends to a map $p : \mathbb{C}_\infty \to \mathbb{C}_\infty$ and $p^{-1}(\infty) = \infty$ with multiplicity d. So by valency theorem 0 also has d preimages counting multiplicity. □

As a consequence we have

Proposition 2.14. *Let $f : \mathbb{C}_\infty \to \mathbb{C}_\infty$ be an nonconstant analytic map. Then we can write f as a rational function*

$$f(z) = c\frac{(z-a_1)\cdots(z-a_m)}{(z-b_1)\cdots(z-b_n)}$$

where $a_i, b_j \in \mathbb{C}, c \in \mathbb{C}^*$.

Proof. Assume wlog $f^{-1}(\infty) \subseteq \mathbb{C}$, so that $f^{-1}(\infty) = \{b_1, \dots, b_n\}$. f analytic at b_i is equivalent to saying that $\frac{1}{f}$ is an analytic function on a neighbourhood of b_i, i.e.

$$\frac{1}{f(z)} = (z-b_i)^{m_f(b_i)}g(z)$$

where g is nonzero analytic at b_i, so f has Laurent series

$$f(z) = \sum_{j=-k_i}^{\infty} a_{j,i}(z-b_i)^j$$

so the function

$$f(z) - \sum_{i=1}^{n}\left(\sum_{j=-k_i}^{-1} a_{j,i}(z-b_i)^j\right)$$

has no preimage of ∞ so is constant. □

Remark. If $f(\infty) \neq \infty$ then $m \leq n$, in which case $\deg f = n$. In general, by considering $f^{-1}(\infty)$ and $f^{-1}(0)$ to see that

$$\deg f = \max\{m, n\}.$$

Corollary 2.15. *The analytic isomorphisms of $\mathbb{C}_\infty$ are Möbius transformations.*

3 Riemann-Hurwitz formula

3.1 Triangulation and Euler characteristic

Let S be a compact Riemann surface. We say $T \subseteq S$ is a *topological triangle* if it is the homeomorphic image of a closed triangle in $\mathbb{R}^2$.

Definition (triangulation)**.** A *triangulation* of S is a finite collection of topological triangles $\{T_1, \dots, T_n\}$ in S such that

1. $\bigcup_{i=1}^n T_i = S$,
2. If $T_i \cap T_j \neq \emptyset$ then $T_i \cap T_j$ is a common edge or a common vertex,
3. every edge is the edge of exactly two triangles.

Definition (Euler characteristic)**.** The *Euler characteristic* of S is

$$\chi(S) = F - E + V$$

where F, E, V are the number of faces, edges and vertices respectively for any choice of triangulation of S.

We state without proof the following results:

Fact.

1. $\chi(S)$ is independent of choice of triangulation (to check this suffices to check it is invariant under refinement).
2. (corollary of classification of compact surfaces) every compact Riemann surface is homeomorphic to a surface with handles. The number of handles is the *genus* of the surface.
3. Every compact Riemann surface can be triangulated and $\chi(S) = 2 - 2g$ where g is the genus of S. It is possible to check this by assuming 2 and induct on g.

Example. Let $S = \mathbb{C}_\infty$. Take three orthogonal great cricles. Then S is divided into 8 topological triangles. We have

$$F = 8, V = 6, E = 12$$

so

$$\chi(S) = 8 - 12 + 6 = 2$$

which agrees with $2 - 2g = 2$ as S has genus 0.

Example. Let S be a complex torus and triangulate the fundamental parallelogram. Triangulate it into 18 triangles. Have

$$F = 18, E = 27, V = 9$$

so

$$\chi(S) = 0$$

which agrees with $2 - 2g$ as S has genus 1.

Remark. The topological torus admits infinitely many nonisomorphic conformal structures. See example sheet 2. For future reference, the collection for a fixed surface of the conformal structures it admits is known as the *Teichmüller space*. It is the key object in the advanced study of Riemann surfaces.

Theorem 3.1 (Riemann-Hurwitz formula)**.** *Let $f : R \to S$ be a nonconstant analytic map of compact Riemann surfaces of degree $n \geq 1$. Then*

$$\chi(R) = n\chi(S) - \sum_{P \in R} (e_P - 1)$$

where $e_P = m_f(P)$, the ramification index of f at P.

Intuitively, the first term on RHS says that in a covering every sufficiently small triangle in S have n homeomorphic preimages in R. The second terms add a correction term in case of ramification, as at a branch point P, e_P vertices, each from a preimage, are mapped to a single point.

