University of
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Mathematics Tripos

Part II

Representation Theory

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Contents

0 Introduction

Representation theory is the theory of how *groups* act as groups on *vector spaces*. Here

1. groups are either finite or compact topological groups,
2. vector spaces are finite-diemnsional and usually over $\mathbb{C}$,
3. actions are linear.

1 Group actions

Notation.

1. $\mathbb{F}$ is a field, usually $\mathbb{C}$, $\mathbb{R}$ or $\mathbb{Q}$. In particular $\mathbb{F}$ is a field of characteristic zero. Thus in this course we mostly deal with what is known as *ordinary representation theory*. Sometimes $\mathbb{F} = \mathbb{F}_p$ or $\overline{\mathbb{F}_p}$, and the study of which is known as *modular representation theory*.

2. V is a vector space over $\mathbb{F}$ and will always be finite-dimensional.

3. $\mathrm{GL}(V) = \{\theta : V \to V \text{ linear invertible}\}$.

1.1 Review of linear algebra

If $\dim_{\mathbb{F}} V = n$, choose basis $e_1, \dots, e_n$ over $\mathbb{F}$ so we can identify it with $\mathbb{F}^n$. Then $\theta \in \mathrm{GL}(V)$ correponds to an $n \times n$ matrix $A_\theta = (a_{ij})$, where

$$\theta(e_j) = \sum_i a_{ij} e_i$$

for $1 \le j \le n$. In fact we have $A_\theta \in \mathrm{GL}_n(\mathbb{F})$, the *general linear group*. Thus

Proposition 1.1. *The map*

$$\begin{aligned} \mathrm{GL}(V) &\to \mathrm{GL}_n(\mathbb{F}) \\ \theta &\mapsto A_\theta \end{aligned}$$

is a group isomorphism.

Proof. Check $A_{\theta_1\theta_2} = A_{\theta_1} A_{\theta_2}$ and bijectivity. □

Choosing a different basis gives different isomorphism to $\mathrm{GL}_n(\mathbb{F})$, but

Proposition 1.2. *Matrices A_1, A_2 represent the same element of* $\mathrm{GL}(V)$ *with respect to different basis if and only if they are* conjugate *or* similar*, i.e. exists $X \in \mathrm{GL}_n(\mathbb{F})$ such that $A_2 = X A_1 X^{-1}$.*

Recall that the *trace* of a matrix A is

$$\operatorname{tr} A = \sum_i a_{ii}.$$

Proposition 1.3. *As* $\operatorname{tr}(XAX^{-1}) = \operatorname{tr} A$ *we can define*

$$\operatorname{tr} \theta = \operatorname{tr}(A_\theta)$$

which is independent *of the basis chosen.*

Some notes on diagonalisation:

Example. Let $\alpha \in \mathrm{GL}(V)$ where V is a finite-dimensioanl vector space over $\mathbb{C}$ with $\alpha^m = \mathrm{id}$ for some m. Then α is diagonalisable.

Proposition 1.4. *Let V a finite-dimensional vector space over $\mathbb{C}$ and $\alpha \in \mathrm{End}(V)$. Then α is diagonalisable if and only if there exists a polynomial f with distinct linear factors with $f(\alpha) = 0$.*

Remark. In the previous example take $f(X) = X^m - 1 = \prod_{j=0}^{m-1}(X - \omega^j)$ where $\omega = e^{\frac{2\pi i}{m}}$.

Proposition 1.5. *A finite family of commuting separately diagonalisable non-singular transformations of a $\mathbb{C}$-vector space can be simultaneously diagonalised.*

1.2 Basic group theory

We have an ample supply of basic groups:

1. symmetric group $S_n = \mathrm{Sym}(X)$ on a set $X = \{1, \dots, n\}$ is the set of all permutations of X. $|S_n| = n!$.
2. alternating group A_n with $|A_n| = \frac{n!}{2}$ consists of all even permutations.
3. cyclic group of order n: $C_n = \langle x : x^m = 1 \rangle$. For example $(\mathbb{Z}/m\mathbb{Z}, +)$. It's also
   - the group of mth root of unity in $\mathbb{C}$ (which embeds to $\mathrm{GL}_1(\mathbb{C}) = \mathbb{C}^\times$),
   - the group of rotations, centre 0 of a regular m-gon in $\mathbb{R}^2$ (which embeds to $\mathrm{GL}_2(\mathbb{R})$).
4. diahedral groups: $D_{2m} = \langle x, y : x^m = y^2 = 1, yxy^{-1} = x^{-1} \rangle$ of order $2m$. Think of this as set of rotations and reflections preserving a regular m-gon.
5. quaternion group: $Q_8 = \langle x, y : x^4 = 1, y^2 = x^2, yxy^{-1} = x^{-1} \rangle$ of order 8. In $\mathrm{GL}_2(\mathbb{C})$, can put

$$i = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad j = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad k = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}$$

   then $Q_8 = \{\pm i, \pm j, \pm k, \pm I_2\}$.

Definition (conjugacy class, centraliser)**.** The *conjugacy class* of $g \in G$ is

$$\mathcal{C}_G(g) = \{xgx^{-1} : x \in G\}.$$

Then

$$|\mathcal{C}_G(g)| = |G : C_G(g)|$$

where $C_g(g) = \{x \in g : xg = gx\}$ is the *centraliser* of g in G.

Definition (group action)**.** Let G be a group and X be a set. G *acts* on X

if there exists a map

$$\begin{aligned} G \times X &\to X \\ (g, x) &\mapsto gx \end{aligned}$$

such that

$$\begin{aligned} 1x &= x \text{ for all } x \in X \\ (gh)x &= g(hx) \text{ for all } g, h \in G, x \in X \end{aligned}$$

Proposition 1.6 (permutation representation)**.** *Given an action of G on X, we obtain a homomorphism $\theta : G \to \mathrm{Sym}(X)$, called the* permutation representation *of G.*

Proof. For $g \in G$ the function $\theta_g : X \to X, x \mapsto gx$ is a permutation of X (with inverse $\theta_{g^{-1}}$. Moreover for all $g_1, g_2 \in G$,

$$\theta_{g_1 g_2} = \theta_{g_1} \theta_{g_2}$$

since $(g_1 g_2)x = g_1(g_2 x)$ for all $x \in X$. □

In this course X is often a finite-dimensional vector space over $\mathbb{F}$and the action is required to be *linear*, namely

$$\begin{aligned} g(v_1 + v_2) &= gv_1 + gv_2 \\ g(\lambda v) &= \lambda g(v) \end{aligned}$$

for all $v_1, v_2 \in V, g \in G, \lambda \in \mathbb{F}$.

2 Basic definitions

Let G be a finite group, $\mathbb{F}$ a field.

Definition (representation)**.** Let V be a finite-dimensional vector space over $\mathbb{F}$. A *(linear) representation* of G on V is a group homomorphism

$$\rho = \rho_V : G \to \mathrm{GL}(V).$$

Write ρ_g for $\rho_V(g)$.

So for each $g \in G, \rho_g \in \mathrm{GL}(V), \rho_1 = \mathrm{id}$ and $\rho_{g_1 g_2} = \rho_{g_1}\rho_{g_2}, \rho_{g_1^{-1}} = \rho_{g_1}^{-1}$.

The *dimension* or *degree* of ρ is $\dim_{\mathbb{F}} V$.

Reall that $\ker \rho \trianglelefteq G$ and $G/\ker\rho \cong \rho(G) \leq \mathrm{GL}(V)$. We say ρ is *faithful* if $\ker \rho = \{1\}$.

We repeat what we said in introduction, namely the correspondence between group representation and group action:

Definition (linear action)**.** G acts *linearly* on V if ther exists a linear action $G \times V \to V, (g, v) \mapsto gv$ such that

$$\begin{aligned} (g_1 g_2)v &= g_1(g_2 v), 1v = v \\ g(v_1 + v_2) &= gv_1 gv_2, g(\lambda v) = \lambda g(v) \end{aligned}$$

Now if G acts on V, the map

$$\begin{aligned} G &\to \mathrm{GL}(V) \\ g &\mapsto \rho_g \end{aligned}$$

with $\rho_g : v \mapsto gv$ is a representation. Conversely, given a representation $G \to \mathrm{GL}(V)$ we have a linear action of G on V via

$$gv = \rho(g)(v).$$

Remark. We also say that V is a *G-space* or that V is a *G-module*. This use of "module" might seen unconventional but if fact if you define the *group algebra*

$$\mathbb{F}G = \left\{ \sum_{g \in G} \alpha_g g : \alpha_g \in \mathbb{F} \right\}$$

with natural addition an multiplication, then V is an $\mathbb{F}G$-module. $\mathbb{F}G$ is an example of $\mathbb{F}$-*algebra*, i.e. a ring which is also an $\mathbb{F}$-module such that multiplication is bilinear.