Proof. The idea is to consider preimage of triangulations of S under f and compute its Euler characterisic. Call $\{Q_1, \dots, Q_r\}$ the branch points of f. Choose chart preimages of disks (as in the proof of valency theorem) and use compactness, we can find open sets $U_1, \dots, U_r, U_{r+1}, \dots, U_s$ of S so that

1. if $j > r$ then $f^{-1}(U_j)$ is a disjoint union of preimages $V_1, \dots, V_n$, and $f|_{V_i} : V_i \to U_j$ is an isomorphism,
2. if $1 \leq j \leq r$ then for each component V of $f^{-1}(U_j)$, we have a unique preimage P of Q_j, and $f|_V : V \to U_j$ is an e_P-to-1 map, whose local expression is an e_P-to-1 powering map.

Let $\mathcal{T}$ be a triangulation of S which contains the Q_i's as vertices. We can refine the triangulation to assume wlog that every triangle is contained in some U_j. Given $T \in \mathcal{T}$, if $j > r$ and $T \subseteq U_j$ then $f^{-1}(T)$ is a disjoint union of copies of T. If $1 \leq j \leq r$ and $T \subseteq U_j$, if Q_j is not a vertex of T, refine if necessary so triangles are contained in some $2\pi/e_p$ sector, then again $f^{-1}(T)$ is a disjoint union of triangles, by valency theorem. If, however, Q_j is a vertex of T, again refine if needed, we have e_P triangles as preimage, which have common vertex P.

Thus we have that the preimage of $\mathcal{T}$ is a triangulation of R. Let F', E', V' be the number of faces, edges and vertices of this triangulation. Have

$$F' = nF, E' = nE, V' = nV - \sum_{P \in R} (e_P - 1)$$

so

$$\chi(R) = n\chi(S) - \sum_{P \in R} (e_P - 1).$$

□

Remark. Equivalently we may express Euler characteristic in terms of genus,

$$2g_R - 2 = n(2g_S - 2) + \sum_{P \in R} (e_P - 1).$$

There are lots we can say about this. At the very least, ramification satisfies certain relation modulo 2. In addition as $e_P - 1 \geq 0$, Riemann-Hurwitz restricts the existence of degree n maps in terms of genus of surfaces. We list a few implications here.

Corollary 3.2.

1. $$\sum_{P \in R} (e_P - 1) = 0 \pmod 2.$$
2. $g_R \geq g_S$.
3. *If* $g_S = 0$ *and* $g_R > 1$ *then* f *must be ramified.*
4. *If* f *is unramified and* $g_S > 1$ *then either* $g_R = g_S$ *and* $n = 1$ *or* $g_R > g_S$.
5. *If* R *admits an unramified self-map with degree* $n > 1$ *then* $g_R = 1$.

Example. Let $R' = \{(z, w) : w^2 = z^3 - z\} \subseteq \mathbb{C}^2$. Let $f(z) = z^3 - z$. The ramification points of $\pi_z : R' \to \mathbb{C}$ are precisely $(-1, 0), (0, 0)$ and $(1, 0)$. Charts around these points are given by π_w so for example, the valency of π_z at $(0, 0)$ is the degree of

$$\pi_z \circ \pi_w^{-1}$$

at 0. But $\pi^{-1}(w) = (f^{-1}(w^2), w)$ for some branch of f^{-1} locally so $\pi_z \circ \pi_w^{-1}(w) = f^{-1}(w^2)$. We can show

$$\frac{d}{dw} f^{-1}(w^2) = 0$$
$$\frac{d^2}{dw^2} f^{-1}(w^2) \neq 0$$

so these points ramify with multiplicity 2 each.

We have seen that R' embeds analytically in its compactification R. Claim that π_z extends to some analytic map $\overline{\pi}_z : R \to \mathbb{C}_\infty$ with $\overline{\pi}_z(R \setminus R') = \{\infty\}$: $\frac{1}{\pi_z}$ is a bounded analytic function on a punctured neighbourhood of $P \in R \setminus R'$ with

$$\lim_{Q \to P} \frac{1}{\pi_z(Q)} = 0$$

so P is a removable singularity of $\frac{1}{\pi_z}$. Thus extends to P and takes value 0. This is precisely an analytic map to $\mathbb{C}_\infty$.

Now we have an analytic map $\overline{\pi}_z : R \to \mathbb{C}_\infty$ between compact Riemann surfaces. By considering, for example, that for finite z, $w^2 = z^3 - z$ has two solutions, $\overline{\pi}_z$ has degree 2. Thus the only point P in $R \setminus R'$ ramifies with multiplicity 2.