If we bring in a basis for V, we get yet another equivalent definition:

Definition (matrix representation)**.** R is a *matrix representation* of G of degree n if R is a homomorphism $G \to \mathrm{GL}_n(\mathbb{F})$.

Given a linear representation $\rho : G \to \mathrm{GL}(V)$ with $\dim_F V = n$, fix a basis $\mathcal{B}$ then we get a matrix representation

$$\begin{aligned} G &\to \mathrm{GL}_n(\mathbb{F}) \\ g &\mapsto [\rho(g)]_{\mathcal{B}} \end{aligned}$$

Conversely, given a matrix representation $R : G \to \mathrm{GL}_n(\mathbb{F})$, you get a linear representation

$$\begin{aligned} \rho : G &\to \mathrm{GL}(\mathbb{F}^n) \\ g &\mapsto \rho_g \end{aligned}$$

via $\rho_g(v) = R_g(v)$.

Example. Given any group G, take $V = \mathbb{F}$ (the 1 dimensional space) and

$$\begin{aligned} \rho : G &\to \mathrm{GL}(V) \\ g &\mapsto \mathrm{id}_V \end{aligned}$$

is known as the *trivial representation.* $\deg \rho = 1$.

Example. Let $G = C_4 = \langle x : x^4 = 1 \rangle$. Take $\mathbb{F} = \mathbb{C}$ and let $n = 2$. Then $R : x \mapsto X$ will determine $x^j \mapsto X^j$ and thus the matrix representation R. We need $X^4 = I$. We can take

- either X diagonal: any such with diagonal entries in $\{\pm 1, \pm i\}$ (16 choices),
- or X is not diagonal: then it will be conjugate to a diagonal (by diagonalisability criterion).

2.1 Equivalent representations

Definition (G-homomorphism, G-isomorphism)**.** Fix G and $\mathbb{F}$. Let V and V' be $\mathbb{F}$-vector spaces and $\rho : G \to \mathrm{GL}(V), \rho' : G \to \mathrm{GL}(V')$ be representations of G. The linear map $\varphi : V \to V'$ is a *G-homomorphism* or *intertwining homomorphism* if

$$\varphi \rho(g) = \rho'(g) \varphi.$$

In other words, the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\rho_g} & V \\ \downarrow \varphi & & \downarrow \varphi \\ V' & \xrightarrow{\rho'_g} & V' \end{array}$$

We say φ *intertwines* ρ and ρ'. Write $\mathrm{Hom}_G(V, V')$ for the $\mathbb{F}$-space of all such.

φ is a *G-isomorphism* if φ is also bijective. If such a φ exists, say ρ and ρ' are *isomorphic* or *equivalent.* If φ is a G-isomorphism we can write the intertwining condition as

$$\rho' = \varphi \rho \varphi^{-1}.$$

Lemma 2.1. *Being isomorphic is an equivalence relation on the set of all representations of G over $\mathbb{F}$.*

Proof. Exercise. □

Remark. If ρ and ρ' are isomorphic representation then they have the same dimension. The converse is false: C_4 has four non-isomorphic 1 dimensional representations.

Remark. Given $G, V, \mathbb{F}$ with $\dim_{\mathbb{F}} V = n$ and $\rho : G \to \mathrm{GL}(V)$, fix a basis $\mathcal{B}$ of V. We get an isomorphism

$$\begin{aligned} \varphi : V &\to \mathbb{F}^n \\ v &\mapsto [v]_{\mathcal{B}} \end{aligned}$$

And φ gives a representation $\rho' : G \to \mathrm{GL}(\mathbb{F}^n)$ isomorphic to ρ.

Proposition 2.2.

1. *Transformations in terms of matrix representatives: $R : G \to \mathrm{GL}_n(\mathbb{F}), R' : G \to \mathrm{GL}_n(\mathbb{F})$ are G-isomorphic or G-equivalent if exists $X \in \mathrm{GL}_n(\mathbb{F})$ with*
$$R'(g) = XR(g)X^{-1}$$
for all $g \in G$.

2. *In terms of linear G-actions, the action of G on V, V' are G-isomorphic if there exists $\varphi : V \to V'$ such that*
$$g\varphi(v) = \varphi(gv)$$
for all $g \in G, v \in V$.

2.2 Subrepresentation

Definition (G-subspace)**.** Let $\rho : G \to \mathrm{GL}(V)$ be a representation of G. We say that $W \leq V$ is a *G-subspace* if it is a subspace and it is $\rho(G)$-invariant, i.e. $\rho_g(W) \subseteq W$ for all $g \in G$.

Obviously $\{0\}$ and V are G-subspaces. On the other hand,

Definition (irreducible/simple representation)**.** ρ is said to be *irreducible* or *simple* representation if there are no proper G-subspaces.

Example. Any 1 dimensional representation of G is irreducible. The converse is not true. For example D_8 has a 2 dimensional irreducible representation.

Definition (subrepresentation)**.** If W is a G-subspace then the corresponding map

$$\begin{aligned} G &\to \mathrm{GL}(W) \\ g &\mapsto \rho(g)|_W \end{aligned}$$

is a representation of G, known as a *subrepresentation* of ρ.

Lemma 2.3. *If $\rho : G \to \mathrm{GL}(V)$ is a representation, W is a G-subspace of V and $\mathcal{B} = \{v_1, \dots, v_n\}$ is a basis containing a basis $\{v_1, \dots, v_m\}$ of W, where $0 < m \leq n$, then the matrix of $\rho(g)$ with respect to $\mathcal{B}$ has block upper triangular form*

$$\begin{pmatrix} * & * \\ 0 & * \end{pmatrix}$$

for each $g \in G$.

Example. Let $\mathbb{F} = \mathbb{C}$.

1. Irreducible representation of $C_4 = \langle x : x^4 = 1 \rangle$ are all 1 dimensional and four of them are
$$x \mapsto i, x \mapsto -1, x \mapsto -i, x \mapsto 1.$$
In general C_m has precisely m inequivalent complex irreducible representations, all of degree 1. Actually all complex irreducible representations of a *finite abelian group* are 1 dimensional, by simultaneous diagonalisation and primary decomposition. Alternatively, this follows from Schur's lemma.

2. $G = D_6$: every irreducible $\mathbb{C}$-representation has dimension ≤ 2. Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation of G. Let r be a rotation and s be reflection. Take an eigenvector v of $\rho(r)$ so $\rho(r)v = \lambda v$ for some $\lambda \in \mathbb{C}, \lambda \neq 0$. Let
$$W = \langle v, \rho(s)v \rangle \leq V.$$
Since
$$\begin{aligned} &\rho(s)\rho(s)v = v \\ &\rho(r)\rho(s)v = \rho(s)\rho(r)^{-1}v = \lambda^{-1}\rho(s)v \end{aligned}$$
so W is G-invariant. Since V is irreducible $W = V$.

Definition ((in)decomposable representation, direct sum)**.** We say that $\rho : G \to \mathrm{GL}(V)$ is *decomposable* if there are proper G-invariant subspaces U, W with $V = U \oplus W$. Say ρ is the *direct sum* $\rho_U \oplus \rho_W$. If no such subspaces exist we say ρ is *indecomposable*.

Lemma 2.4. *If $\rho : G \to \mathrm{GL}(V)$ is decomposable, $\mathcal{B} = \{v_1, \dots, v_k, w_1, \dots, w_\ell\}$ is a basis of V consisting of a basis of U and a basis of W, then $\rho(g)$ with respect to $\mathcal{B}$ is block diagonal for all $g \in G$.*

Definition (direct sum)**.** Let $\rho : G \to \mathrm{GL}(V), \rho' : G \to \mathrm{GL}(V')$ be two representations. The *direct sum* of ρ, ρ' is

$$\begin{aligned} \rho \oplus \rho' : G &\to \mathrm{GL}(V \oplus V') \\ (\rho \oplus \rho')(g)(v + v') &= \rho(g)v + \rho'(g)v' \end{aligned}$$

For matrix representations $R : G \to \mathrm{GL}_n(\mathbb{F}), R' : G \to \mathrm{GL}_{n'}(\mathbb{F})$, define $R \oplus R' : G \to \mathrm{GL}_{n+n'}(\mathbb{F})$ is given by

$$g \mapsto \begin{pmatrix} R(g) & 0 \\ 0 & R'(g) \end{pmatrix}$$

for all g.