Suppose we merely knew $\overline{\pi}_z : R \to \mathbb{C}_\infty$ existed but didn't know how many points over ∞ were in $R \setminus R'$. Must have $\overline{\pi}^{-1}(\infty) = R \setminus R'$ so either there are two points in $R \setminus R'$ each with degree 1 or one point with degree 2. By Riemann-Hurwitz,

$$2g_R - 2 = 2(0-2) + \sum_{P \in R} (e_P - 1)$$

Reduce mod 2, there must be ramification above ∞, and so there is a single point in $R \setminus R'$, mapped with degree 2 to ∞ and $2g_R - 2 = -4 + 4$ so $g_R = 1$.

Example. Let R and R' be as above and $X' = \{(x, y) : y^2 = x^4 - 1\} \subseteq \mathbb{C}^2$. X' admits a complex structure via π_x, π_y, and a compactification X via topological gluing such that both π_x and π_y extend to X. There exists a map

$$\begin{aligned} X' &\to R' \\ (x, y) &\mapsto (x^2, xy) \end{aligned}$$

which extends to an analytic map $f : X \to R$. This map has degree 2, and is ramified if and only if $x = -x$ and $y = -y$, so in particular f is unramified on X'. By Riemann-Hurwitz,

$$2g_X - 2 = 2(2 \cdot 1 - 2) + \sum_{P \in X} (e_P - 1).$$

The points of $X \setminus X'$ are mapped to $R \setminus R'$. Again reduce mod 2, there are two points of $X \setminus X'$ and f is unramified at both. $g_X = 1$.

Example (Fermat curve)**.** For $d \geq 3$, define the *Fermat curve*

$$F'_d = \{(x, y) \in \mathbb{C}^2 : x^d + y^d = 1\}.$$

By example sheet 3 Q13 there exists a compactification F_d of F'_d by gluing

$$\{(t, u) \in \mathbb{C}^2 : 1 + u^d = t^d\}$$

via $t = \frac{1}{x}, u = \frac{y}{x}$, and π_x, π_y extend to analytic maps $F_d \to \mathbb{C}_\infty$. Note that there are d points in $F_d \setminus F'_d$. π_x has degree d with ramification at $(\zeta_d, 0)$ for all dth roots of unity ζ_d. By Riemann-Hurwitz, as π_x has multiplicity d at such point,

$$2g_{F_d} - 2 = d(2 \cdot 0 - 2) + d(d-1)$$

so

$$g_{F_d} = \frac{(d-1)(d-2)}{2}.$$

Corollary 3.3. *There exist Riemann surfaces of arbitrarily large genus.*

Our next goal is to show complex tori are algebraic, i.e. they are all compactification of $\{(x, y) \in \mathbb{C}^2 : p(x, y) = 0\}$ where p is some polynomial.

Definition (period)**.** Let $f : \mathbb{C} \to \mathbb{C}_\infty$ be nonconstant analytic. $\omega \in \mathbb{C}$ is a *period* of f if

$$f(z + \omega) = f(z)$$

for all $z \in \mathbb{C}$.

It is immediate by principle of isolate zeros that periods of f consists of isolated point and they form an additive group. By example sheet 3 Q1, let Λ be the set of periods of f, then exactly one of the following happens:

1. $\Lambda = \{0\}$,
2. $\Lambda = \mathbb{Z}\omega$ for some $\omega \neq 0$,
3. $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$ with $\frac{\omega_2}{\omega_1} \notin \mathbb{R}$.

In case 2 we say f is *simply periodic* and in case 3 f is *doubly periodic*, or *elliptic*.

Proposition 3.4. *Suppose f is simply periodic. By composing with scalar, assuming wlog $\Lambda = \mathbb{Z}$. Then there exists analytic map $\tilde{f} : \mathbb{C}^\times \to \mathbb{C}_\infty$ such that*

$$\tilde{f}(e^{2\pi i z}) = f(z).$$

Proof. Since $\Lambda = \mathbb{Z}$ there is a well-defined function $\tilde{f} : \mathbb{C}^\times \to \mathbb{C}_\infty$ via $\tilde{f}(e^{2\pi i z}) = f(z)$. Left to show this is analytic.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{f} & \mathbb{C}_\infty \\ {\scriptstyle z\mapsto e^{2\pi i z}}\downarrow & \nearrow_{\tilde{f}} & \\ \mathbb{C}^\times & & \end{array}$$

$\tilde{f}$ is continuous as $e^{2\pi i z}$ and f are continuous and open. Locally $\tilde{f}(w) = f(\frac{\log w}{2\pi i})$ so f is analytic. ☐

Let f be doubly periodic with $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$, so that f takes all its values on a fundamental parallelogram

$$P_z = \{z + t_1\omega_1 + t_2\omega_2 : t_1, t_2 \in [0,1)\}.$$

If f has no pole then f is bounded on $\mathbb{C}$ so constant by Liouville.