3 Complete reducibility and Maschke's theorem

Given $G, \mathbb{F}$ as usual.

Definition (completely reducible/semisimple representation)**.** A representation $\rho : G \to \mathrm{GL}(V)$ is *completely reducible* or *semisimple* if it is a direct sum of irreducible representations.

Remark. Irreducible implies completely reducible. The converse is not true. See example sheet 1 question 3.

From now on take G to be finite and $\operatorname{ch} F = 0$ throughout this chapter.

Theorem 3.1 (complete reducibility theorem)**.** *Every finite-dimensional representation V of a finite group over a field of characteristic 0 is completely reducible, i.e. $V = V_1 \oplus \cdots \oplus V_r$ is a direct sum of representations with each V_i irreducible.*

In fact it is enough to prove

Theorem 3.2 (Maschke)**.** *Suppose G is finite and $\rho : G \to \mathrm{GL}(V)$ is a representation with V finite-dimensional,* $\operatorname{ch} F = 0$*. If W is a G-subspace of V then there exists a G-subspace U of V such that $V = W \oplus U$, a direct sum of G-subspaces.*

Proof. Let W' be any complementary subspace of W in V, i.e. $V = W \oplus W'$. Let $q : V \to W$ be the projection of V onto W along W', i.e. if $v = w + w'$ then $q(v) = w$. Define

$$\overline{q} : v \mapsto \frac{1}{|G|} \sum_{g \in G} gq(g^{-1}(v)),$$

the "average of q over G". Note that we've dropped the ρ in $\rho(g)$ and $\rho(g^{-1})$ to avoid excessive notations.

Claim that $\overline{q} : V \to W$: for $v \in V$, $q(g^{-1}(v)) \in W$ and $g(W) \subseteq W$. Also $\overline{q}(w) = w$ for $w \in W$ as

$$\overline{q}(w) = \frac{1}{|G|} \sum_{g \in G} gq(g^{-1}w) = \frac{1}{|G|} \sum_{g \in G} g(g^{-1}w) = \frac{1}{|G|} \sum_{g \in G} w = w$$

Thus $\overline{q}$ projects V onto W.

As $\overline{q}$ is a projection we can write $V = \operatorname{im} \overline{q} \oplus \ker \overline{q} = W \oplus \ker \overline{q}$. Need to show $\ker \overline{q}$ is G-invariant. Note that if $h \in G$

$$\begin{aligned} h\overline{q}(v) &= h \frac{1}{|G|} \sum_g gq(g^{-1}v) \\ &= \frac{1}{|G|} \sum_g hgq(g^{-1}v) \\ &= \frac{1}{|G|} \sum_g (hg)q((hg)^{-1}hv) \\ &= \frac{1}{|G|} \sum_g gq(g^{-1}(hv)) \\ &= \overline{q}(hv). \end{aligned}$$

Thus if $v \in \ker \overline{q}, h \in G$ then

$$h\overline{q}(v) = 0 = \overline{q}(hv)$$

so $hv \in \ker \overline{q}$. Therefore

$$V = \operatorname{im} \overline{q} \oplus \ker \overline{q} = W \oplus \ker \overline{q}$$

which is a G-subspace decomposition. □

In fact, we only need $\operatorname{ch} \mathbb{F} \nmid |G|$.

Remark. Complements are not unique. For example, take $G = 1$. Then a representation of G is just a vector space. Take $V = \mathbb{C}^2$. Then any proper subspace $W \leq V$ will do.

Exercise. Deduce complete reducibility theorem from Maschke by induction on dimension.

We'll present another proof using inner product. This will generalise easily to compact Lie groups. Take $\mathbb{F} = \mathbb{C}$.

Recall that for V a $\mathbb{C}$-vector space. $\langle \cdot, \cdot \rangle$ is a *Hermitian inner product* if

1. $\langle w, v \rangle = \overline{\langle v, w \rangle}$ for all v, w.
2. sesquilinear: linear in second argument.
3. positive definite: $\langle v, v \rangle > 0$ if $v \neq 0$.

Furthermore $\langle \cdot, \cdot \rangle$ is *G-invariant* if

$$\langle gv, gw \rangle = \langle v, w \rangle$$

for all $v, w \in V, g \in G$.

If W is a G-invariant subspace of V (with a G-invariant inner product) then $W^\perp$ is also G-invariant and $W = W \oplus W^\perp$: enough to show for all $v \in W^\perp, g \in G$, have $gv \in W^\perp$. But by definition $\langle v, w \rangle = 0$ for all $w \in W$. Thus by G-invariance $\langle gv, gw \rangle = 0$ for all g. Certainly $\langle gv, w' \rangle = 0$ for all $w' \in W$ as we can choose $w = g^{-1}w' \in W$. The result thus follows.

Therefore if there is a G-invariant inner product on any complex G-space then we get another proof of Maschke's theorem.

> **Lemma 3.3** (Weyl's unitary trick). *Let ρ be a complex representation of a finite group G on the $\mathbb{C}$-vector space V. Then there is a G-invariant inner product on V.*

Proof. There exists an inner product on V: take basis $e_1, \dots, e_n$ and define $(e_i, e_j) = \delta_{ij}$. Extend sesquilinearly. Now define

$$\langle v, w \rangle = \frac{1}{|G|} \sum_g (gv, gw).$$

Easy exercise that $\langle \cdot, \cdot \rangle$ is a G-invariant inner product. For example for G-invariance, for all $h \in G$,

$$\begin{aligned} \langle hv, hw \rangle &= \frac{1}{|G|} \sum_g ((gh)v, (gh)w) \\ &= \frac{1}{|G|} \sum_{g'} (g'v, g'w) \\ &= \langle v, w \rangle \end{aligned}$$

□

Corollary 3.4. *Every finite subgroup of* $\mathrm{GL}_n(\mathbb{C})$ *is conjugate to a subgroup of* $U(n)$*.*

Proof. Example sheet 1 Q5, Q12. □

Definition (regular representation)**.** Recall group algebra of G is the $\mathbb{F}$-space

$$\mathbb{F}G = \mathrm{span}\{e_g : g \in G\}.$$

There is a linear G-action

$$h. \sum_g a_g e_g = \sum_g a_g e_{hg} = \sum_{g'} a_{h^{-1}g'} e_{g'}.$$

This is known as *regular representation* of G, denoted ρ_{reg}.

This is a faithful representation of dimension $|G|$. We call $V = \mathbb{F}G$ (sometimes also written as $\mathbb{F}[G]$) the *regular module.*

It turns out that every irreducible representation of G is a subrepresentation of ρ_{reg}:

Proposition 3.5. *Let ρ be an irreducible representation of G over a field of characteristic* 0*. Then ρ is isomorphic to a subrepresentation of ρ_{reg}.*

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be irreducible and let $v \in V$ nonzero. Consider

$$\begin{aligned} \theta : \mathbb{F}G &\to V \\ \sum_g a_g e_g &\mapsto \sum_g a_g gv \end{aligned}$$

This is a G-homomorphism. Now V is irreducible and $\mathrm{im}\,\theta = V$ since $\mathrm{im}\,\theta$ is a G-subspace. Then $\ker\theta$ is a G-subspace of $\mathbb{F}G$. Let W be a G-complement of $\ker\theta$ in $\mathbb{F}G$. Thus

$$W \cong \mathbb{F}G/\ker\theta \cong \mathrm{im}\,\theta = V.$$

□

More generally,

Definition (permutation representation)**.** Let G act on a set X. Let $\mathbb{F}X = \text{span}\{e_x : x \in X\}$ with G action

$$g.\sum_x a_x e_x = \sum_x a_x e_{gx}$$

so we have a G-space $\mathbb{F}X$. The representation $G \to \text{GL}(\mathbb{F}X)$ is the corresponding *permutation representation*.

4 Schur's lemma

Theorem 4.1 (Schur's lemma)**.**

1. *Assume V and W are irreducible G-spaces (over field $\mathbb{F}$). Then any G-homomorphism $\theta : V \to W$ is either 0 or a G-isomorphism.*

2. *Assume $\mathbb{F}$ is algebraically closed and let V be an irreducible G-space. Then any G-endomorphism $V \to V$ is a scalar multiple of the identity map 1_V (a* homothety*).*

Proof.

1. Let $\theta : V \to W$ be a G-homomorphism. Then $\ker\theta$ is a G-subspace of V. Since V is irreducible either $\ker\theta = 0$ or $\ker\theta = V$. Similarly $\operatorname{im}\theta = 0$ or $\operatorname{im}\theta = W$. Hence either $\theta = 0$ or θ is injective and surjective.

2. Since $\mathbb{F}$ is algebraically closed, θ has an eigenvalue λ. Then $\theta - \lambda 1_V$ is a singular G-endomorphism on V, so must be 0.

□

Recall the $\mathbb{F}$-space $\operatorname{Hom}_G(V, W)$ of all G-homomorphisms $V \to W$, we can restate Schur's lemma

Corollary 4.2. *If V and W are irreducible complex G-spaces then*

$$\dim_{\mathbb{C}} \operatorname{Hom}_G(V, W) = \begin{cases} 1 & \text{if } V, W \text{ are } G\text{-isomorphic} \\ 0 & \text{otherwise} \end{cases}$$

Proof. If V and W are not isomorphic then the only G-homomorphism $V \to W$ is 0. Assume $V \cong_G W$ and $\theta_1, \theta_2 \in \operatorname{Hom}_G(V, W)$, both nonzero. Then θ_2 is invertible and $\theta_2^{-1}\theta_1 \in \operatorname{End}_G(V)$ and nonzero, so $\theta_2^{-1}\theta_1 = \lambda 1_V$. Then $\theta_1 = \lambda\theta_2$. □

Corollary 4.3. *If G has a faithful complex irreducible representation then $Z(G)$ is cyclic.*

Remark. The converse is false. See example sheet Q10.

Proof. Let $\rho : G \to \mathrm{GL}(V)$ be a faithful representation over $\mathbb{C}$. Let $z \in Z(G)$, then $\varphi_z : v \mapsto zv$ is a G-endomorphism, hence multiplication by a scalar, say μ_z. Then

$$\begin{aligned} Z(G) &\to \mathbb{C}^\times \\ g &\mapsto \mu_g \end{aligned}$$

is a representation of $Z(G)$ and is faithful since ρ is. Thus $Z(G)$ is isomorphic to a finite subgroup of $\mathbb{C}^\times$ so cyclic. □

This is our first group theoretic result based on representation theory. This is a recurring theme in representation theory.

Corollary 4.4. *The irreducible $\mathbb{C}$-representations of a finite abelian group G are all 1 dimensional.*

Proof. One can use Proposition 1.5 to invoke simultaneous diagonalisation: if v is an eigenvector for each $g \in G$ and if V is irreducible then $V = \langle v \rangle$.

Alternatively, let V be an irreducible representation. Given $g \in G$, the map

$$\begin{aligned} \theta_g : V &\to V \\ v &\mapsto gv \end{aligned}$$

is a G-endomorphism of V. Hence $\theta_g = \lambda_g 1_V$ for some $\lambda_g \in \mathbb{C}$. Thus $gv = \lambda_g v$ for any $g \in G$. Thus as $V \neq 0$ is irreducible, $V = \langle v \rangle$. □

Remark. This fails for $\mathbb{R}$. For example C_3 has two irreducible $\mathbb{R}$-representations, one of dimension 1 and one of dimension 2.

Recall that every finite abelian group G is isomorphic to a product of cyclic groups. In fact it can be written as product of C_{p^α} for various primes p and $\alpha \geq 1$. The elements are uniquely determined up to order.

Proposition 4.5. *The finite abelian group $G \cong C_{n_1} \times \dots \times C_{n_r}$ has precisely $|G|$ irreducible $\mathbb{C}$-representations as described below.*

Proof. Write $G = \langle x_1 \rangle \times \dots \times \langle x_r \rangle$ where $|x_j| = n_j$. Suppose ρ is irreducible so it is 1 dimensional. Let $\rho(1, \dots, x_j, \dots, 1) = \lambda_j$. Then $\lambda_j^{n_j} = 1$ so λ_j is an n_jth root of unity. Now the values $(\lambda_1, \dots, \lambda_r)$ determine ρ, and no two are equivalent. □

Note that however, there is no canonical bijective correspondence between the elements of G and the representations of G. If you choose an isomorphism $G \cong C_{a_1} \times \dots C_{a_r}$ then we can identify the two sets, but it depends on the choice of isomorphism.

4.1 Isotypical decompositions

We know that in characteristic 0, every representation V of G decomposes as $\bigoplus V_i$ where each V_i is irreducible. How unique is this?

A wishlist of properties:

1. uniqueness: for each V there is only one way to decompose $V = \bigoplus V_i$ with V_i irreducible.

2. uniqueness of isotypes: for each V there exist unique subrepresentations $U_1, \dots, U_k$ such that $V = \bigoplus U_i$ and if $V_i \leq U_i, V'_j \leq U_j$ irreducible subrepresentations then $V_i \cong V'_j$ if and only if $i = j$.

3. uniqueness of factors: if $\bigoplus_{i=1}^k V_i \cong \bigoplus_{i=1}^{k'} V'_i$ and V_i, V'_i are irreducible then $k = k'$ and there exists $\pi \in S_k$ such that $V'_{\pi(i)} \cong V_i$.

Evidently 1 is too strong ($G = 1$ acting on any V with dimension > 1). However 2 and 3 do work. We will skip the proof and refer the reader to Teleman §5. However, we shall discuss how to calculate multiplicities of simples in the isotypes.

Lemma 4.6. *Let V, V_1, V_2 be G-spaces.*

1. $\mathrm{Hom}_G(V, V_1 \oplus V_2) \cong \mathrm{Hom}_G(V, V_1) \oplus \mathrm{Hom}_G(V, V_2)$.
2. $\mathrm{Hom}_G(V_1 \oplus V_2, V) \cong \mathrm{Hom}_G(V_1, V) \oplus \mathrm{Hom}_G(V_2, V)$.

Proof. Let $\pi_i : V_1 \oplus V_2 \to V_i$ be the G-linear projections in V_i with kernel V_{3-i}. Then

$$\begin{aligned} \mathrm{Hom}_G(V, V_1 \oplus V_2) &\to \mathrm{Hom}_G(V, V_1) \oplus \mathrm{Hom}_G(V, V_2) \\ \varphi &\mapsto (\pi_1\varphi, \pi_2\varphi) \end{aligned}$$

has inverse $(\psi_1, \psi_2) \mapsto \psi_1 + \psi_2$.

Also the map

$$\begin{aligned} \mathrm{Hom}_G(V_1 \oplus V_2, V) &\to \mathrm{Hom}_G(V_1, V) \oplus \mathrm{Hom}_G(V_2, V) \\ \varphi &\mapsto (\varphi|_{V_1}, \varphi|_{V_2}) \end{aligned}$$

has inverse $(\psi_1, \psi_2) \mapsto \psi_1\pi_1 + \psi_2\pi_2$. □

Corollary 4.7. *Suppose $\mathbb{F}$ is algebraically closed and $V = \bigoplus_{i=1}^n V_i$ is a decomposition into irreducibles. Then for each irreducible representation S of G,*

$$\#\{j : V_j \cong S\} = \dim \mathrm{Hom}_G(S, V).$$

This is known as the multiplicity *of S in V.*

Proof. By induction on n. Obvious for $n = 0, 1$. For $n > 1$, write

$$V = (\bigoplus_{i=1}^{n-1} V_i) \oplus V_n.$$

Then

$$\dim \mathrm{Hom}_G(S, (\bigoplus_{i=1}^{n-1} V_i) \oplus V_n) = \dim \mathrm{Hom}_G(S, \bigoplus_{i=1}^{n-1} V_i) + \dim \mathrm{Hom}_G(S, V_n)$$

and use Schur's lemma. □

Definition (canonical decomposition)**.** A decomposition $V = \bigoplus W_i$ where each W_j is isomorphic to n_j copies of irreducible representation S_j (each non-isomorphic for each j) is the *canonical decomposition* or the *decomposition into isotypical components* W_j.