Proposition 3.5. *Let f be doubly periodic with periods Λ. Then there exists $\tilde{f} : \mathbb{C}/\Lambda \to \mathbb{C}_\infty$ nonconstant analytic so that $f = \tilde{f} \circ \pi$ where $\pi : \mathbb{C} \to \mathbb{C}/\Lambda$ is the quotient.*

Proof. Ditto.

$$\begin{array}{ccc} \mathbb{C} & \xrightarrow{f} & \mathbb{C}_\infty \\ {\scriptstyle \pi}\downarrow & \nearrow_{\tilde{f}} & \\ \mathbb{C}/\Lambda & & \end{array}$$

☐

Corollary 3.6. *If f is nonconstant elliptic then exists $n \geq 1$ such that* $\deg f = n$*, i.e. every point in $\mathbb{C}_\infty$ has n preimages, counting multiplicity, on any period parallelogram.*

Proof. Immediate from valency theorem. ☐

Here we say f has degree n to mean $\tilde{f} : \mathbb{C}/\Lambda \to \mathbb{C}_\infty$ has degree n.

Corollary 3.7. *If f is nonconstant elliptic of degree n then $n \geq 2$.*

Proof. If $n = 1$ then $\tilde{f}$ is a conformal isomorphism. But $\mathbb{C}/\Lambda$ and $\mathbb{C}_\infty$ are not even homeomorphic.

Alternatively, choose a period parallelogram P for Λ with no zeros or poles of f on its boundary (exists by principle of isolated zeros and discreteness of lattice). Then by residue theorem,

$$\sum_{z \in P} \operatorname{res}_z(f) = \oint_{\partial P} f(z) dz = 0$$

where the last equality is because f is doubly periodic. Thus there are at least 2 poles of f counting multiplicity. □

3.2 Weierstrass p-function

We exhibit a degree 2 elliptic function asssociated to each lattice.

Definition (Weierstrass $\wp$-function)**.** Let Λ be a lattice in $\mathbb{C}$. The *Weierstrass $\wp$-function* associated to Λ is

$$\wp(z) = \wp_\Lambda(z) = \frac{1}{z^2} + \sum_{\omega \in \Lambda \setminus \{0\}} \left(\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2} \right).$$

To show that we have written down a sensible thing we should check this converges. We use the following lemma:

Lemma 3.8. *Let Λ be a lattice. Then $\sum_{\omega \in \Lambda \setminus \{0\}} \frac{1}{|\omega|^t}$ converges if and only if $t > 2$.*

As a comment, in general when trying to understand a series defined in terms of a lattice Λ, we always relate Λ to the square lattice $\mathbb{Z} \oplus \mathbb{Z}i$.

Proof. Let $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$. Consider the function $(t_1, t_2) \mapsto |t_1\omega_1 + t_2\omega_2|$. This is continuous, and since $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$, this is nonzero on $\mathbb{R}^2 \setminus \{0\}$, and so achieves positive bounds c_1, c_2 on the compact set $\{(t_1, t_2) : |t_1| + |t_2| = 1\}$, i.e.

$$0 < c_1 \leq |t_1\omega_1 + t_2\omega_2| \leq c_2$$

on this set.

Given $(k, \ell) \in \mathbb{Z}^2 \setminus \{(0, 0)\}$, let

$$t_1 = \frac{k}{|k| + |\ell|}$$
$$t_2 = \frac{\ell}{|k| + |\ell|}$$

so that

$$c_1(|k| + |\ell|) \leq |k\omega_1 + \ell\omega_2| \leq c_2(|k| + |\ell|).$$

So $\sum_{\omega\in\Lambda\setminus\{0\}} \frac{1}{|\omega|^2}$ converges if and only if $\sum_{(k,\ell)\in\mathbb{Z}\setminus\{0\}} \frac{1}{(|k|+|\ell|)^t}$ converges but

$$\sum_{(k,\ell)\in\mathbb{Z}\setminus\{0\}} \frac{1}{(|k|+|\ell|)^t} = \sum_{q=1}^{\infty} \sum_{|k|+|\ell|=q} \frac{1}{q^t} = \sum_{q=1}^{\infty} \frac{4q}{q^t}$$

which converges if and only if $t > 2$. □

Proposition 3.9. *$\wp$ converges to an elliptic function with period lattice Λ. Moreover $\wp$ is an even function of degree 2.*