For $\mathbb{F}$ closed, the above lemma says that $n_j = \dim \mathrm{Hom}_G(S_j, V)$, i.e. n_j is detectable at G-homomorphism level.

Example. Teleman §5 gives an example on D_6.

If G is finite abelian then every complex representation V of G has unique isotypical decomposition.

5 Character theory

We want to attach invariants to a representation ρ of a finite group G on V. Matrix coefficients of $\rho(g)$ are basis-dependent so not true invariants. det is an invariant but not a very useful one, as lots of inequivalent representations have determinant 1. Instead we'll use trace.

Let $\mathbb{F} = \mathbb{C}$ and let $\rho = \rho_V : G \to \mathrm{GL}(V)$ be a representation.

Definition (character)**.** The *character* $\chi_\rho = \chi_V = \chi$ is defined as

$$\begin{aligned} \chi : G &\to \mathbb{C} \\ g &\mapsto \operatorname{tr} \rho(g) \end{aligned}$$

The *degree* of χ_V is $\dim V$.

χ is *linear* if $\dim V = 1$, in which case χ is a homomorphism $G \to \mathbb{C}^\times$. χ is *irreducible/faithful/trivial (or principal)* if ρ is. In the last case we also write $\chi = 1_G$.

It turns out that χ is a complete invariant in the sense that it determines ρ up to isomorphism. We'll prove this later.

Theorem 5.1.

1. $\chi_V(1) = \dim V$.
2. χ_V *is a* class function*, namely it is conjugation invariant. Thus* χ_V *is constant on conjugacy classes of* G*.*
3. $\chi_V(g^{-1}) = \overline{\chi_V(g)}$.
4. *For two representations* V *and* W*,*

$$\chi_{V\oplus W} = \chi_V + \chi_W.$$

Proof.

1. Clearly $\operatorname{tr} I_n = n$.
2. $\chi(hgh^{-1}) = \operatorname{tr}(R_h R_g R_h^{-1}) = \operatorname{tr} R_g = \chi(g)$.
3. $g \in G$ has finite order so diagonalisable so can assume $\rho(g)$ is represented by diagonal matrix

$$\begin{pmatrix} \lambda_1 & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & \lambda_n \end{pmatrix}$$

so $\chi(g) = \sum \lambda_i$. Now g^{-1} is represented by

$$\begin{pmatrix} \lambda_1^{-1} & \cdots & 0 \\ & \ddots & \\ 0 & \cdots & \lambda_n^{-1} \end{pmatrix}$$

hence

$$\chi(g^{-1}) = \sum \lambda_i^{-1} = \sum \overline{\lambda}_i = \overline{\sum \lambda_i} = \overline{\chi(g)}.$$

4. Suppose $V = V_1 \oplus V_2, \rho_i : G \to \mathrm{GL}(V_i), \rho : G \to \mathrm{GL}(V)$. Take basis $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$, where $\mathcal{B}_1$ and $\mathcal{B}_2$ are basis for V_1 and V_2 respectively, of V. With respect to $\mathcal{B}$ $\rho(g)$ has matrix
$$\begin{pmatrix} [\rho_1(g)]_{\mathcal{B}_1} & 0 \\ 0 & [\rho_2(g)]_{\mathcal{B}_2} \end{pmatrix}$$
and so
$$\chi(g) = \operatorname{tr}\rho_1(g) + \operatorname{tr}\rho_2(g) = \chi_1(g) + \chi_2(g).$$

□

Remark. We'll see later that if χ_1, χ_2 are characters of G then $\chi_1\chi_2$ is also a character of G (spoiler: tensor product).

Lemma 5.2. *Let $\rho : G \to \mathrm{GL}(V)$ be a (complex) representation affording the character χ. Then for $g \in G$, $|\chi(g)| \leq \chi(1)$ with equality if and only if $\rho(g) = \lambda I$ for some $\lambda \in \mathbb{C}$ a root of unity. Moreover $\chi(g) = \chi(1)$ if and only if $g \in \ker\rho$. In other words, the* kernel *of χ* $\ker\chi$ *is*
$$\ker\rho = \{g \in G : \chi(g) = \chi(1)\}.$$

Proof. Example sheet 2 Q1. □

Lemma 5.3.

1. *If χ is a (complex irreducible, respectively) character of G then so is $\overline{\chi}$.*
2. *If χ is a (complex irreducible, respectively) character of G then so is $\varepsilon\chi$ for any linear (i.e. 1 dimensional) character ε of G.*

Proof. If $R : G \to \mathrm{GL}_n(\mathbb{C})$ is a (complex irreducible) representation then so is
$$\begin{aligned} R : G &\to \mathrm{GL}_n(\mathbb{C}) \\ g &\mapsto \overline{R(g)} \end{aligned}$$
Similarly $r' : g \mapsto \varepsilon(g)R(g)$. Check the details. □

Definition (class function, class number)**.** Define
$$\mathcal{C}(G) = \{f : G \to \mathbb{C} : f(hgh^{-1}) = f(g) \text{ for all } h, g \in \mathbb{C}\},$$
the complex space of *class functions*. It is a $\mathbb{C}$-vector space.

Let $k = k(G)$ be the *class number* of G, i.e. number of conjugacy classes of G. List conjugacy classes as $\mathcal{C}_1 = \{1\}, \mathcal{C}_2, \dots, \mathcal{C}_k$. Choose $g_1 = 1, g_2, \dots, g_k$ representatives of the classes. Note that $\dim \mathcal{C}(G) = k$, as the characteristic functions δ_j of the conjugacy classes form a basis.

Define a Hermitian inner product on $\mathcal{C}(G)$ as follow:

$$\begin{aligned}\langle f, f'\rangle &= \frac{1}{|G|}\sum_{g\in G}\overline{f(g)}f'(g)\\ &= \frac{1}{|G|}\sum_{j=1}^{k}|\mathcal{C}_j|\overline{f(g_j)}f'(g_j)\\ &= \sum_{j=1}^{k}\frac{1}{|C_G(g_j)|}\overline{f(g_j)}f'(g_j)\end{aligned}$$

For characters we have

$$\langle \chi, \chi'\rangle = \sum_{j=1}^{k}\frac{1}{|C_G(g_j)|}\chi(g_j^{-1})\chi'(g_j)$$

which is a real symmetric form (in fact we will show it is an integer).

Theorem 5.4 (completeness of characters)**.** *The $\mathbb{C}$-irreducible characters of G form an orthonormal basis of $\mathcal{C}(G)$. More precisely,*

1. *if $\rho : G \to \mathrm{GL}(V), \rho' : G \to \mathrm{GL}(V')$ are irreducible representations of G, affording characters χ and χ' then*

$$\langle \chi, \chi'\rangle = \begin{cases} 1 & \text{if } \rho, \rho' \text{ are isomorphic}\\ 0 & \text{otherwise}\end{cases}$$

   This is called row orthogonality.

2. *each class function of G is a linear combination of irreducible characters of G.*

Proof. See chapter 6. □

Corollary 5.5. *Complex representations of finite groups are characterised by their characters.*

Note the finiteness condition. For counterexample otherwise take $G = \mathbb{Z}$, $1 \mapsto I$ and $1 \mapsto \left(\begin{smallmatrix}1&1\\0&1\end{smallmatrix}\right)$.

Proof. Let G be a finite group and $\rho : G \to \mathrm{GL}(V)$ be a representation affording χ. By Maschke's theorem $\rho = m_1\rho_1 \oplus \cdots \oplus m_k\rho_k$ where $\rho_1, \dots, \rho_k$ are irreducible and $m_j \geq 0$. Then $m_j = \langle \chi_j, \chi\rangle$ where χ_j is afforded by ρ_j: for $\chi = m_1\chi_1 + \cdots + m_k\chi_k$ and thus

$$\langle \chi_j, \chi\rangle = \langle \chi_j, m_1\chi_1 + \cdots + m_j\chi_j\rangle = m_j$$

□

Corollary 5.6 (irreducibility criterion)**.** *If ρ is a $\mathbb{C}$-representation of G affording χ then ρ is irreducible if and only if $\langle\chi,\chi\rangle = 1$.*

Proof. $\implies$ is row orthogonality. For $\impliedby$, suppose $\langle\chi,\chi\rangle = 1$. Complete reducibility says that $\chi = \sum m_j\chi_j$ where χ_j's are irreducible and $m_j \geq 0$. Then $\sum m_j^2 = 1$ so $\chi = \chi_j$ for some j, so χ is irreducible. □

Theorem 5.7. *If the irreducible $\mathbb{C}$-representations of G, $\rho_1, \dots, \rho_k$ have dimensions $n_1, \dots, n_k$ then*

$$|G| = \sum_{i=1}^{k} n_i^2.$$

Proof. Recall $\rho_{\text{reg}} : G \to \mathrm{GL}(\mathbb{C}G)$, the regular representation of G of dimension $|G|$. Let π_{reg} be its character, the *regular character* of G. Note that

$$\pi_{\text{reg}}(g) = \begin{cases} |G| & g = 1 \\ 0 & \text{otherwise} \end{cases}$$

Also claim that $\pi_{\text{reg}} = \sum_j n_j\chi_j$ with $n_j = \chi_j(1)$:

$$n_j = \langle\pi_{\text{reg}}, \chi_j\rangle = \frac{1}{|G|}\sum_{g\in G}\overline{\pi_{\text{reg}}(g)}\chi_j(g) = \frac{1}{|G|}|G|\chi_j(1) = \chi_j(1).$$

□

Corollary 5.8. *The number of irreducible characters of G (up to equivalence) equals to the class number.*

Corollary 5.9. *Elements $g_1, g_2 \in G$ are conjugate if and only if $\chi(g_1) = \chi(g_2)$ for all irreducible characters χ of G.*

Proof. $\implies$: characters are class functions. $\impliedby$: if $\chi(g_1) = \chi(g_2)$ for all irreducible characters χ then $f(g_1) = f(g_2)$ for all class fucntions of G. In particular this is true for the characteristic function δ taking 1 on conjugacy class of g_1 and 0 otherwise. □

Recall the inner prodcut on $\mathcal{C}(G)$ and the real symmetric form $\langle\cdot,\cdot\rangle$ for characters.

Definition (character table)**.** Let G be a finite group and $\mathbb{F} = \mathbb{C}$. The *character table* of G is the $k \times k$ matrix $X = [\chi_i(g_j)]$ where $1 = \chi_1, \dots, \chi_k$ are the irreducible characters of G and $\mathcal{C}_1 = \{1\}, \dots, \mathcal{C}_k$ are the conjugacy classes with $g_j \in \mathcal{C}_j$.

Example. $G = S_3 = D_6 = \langle r, s : r^3 = s^2 = 1, srs^{-1} = r^{-1}\rangle$. The conjugacy classes are

$$\mathcal{C}_1 = \{1\}, \mathcal{C}_2 = \{s, sr, sr^2\}, \mathcal{C}_3 = \{r, r^{-1}\}.$$

Thus from the corollary there are three representations. It is easy to write down two of them: the trivial representation 1 and the sign S of permutation. Think geometrically, it's not hard to come up with a 2 dimensional irreducible representation W of symmetry of an equilateral triangle. sr^j acts by matrix with eigenvalues ± 1 so $\chi(sr^j) = 0$ for all j. r^k acts by the matrix

$$\begin{pmatrix} \cos \frac{2k\pi}{3} & -\sin \frac{2k\pi}{3} \\ \sin \frac{2k\pi}{3} & \cos \frac{2k\pi}{3} \end{pmatrix}$$

so $\chi(r^k) = 2\cos \frac{2k\pi}{3} = -1$ for all k. Thus we have character table

	1	$\mathcal{C}_1$	$\mathcal{C}_2$
1	1	1	1
S	1	-1	1
W	2	0	-1

We can do a few sanity checks: the sum of squares of the first column is 6, which equals to the order of G. Also

$$\langle \chi_W, \chi_W \rangle = \frac{2^2}{6} + \frac{0^2}{2} + \frac{(-1)^2}{3} = 1$$

so indeed it is irreducible.

6 Proof of orthogonality

Proof of completeness of characters 1. Fix bases of V and V'. Write $R(g), R'(g)$ for matrices of $\rho(g)$ and $\rho'(g)$ with respect to these bases respectively. Then

$$\langle \chi', \chi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi'(g^{-1}) \chi(g) = \frac{1}{|G|} \sum_{\substack{g \in G \\ 1 \le i,j \le n}} R'(g^{-1})_{ii} R(g)_{jj}.$$

Let $\varphi : V \to V'$ be linear and define its "average"

$$\begin{aligned} \tilde{\varphi} : V &\to V' \\ v &\mapsto \frac{1}{|G|} \sum_{g \in G} \rho'(g^{-1}) \varphi \rho(g) v \end{aligned}$$

then $\tilde{\varphi}$ is a G-homomorphism. To see this, if $h \in G$ then

$$\begin{aligned} \rho'(h^{-1}) \tilde{\varphi} \rho(h)(v) &= \frac{1}{|G|} \sum_{g \in G} \rho'((gh)^{-1}) \varphi \rho(gh)(v) \\ &= \frac{1}{|G|} \sum_{g' \in G} \rho'(g'^{-1}) \varphi \rho(g')(v) \\ &= \tilde{\varphi}(v) \end{aligned}$$

Case 1: ρ, ρ' are not isomorphic Schur's lemma says $\tilde{\varphi} = 0$ for any $\varphi : V \to V'$ linear. Take $\varphi = \varepsilon_{\alpha\beta}$, having matrix $E_{\alpha\beta}$ with respect to our basis with 0 everywhere except 1 in (α, β)th entry. Then

$$0 = \tilde{\varepsilon}_{\alpha\beta} = \frac{1}{|G|} \sum_{g \in G} (R'(g^{-1}) E_{\alpha\beta} R(g))_{ij}$$

so

$$\frac{1}{|G|} \sum_{g \in G} R(g^{-1})_{i\alpha} R(g)_{\beta j} = 0$$

for all i, j. Specialise to $i = \alpha, j = \beta$ and sum over i, j to get

$$\langle \chi', \chi \rangle = 0.$$

Case 2: ρ, ρ' are isomorphic $\chi = \chi'$. Take $V = V', \rho = \rho'$. If $\varphi : V \to V$ is linear endomorphism then $\tilde{\varphi} \in \operatorname{End}_G(V)$. Now $\operatorname{tr} \varphi = \operatorname{tr} \tilde{\varphi}$:

$$\operatorname{tr} \tilde{\varphi} = \frac{1}{|G|} \sum_g \operatorname{tr}(\rho(g^{-1}) \varphi \rho(g)) = \frac{1}{|G|} \sum_g \operatorname{tr} \varphi = \operatorname{tr} \varphi$$

By Schur, $\tilde{\varphi} = \lambda \operatorname{id}_V$ for some $\lambda \in \mathbb{C}$. Then $\lambda = \frac{1}{n} \operatorname{tr} \varphi$ where n is the dimension of V.

Let $\varphi = \varepsilon_{\alpha\beta}$ so $\operatorname{tr} \varphi = \delta_{\alpha\beta}$. Hence

$$\tilde{\varphi}_{\alpha\beta} = \frac{1}{n} \delta_{\alpha\beta} \operatorname{id} = \frac{1}{|G|} \sum_g \rho(g^{-1}) \varepsilon_{\alpha\beta} \rho(g)$$

In terms of matrices, take (i,j)th entry:

$$\frac{1}{|G|}\sum_g R(g^{-1})_{i\alpha}R(g)_{\beta j} = \frac{1}{n}\delta_{\alpha\beta}\delta_{ij}$$

and put $\alpha = i, \beta = j$ to get

$$\frac{1}{|G|}\sum_g R(g^{-1})_{ii}R(g)_{jj} = \frac{1}{n}\delta_{ij}.$$

Finally sum over i, j to get

$$\langle \chi, \chi \rangle = 1.$$

□

Before proving 2, let's prove column orthogonality, assuming Corollary 5.8.