Proof. We show $\wp$ converges on compact sets: choose $R \gg 1$ and let $|z| \leq R$. There exist finitely many points $\Lambda \cap D(0, 2R)$ and if $|\omega| > 2R$ for $\omega \in \Lambda$,

$$\left|\frac{1}{(z-\omega)^2} - \frac{1}{\omega^2}\right| = \left|\frac{2\omega z - z^2}{\omega^2(z-\omega)^2}\right| \leq \frac{R|2\omega - z|}{|\omega|^4 \cdot \frac{1}{4}} \leq \frac{12R}{|\omega|^3}$$

so by the lemma we have convergence.

Thus $\wp$ is meromorphic with well-define derivative

$$\wp'(z) = \sum_{\omega\in\Lambda} \frac{-2}{(z-\omega)^3}.$$

$\wp'$ has all $\omega \in \Lambda$ as periods so $\wp(z+\omega) - \wp(z)$ is constant. Evaluate at, for example, $z = -\frac{\omega}{2}$, we get $\wp(\frac{\omega}{2}) - \wp(-\frac{\omega}{2})$. But $\wp$ is manifestly even so this constant is 0. Thus every $\omega \in \Lambda$ is a period for $\wp$. Moreover since these are the only poles, they are the only periods for $\wp$.

Finally since $0 \mapsto \infty$ with degree 2, and we can choose a period parallelogram with no other lattice points so no other pole of $\wp$, by a previous corollary $\deg \wp = 2$. □

Remark.

1. Using factorisation through quotient, we can show that $\wp$ is the unique meromorphic function that satisfy the following:

   (a) elliptic with periods Λ,

   (b) have poles only in Λ,

   (c) $\wp(z) - \frac{1}{z^2} \to 0$ as $z \to 0$.

2. $\wp'$ has degree 3, with a pole of degree 3 at lattice points, $\wp'$ is odd and $\wp'(\frac{\omega}{2}) = \wp'(-\frac{\omega}{2})$ for $\omega \in \Lambda$ by periodicity. Thus $\wp'(\frac{\omega}{2}) = 0$, i.e. $\wp' = 0$ at the half-lattice points. There are 3 of these, so these are the only zeros of $\wp'$. So $\wp$ ramifies at the lattice points and half-lattice points. Because $\deg \wp = 2$, the multiplicity is 2 at all such points. Additionally, the branch points $\infty = \wp(0), e_1, e_2, e_3$ are distinct.

   Note that Riemann-Hurwitz is satisfied on $\mathbb{C}/\Lambda$: $\wp$ induces an analytic $\tilde{\wp} : \mathbb{C}/\Lambda \to \mathbb{C}_\infty$ of degree 2 so

   $$2g_{\mathbb{C}/\Lambda} - 2 = 2(2g_{\mathbb{C}_\infty} - 2) + 4.$$

Proposition 3.10. *Let Λ be a lattice. There exist constants g_2, g_3 (depending on Λ) such that $\wp_\Lambda$ satisfies*

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3.$$

Proof. Locally around 0, we have Laurent series

$$\wp(z) = \frac{1}{z^2} + az^2 + ...$$

because $\wp(z) - \frac{1}{z^2} = 0$ at $z = 0$ and the first order term vanishes because $\wp$ is even. So

$$\wp'(z) = -\frac{2}{z^3} + 2az + ...$$

square and set $g_2 = 20a$,

$$(\wp')^2 - 4\wp^3 = \frac{-g_2}{z^2} + \text{ analytic}$$

so

$$(\wp')^2 - 4\wp^3 + g_2\wp(z)$$

is analytic so constant as it has no poles. Thus

$$(\wp')^2 = 4\wp^3 - g_2\wp - g_3$$

as required. □

Note.

1. Note that $4\wp^3 - g_2\wp - g_3 = 4(\wp - e_1)(\wp - e_2)(\wp - e_3)$ where e_1, e_2, e_3 are the branch points of $\wp$. In particular, the sum $e_1 + e_2 + e_3 = 0$.
2. The ramification points of $\wp$ are precisely the elements of the group $\mathbb{C}/\Lambda$ which are 2-torsion, i.e. $2P = 0$.