Corollary 6.1 (column orthogonality relations)**.**

$$\sum_{i=1}^k \overline{\chi_i(g_j)}\chi_i(g_\ell) = \delta_{j\ell}|C_G(g_j)|.$$

This has an easy corollary:

Theorem 6.2.

$$|G| = \sum_{i=1}^k \chi_i^2(1).$$

Proof.

$$\delta_{ij} = \langle \chi_i, \chi_j \rangle = \sum_{\ell=1}^k \frac{1}{|C_G(g_\ell)|}\overline{\chi_i(g_\ell)}\chi_j(g_\ell)$$

Consider the character table $X = (\chi_i(g_j))$. Then

$$\overline{X}D^{-1}X^t = I_k$$

where

$$D = \begin{pmatrix} |C_G(g_1)| & & 0 \\ & \ddots & \\ 0 & & |C_G(g_k)| \end{pmatrix}$$

Since X is square, it follows that $D^{-1}\overline{X}^t$ is the inverse of X so $\overline{X}^t X = D$. □

Proof of completeness of characters 2. List all the irreducible characters $\chi_1, \dots, \chi_\ell$ of G. Claim these generate $\mathcal{C}(G)$, the $\mathbb{C}$-space of class functions on G. It's enough to show that the orthogonal complement to $\operatorname{span}(\chi_1, \dots, \chi_\ell)$ in $\mathcal{C}(G)$ is 0. To see this let $f \in \mathcal{C}(G)$ with $\langle f, \chi_j \rangle = 0$ for all χ_j irreducible. Let $\rho : G \to \mathrm{GL}(V)$ be irreducible representation affording $\chi \in \{\chi_1, \dots, \chi_\ell\}$. Then $\langle f, \chi \rangle = 0$.

Consider the G-endormophism

$$\frac{1}{|G|}\sum_g \overline{f(g)}\rho(g) : V \to V$$

so as ρ is irreducible it must be $\lambda \operatorname{id}_V$ for some $\lambda \in \mathbb{C}$. Take trace,

$$n\lambda = \operatorname{tr} \frac{1}{|G|} \sum_g \overline{f(g)} \rho(g) = \frac{1}{|G|} \sum_g \overline{f(g)} \chi(g) = \langle f, \chi \rangle = 0$$

so $\lambda = 0$. Hence $\sum \overline{f(g)} \rho(g) = 0$ for all representation ρ by complete reducibility. Take $\rho = \rho_{\text{reg}}$ so

$$\sum_g \overline{f(g)} \rho_{\text{reg}}(g)(e_1) = \sum_g \overline{f(g)} e_g = 0$$

so $f(g) = 0$ for all g. $\square$

7 Permutation representations

Let G be a finite group acting on $X = \{x_1, \dots, x_n\}$. Recall that $\mathbb{C}X$ is the free $\mathbb{C}$-space generated by X. The corresponding permutation representation

$$\begin{aligned} \rho_X : G &\to \mathrm{GL}(\mathbb{C}X) \\ g &\mapsto \rho(g) \end{aligned}$$

is given by $\rho(g) : e_{x_j} \mapsto e_{gx_j}$. We call ρ_X the *permutation representation* corresponding to the action of G on X. Matrices of $\rho_X(g)$ with respect to basis $\{e_x\}_{x\in X}$ are permutation matices: 0 except for one 1 in each row and column and $(\rho(g))_{ij} = 1$ when $gx_j = x_i$. The corresponding *permutation character* π_X is

$$\pi_X(g) = |\mathrm{fix}_X(g)| = |\{x \in X : gx = x\}|.$$

Lemma 7.1. *π_X always contains 1_G.*

Proof. $\mathrm{span}(e_{x_1} + \dots + e_{x_n})$ is a trivial G-subspace of $\mathbb{C}X$ with G-invariant complement $\mathrm{span}(\sum_{x\in X} a_x e_x : \sum a_x = 0)$. ☐

Lemma 7.2.

$$\langle \pi_X, 1_G \rangle = \#G\text{-orbits of } G \text{ on } X.$$

Proof. If $X = X_1 \cup \dots \cup X_\ell$ is the disjoint union of orbits then

$$\pi_X = \pi_{X_1} + \dots + \pi_{X_\ell}$$

with π_{X_j} the permutation character of G on X_j. So prove the lemma, it is enough to show that if G acts transitively on X then $\langle \pi_X, 1 \rangle = 1$. Assume G is transitive on X,

$$\begin{aligned} \langle \pi_X, 1 \rangle &= \frac{1}{|G|} \sum_g \pi_X(g) \\ &= \frac{1}{|G|} |\{(g, x) \in G \times X : gx = x\}| \\ &= \frac{1}{|G|} \sum_{x\in X} |G_x| \\ &= \frac{1}{|G|} |X||G_X| \\ &= \frac{1}{|G|} |G| \quad \text{orbit-stabiliser} \\ &= 1 \end{aligned}$$

The whole proof can be seen as different ways to write fixed points of G. ☐

Lemma 7.3. *Let G act on the sets X_1, X_2. Then G acts on $X_1 \times X_2$ via $g(x_1, x_2) = (gx_1, gx_2)$. The character $\pi_{X_1 \times X_2} = \pi_{X_1}\pi_{X_2}$ and so*

$$\langle \pi_{X_1}, \pi_{X_2} \rangle = \#\{\textit{orbits of } G \textit{ on } X_1 \times X_2\}$$

Proof. If $g \in G$ then $\pi_{X_1 \times X_2}(g) = \pi_{X_1}(g)\pi_{X_2}(g)$. And

$$\langle \pi_{X_1}, \pi_{X_2} \rangle = \langle \pi_{X_1}\pi_{X_2}, 1 \rangle = \langle \pi_{X_1 \times X_2}, 1 \rangle = \#\{\text{orbits of } G \text{ on } X_1 \times X_2\}$$

□

Definition (2-transitive)**.** Let G act on X, $|X| > 2$. Then G is 2-*transitive* on X if G has exactly two orbits on $X \times X$: $\{(x, x) : x \in X\}$ and $\{(x_1, x_2) : x_i \in X, x_1 \neq x_2\}$.

Lemma 7.4. *Let G act on X with $|X| > 2$. Then*

$$\pi_X = 1 + \chi$$

with χ irreducible if and only if G is 2-transitive on X.

Proof. Write

$$\pi_X = m_1 1 + m_2 \chi_2 + \dots + m_\ell \chi_\ell$$

with $1, \chi_2, \dots, \chi_\ell$ distinct irreducibles and $m_i \in \mathbb{N}$. Then

$$\langle \pi_X, \pi_X \rangle = \sum_{i=1}^{\ell} m_i^2.$$

Hence G is 2-transitive if and only if $\ell = 2, m_1 = m_2 = 1$. □

Example. S_n acting on $X = \{1, \dots, n\}$ is 2-transitive. Hence $\pi_X = 1 + \chi$ with χ irreducible of degree $n - 1$. Similar for A_n, $n > 3$.

Example. Let's write down the table of $G = S_4$:

	1	3	8	6	6
	1	(12)(34)	(123)	(1234)	(12)
χ_1	1	1	1	1	1
$\text{sgn} = \chi_2$	1	1	1	-1	-1
$\pi_X - 1 = \chi_3$	3	-1	0	-1	1
$\chi_3\chi_2 = \chi_4$	3	-1	0	1	-1
χ_5	2	x	y	z	w

By column orthogonality, $x = 2, y = -1, z = w = 0$. Alternatively, we can use

$$\chi_{\text{reg}} = \chi_1 + \chi_2 + 3\chi_3 + 3\chi_4 + 2\chi_5$$

to deduce χ_5. It is the lifting character of $S_4/V_4 \cong S_3$. See next chapter.

7.1 Alternating groups

Suppose $g \in A_n$ then

$$|\mathcal{C}_{S_n}(g)| = |S_n : C_{S_n}(g)|$$
$$|\mathcal{C}_{A_n}(g)| = |A_n : C_{A_n}(g)|$$

$C_{A_n}(g)$ is contained in $C_{S_n}(g)$ but they are not necessarily equal. For example, let $g = (123) \in A_3$. $\mathcal{C}_{A_3}(g) = g$ but $\mathcal{C}_{S_3}(g) = \{g, g^{-1}\}$. Recall from IA Groups

Lemma 7.5.

1. *If g commutes with some odd permutation in S_n then $\mathcal{C}_{S_n}(g) = \mathcal{C}_{A_n}(g)$.*
2. *If g does not commute with any odd permutation then $\mathcal{C}_{S_n}(g)$ splits into two conjugacy classes in A_n of equal size.*

Exercise. Character table for A_5. See Teleman §12.