Corollary 3.11. *Let $\mathbb{C}/\Lambda$ be a complex torus and g_2, g_3 as in the previous proposition. Then $\mathbb{C}/\Lambda$ is conformally isomorphic to the Riemann surface X compactifying*

$$X' = \{(z, w) \in \mathbb{C}^2 : w^2 = 4z^3 - g_2 z - g_3\}.$$

Every complex torus is algebraic.

Proof. Exercise: As the e_i's are distinct, the coodinates define a Riemann surface, and add a single point via gluing to give X with analytic embedding.

Define

$$\begin{aligned} F : \mathbb{C}/\Lambda &\to X \\ z &\mapsto (\wp(z), \wp'(z)) \end{aligned}$$

Claim F has degree 1, which will imply that F is an isomorphism by valency theorem. Let P be the period parallelogram for Λ centred at 0. For z in interior of P, $\wp(z) = \wp(w)$ if and only if $z = \pm w$ for w in the interior of P. If $z = -w$ then $\wp'(z) = \wp'(-z)$, and since it is odd, $\wp'(z) = -\wp'(z) = 0$. Thus $z \neq 0$ is the unique preimage under F of $F(z)$, i.e. $\deg F = 1$. □

Remark. In example sheet we show $\mathbb{C}/(\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2) \cong \mathbb{C}/(\mathbb{Z}\oplus\tau\mathbb{Z})$ where $\tau = \frac{\omega_2}{\omega_1}$, and $\mathbb{C}/(\mathbb{Z} \oplus \tau_1\mathbb{Z}) \cong \mathbb{C}/(\mathbb{Z} \oplus \tau_2\mathbb{Z})$ if and only if τ_1, τ_2 are in the same orbit of action of $\mathrm{SL}_2(\mathbb{Z})$. Algebraically, g_2, g_3 do *not* quite determine $\mathbb{C}/\Lambda$, rather we have the *j-invariant* defined by

$$j(\Lambda) = \frac{1728 g_2^3}{g_2^3 - 27 g_3^2}$$

and $j(\Lambda_1) = j(\Lambda_2)$ if and only if $\mathbb{C}/\Lambda_1 \cong \mathbb{C}/\Lambda_2$.

Theorem 3.12. *Let f be elliptic with periods Λ. Then*

$$f = Q_1(\wp) + \wp' Q_2(\wp)$$

for some Q_1, Q_2 rational. Moreover, if f is even then we can take $Q_2 = 0$.

Compare this with the statement that meromorphic functions on $\mathbb{C}_\infty$ are precisely rationals.

Proof. First assume f is even. Let

$$E = \{z \in \mathbb{C} : z \in \frac{1}{2}\Lambda \text{ or } f'(z) = 0\}$$

so to avoid branch points of $\wp$. As $f(E)$ is finite, we can find $c \neq d$ in $\mathbb{C} \setminus f(E)$ so that

$$g(z) = \frac{f(z) - d}{f(z) - c}$$

has only simple zeros and poles. Then in a period parallelogram centred at 0, we can write the zeros of g as $\{a_1, \dots, a_n, -a_1, \dots, -a_n\}$ and poles as $\{b_1, \dots, b_n, -b_1, \dots, -b_n\}$. Define

$$h(z) = \frac{(\wp(z) - \wp(a_1)) \cdots (\wp(z) - \wp(a_n))}{(\wp(z) - \wp(b_1)) \cdots (\wp(z) - \wp(b_n))}$$

so that h has the same poles and zeros (counting multiplicity) as g. Thus $g(z) = kh(z)$ for some constant k, so that

$$f = Q_1(\wp)$$

for some rational Q_1.

If f is odd then $\frac{f}{\wp'}$ is even so

$$f = \wp' Q_2(\wp)$$

by the same argument. Any f can be written as sum of an even and odd function. □

4 Quotients of Riemann surfaces

Definition (properly discontinuous action)**.** Given a group G of homeomorphisms of a topological space X, we say G acts *properly discontinuously* if for every $x \in X$ there exists a neighbourhood U of x such that if $g(U) \cap h(U) \neq \emptyset$ then $g = h$.

Remark.

1. If there exists $g \in G$ nontrivial with a fixed point then G does not act properly discontinuously.
2. If G is finite, G acts properly discontinuously implies that all stabilisers are trivial so all orbits have size $|G|$.

Given such a group action, we can form the quotient X/G and equip it with quotient topology via $\pi : X \to X/G$. π is a local homeomorphism, X is path-connected so π is a regular cover. Note that if G is finite then π has well-defined degree $|G|$.

Lemma 4.1. *If X is a Riemann surface and $G \leq \operatorname{Aut}(X)$ acting properly discontinuously, then X/G is a Riemann surface via π^{-1} together with charts of X. Moreover the transition maps are in G.*

Proof. Easy. □

Example. $\mathbb{C}/\Lambda$ is the lattice resulted from translation action.

Proposition 4.2 (Hurwitz)**.** *Let X be a compact Riemann surface of genus $g_X \geq 2$. Let $G \leq \operatorname{Aut}(X)$ act properly discontinuously on X. Then G is finite and*

$$|G| \leq g_X - 1.$$

Proof. Suppose G is not finite. Fix $P_0 \in X$. Then $\{g(P_0) : g \in G\}$ is infinite. By compactness of X it has a converging subsequence $g_n(P) \to Q$. For any neighbourhood V of Q and $n, m \gg 1$, we have

$$P_0 \in g_n^{-1}(V) \cap g_m^{-1}(V).$$

Absurd.

By previous remark $\pi : X \to X/G$ is a degree $|G|$ map of compact Riemann surfaces so by Riemann-Hurtwitz

$$2g_X - 2 = |G|(2g_{X/G} - 2)$$

as there is no ramification (π is a local homeomorphism). As both sides are positive and $2g_{X/G} - 2 \geq 2$,

$$|G| \leq g_X - 1.$$

□

Remark. There is no such bound on $|G|$ for $g_X = 1$: complex tori admit translations via the group structure so choosing an arbitrarily large discrete subgroup of $\mathbb{C}/\Lambda$ to translate by, we obtain arbitrarily large $|G|$. For example let G be the points $P \in \mathbb{C}/\Lambda$ such that

$$[n]P = \mathrm{id}_{\mathbb{C}/\Lambda}.$$

4.1 Uniformisation theorem and consequences

Theorem 4.3 (uniformisation theorem). *Let R be a simply connected Riemann surface. Then R is conformally isomorphic to one of $\mathbb{C}, \mathbb{C}_\infty, \mathbb{D}$ (or $\mathbb{H}$, the upper half plane).*

Proof. Non-examinable. Omitted. □

Fact. Any Riemann surface X is the quotient $\pi : \tilde{X} \to X$ of a simply connected Riemann surface $\tilde{X}$ by the deck transformation of π, i.e. automorphisms $p : \tilde{X} \to \tilde{X}$ such that $\pi \circ p = \pi$. This group acts properly discontinuously. Note then that π is regular.

Definition (universal cover). $\pi : \tilde{X} \to X$ is the *universal cover* of X.

Remark. $\mathbb{C}, \mathbb{C}_\infty, \mathbb{D}$ are distinct: $\mathbb{C}_\infty$ is the only compact one, and if there is an isomorphism $\mathbb{C} \cong \mathbb{D}$ then it is bounded and entire, contradicting Liouville's theorem.

Let's discuss different cases.

1. X has $\mathbb{C}_\infty$ as universal cover: i.e. there exists $G \leq \mathrm{Aut}(\mathbb{C}_\infty)$ acting properly discontinuously such that $X \cong \mathbb{C}_\infty/G$. We've already figured out that $\mathrm{Aut}(\mathbb{C}_\infty)$ is the set of Möbius transformations. Since any Möbius transformation has a fixed point in $\mathbb{C}_\infty$, G is trivial so $X \cong \mathbb{C}_\infty$.

   This also agrees with Riemann-Hurwitz: we can only decrease genus.

2. X has $\mathbb{C}$ as universal cover. As any automorphism of $\mathbb{C}$ extends to an automorphism of $\mathbb{C}_\infty$, we concluded that $\mathrm{Aut}(\mathbb{C}) = \{az+b : a \neq 0, b \in \mathbb{C}\}$. If $a \neq 1$ then $z \mapsto az + b$ has a fixed point so $G \leq \{z + b : b \in \mathbb{C}\}$. Identify $z \mapsto z + b$ with b, G must consist of isolated points. By example sheet 3 Q1, G is one of $0, \mathbb{Z}\omega$ or $\mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$. So one of the following happens:

   $$X \cong \mathbb{C}$$
   $$X \cong \mathbb{C}/2\pi\mathbb{Z} \cong \mathbb{C}^*$$
   $$X \cong \mathbb{C}/\Lambda$$

   where Λ is a lattice.

   Remark. If X is compact with $\mathbb{C}_\infty$ or $\mathbb{C}$ as universal cover, $g_X \in \{0, 1\}$. Equivalently if $g_X \geq 2$ then X must have $\mathbb{D}$ as universal cover.

3. X has $\mathbb{D}$ as universal cover: we can only barely scratch the surface the final, and most interesting family. Recall (or note) that

   $$\mathrm{Aut}(\mathbb{D}) = \{z \mapsto e^{i\theta}\frac{z-a}{1-\overline{a}z}\}.$$

   Alternatively,

   $$\mathrm{Aut}(\mathbb{H}) = \mathrm{PSL}_2(\mathbb{R}) = \mathrm{SL}_2(\mathbb{R})/\{\pm 1\}.$$

   The subgroups of $\mathrm{PSL}_2(\mathbb{R})$ which act properly discontinuously are *Fuschsian groups*, studied in *hyperbolic geometry*.

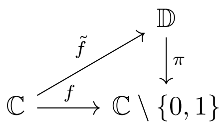
Corollary 4.4. *If X is uniformised by $\mathbb{D}$ then X is a metric space.*

Proof. $\mathrm{Aut}(\mathbb{D})$ are isometries for the hyperbolic metric. □

Corollary 4.5 (Picard)**.** *If $f : \mathbb{C} \to \mathbb{C} \setminus \{0, 1\}$ is analytic then f is constant.*

Proof. Claim that $\mathbb{C} \setminus \{0, 1\}$ has $\mathbb{D}$ as universal cover: if not then as it is non-compact it is isomorphic to either $\mathbb{C}$ or $\mathbb{C}^*$. Suppose $\varphi : \mathbb{C} \setminus \{0, 1\} \to \mathbb{C}$ is an isomorphism, then by Liouville this is unbounded near ∞. If the singularity at ∞ is essential then by Casorati-Weierstrass $\deg \varphi > 1$. So there is a pole of order 1 at ∞, so φ extends to an isomorphism $\mathbb{C}_\infty \setminus \{0, 1\} \to \mathbb{C}_\infty$. Similar for $\mathbb{C}^*$.

Given such an f, as $\mathbb{C}$ is simply connected it can be lifted to $\tilde{f} : \mathbb{C} \to \mathbb{D}$. Can check $\tilde{f}$ is analytic so constant, so f is too.



□

Corollary 4.6 (Riemann mapping theorem)**.** *Let $U \subsetneq \mathbb{C}$ be a domain. If U is simply connected then $U \cong \mathbb{D}$.*

Proof. Similar to above, suffices to show $U \ncong \mathbb{C}$. If it were we would have $\mathbb{C}_\infty \cong U \cup \{\mathrm{pt}\}$, contradicting compactness. □

5 Non-examinable collection

Let

$$X' = \{(x, y) \in \mathbb{C}^2 : y^2 = (x - \alpha_1) \cdots (x - \alpha_{2g+2})\}$$

where $\alpha_1, \dots, \alpha_{2g+2}$ are distinct points in $\mathbb{C}$. X' is a Riemann surface via π_x, π_y, and can be compactified via gluing to

$$Y' = \{(z, w) \in \mathbb{C}^2 : w^2 = (1 - \alpha_1 z) \cdots (1 - \alpha_{2g+2} z)\}$$

and $(x, y) \mapsto (\frac{1}{x}, \frac{y}{x^{g+1}})$. Call the compactification X and note that $X \setminus X'$ contains 2 points. π_x extends to X with $\pi_x(X \setminus X') = \{\infty\}$, has degree 2 so by Riemann-Hurwitz

$$2g_X - 2 = 2(-2) + (2g + 2)$$

so $g_X = g$. This is a natural generalisation of Fermat curve and in particular shows that we can construct a Riemann surface with arbitrary genus.

Define $i_h : X \to X$ by $(x, y) \mapsto (x, -y)$ on X' and $(z, w) \mapsto (z, -w)$ on Y'. Check it is well-defined.

$$G = \langle i_h \rangle \leq \mathrm{Aut}(X)$$

does *not* act properly discontinuously as $(\alpha_i, 0)$ is a fixed point for all i. Nonetheless we have a topological covering $\pi_x : X \to X/G$, which is isomorphic to $\mathbb{C}_\infty$. We study Riemann surfaces via understanding the collection of such quotients.

Index