8 Normal subgroups and lifting characters

Lemma 8.1 (lifting). *Let $N \trianglelefteq G$ and let $\tilde{\rho} : G/N \to \mathrm{GL}(V)$ be a representation of G/N. Then*

$$\rho : G \to G/N \xrightarrow{\tilde{\rho}} \mathrm{GL}(V)$$

is a representation of G where $\rho(g) = \tilde{\rho}(gN)$. Moreover ρ is irreducible if $\tilde{\rho}$ is. The corresponding characters satisfy

$$\chi(g) = \tilde{\chi}(gN)$$

and $\deg \chi = \deg \tilde{\chi}$. *We say that $\tilde{\chi}$* lifts *to χ.*

Lifting $\tilde{\chi} \to \chi$ is a bijection between

$$\{\textit{irreducible reps of } G/N\} \leftrightarrow \{\textit{irreducible reps of } G \textit{ with } N \textit{ lying in kernel}\}.$$

Proof. Example sheet 1 Q4. □

Lemma 8.2. *The derived subgroup $G' = \langle [a,b] : a, b \in G \rangle$ of G is the unique minimal normal subgroup of G such that G/G' is abelian. G has precisely $\ell = |G/G'|$ representations of dimension 1, all with kernel containing G' and obtained by lifting from G/G'. In particular $\ell \mid |G|$.*

Proof. Easy to check $G' \trianglelefteq G$ and given $N \trianglelefteq G$, $G' \le N$ if and only if G/N is abelian. By Proposition 4.5, G/G' has exactly ℓ characters $\tilde{\chi}_1, \dots, \tilde{\chi}_\ell$, all of degree 1. The lifts of these to G also have degree 1 and thus by Lemma 8.1 these are precisely the irreducible characters χ of G such that $G' \le \ker \chi$. But any linear character of G is a homomorphism $\chi : G \to \mathbb{C}^\times$, hence $\chi(g^{-1}h^{-1}gh) = 1$. Thus $G' \le \ker \chi$. Thus $\chi_1, \dots, \chi_\ell$ are all the linear characters of G. □

Example.

1. $G = S_n$. Show $S_n' = A_n$. Since $G/G' \cong C_2$, S_n must have exactly 2 linear characters.
2. $G = A_4$. Let $V = \{1, (12)(34), (13)(24), (14)(23)\} \trianglelefteq G$ and $G^{\mathrm{ab}} = G/V \cong C_3$. Hence there are three linear characters, all of them trivial on V. Thus A_4 has character table

	1 1	3 (12)(34)	4 (123)	4 (132)
1_G	1	1	1	1
χ_2	1	1	ω	ω^2
χ_3	1	1	ω^2	ω
χ_4	3	-1	0	0

where the last row is from orthogonality.

Lemma 8.3. *G is not simple if and only if $\chi(g) = \chi(1)$ for some irreducible character $\chi \ne 1_G$ and some $1 \ne g \in G$. Moreover any normal subgroup of G is the intersection of the kernels of some of the irreducible characters of*

G.

Proof. If $\chi(g) = \chi(1)$ for some non-principal character χ (afforded by ρ) then $g \in \ker\rho$ by Lemma 5.2. So if $g \neq 1$ then $\ker\rho$ is a nontrivial proper normal subgroup of G. If N is a nontrivial proper normal subgroup, take non-principal irreducible $\tilde{\chi}$ of G/N. Lift to get an irreducible χ, afforded by ρ of G, then $N \leq \ker\rho \trianglelefteq G$. Hence $\chi(g) = \chi(1)$ for all $g \in N$.

Claim that if $1 \neq N \trianglelefteq G$ then N is the intersection of the kernels of the lifts of all the irreducibles of G/N: $\leq$ is clear. For $\geq$, if $g \in G \setminus N$ then $gN \neq N$ so $\tilde{\chi}(gN) \neq \tilde{\chi}(N)$ for some irreducible $\tilde{\chi}$ of G/N. Lifting $\tilde{\chi}$ to χ we have $\chi(g) \neq \chi(1)$. □

9 Dual spaces & tensor products

Recall that $\mathcal{C}(G)$ is the $\mathbb{C}$-space of class functions with dimension k. It has an orthonormal basis $\chi_1, \dots, \chi_k$ of irreducible characters of G. There exists an involution $f \mapsto f^*$ where $f^*(g) = f(g^{-1})$.

9.1 Duality

Lemma 9.1 (dual representation)**.** *Let $\rho : G \to \mathrm{GL}(V)$ be a representation over $\mathbb{F}$ and let $V^* = \mathrm{Hom}_{\mathbb{F}}(V, \mathbb{F})$, the dual space of V. Then V^* is a G-space under*

$$(\rho^*(g)\varphi)(v) = \varphi(\rho(g^{-1})),$$

the dual representation *to ρ. Its character is*

$$\chi_{\rho^*}(g) = \chi_\rho(g^{-1}).$$

Proof. First show $\rho^* : G \to \mathrm{GL}(V^*)$ is indeed a representation:

$$\begin{aligned} \rho^*(g_1)(\rho^*(g_2)\varphi)(v) &= (\rho^*(g_2\varphi))(\rho(g_1^{-1})(v)) \\ &= \varphi(\rho(g_2^{-1})\varphi(g_1^{-1})(v)) \\ &= \varphi(\rho(g_1g_2)^{-1}(v)) \\ &= (\rho^*(g_1g_2)\varphi)(v) \end{aligned}$$

For the character, fix $g \in G$ and let $e_1, \dots, e_n$ be a basis of V of eigenvectors of $\rho(g)$, say

$$\rho(g)e_j = \lambda_j e_j.$$

Let $\varepsilon_1, \dots, \varepsilon_n$ be the dual basis. Then

$$(\rho^*(g)\varepsilon_j)(e_i) = \varepsilon_j(\rho(g^{-1})e_i) = \varepsilon_j \lambda_i^{-1} e_i = \lambda_j^{-1}\varepsilon_j e_i$$

for all i so $\rho^*(g)\varepsilon_j = \lambda_j^{-1}\varepsilon_j$. Thus

$$\chi_{\rho^*}(g) = \sum \lambda_j^{-1} = \chi_\rho(g^{-1}).$$

□

Definition (self-dual)**.** $\rho : G \to \mathrm{GL}(V)$ is *self-dual* if $V \cong_G V^*$. Over $\mathbb{F} = \mathbb{C}$, this holds if and only if

$$\chi_\rho(g) = \chi_\rho(g^{-1}) = \overline{\chi_\rho(g)}$$

if and only if $\chi_\rho(g) \in \mathbb{R}$ for all g.

Example.

1. All irreducible representations of S_n are self-dual: the conjugacy classes are determined by cycle types so g, g^{-1} are always S_n-conjugate. Not always true for A_n.

2. Permutation representations $\mathbb{C}X$ are always self-dual.

9.2 Tensor products

Definition (tensor product)**.** Let V, W be $\mathbb{F}$-spaces with $\dim V = m, \dim W = n$. Fix basis $v_1, \dots, v_m$ of V, $w_1 \dots, w_n$ of W. The *tensor product* space $V \otimes_{\mathbb{F}} W$ or $V \otimes W$ is an nm-dimensional $\mathbb{F}$-space with basis

$$\{v_i \otimes w_j : 1 \leq i \leq m, 1 \leq j \leq n\}.$$

Thus

1. $$V \otimes W = \left\{ \sum \lambda_{ij} v_i \otimes w_j : \lambda_{ij} \in \mathbb{F} \right\}$$
   with obvious addition and multiplication.
2. If $v = \sum \alpha_i v_i \in V, w = \sum \beta_j w_j \in W$ define
   $$v \otimes w = \sum \alpha_i \beta_j (v_i \otimes w_j).$$

Remark. Note not all elements of $V \otimes W$ are of this form — some are combinations, e.g. $v_1 \otimes w_1 + v_2 \otimes w_2$, which can't be further simplified.

Lemma 9.2.

1. *For $v \in V, w \in W, \lambda \in \mathbb{F}$,*
   $$(\lambda v) \otimes w = \lambda(v \otimes w) = v \otimes (\lambda w).$$
2. *If $x, x_1, x_2 \in V, y, y_1, y_2 \in W$ then*
   $$\begin{aligned}(x_1 + x_2) \otimes y &= x_1 \otimes y + x_2 \otimes y \\ x \otimes (y_1 + y_2) &= x \otimes y_1 + x \otimes y_2\end{aligned}$$

Proof. Easy verifications:

1. if $v = \sum \alpha_i v_i, w = \sum \beta_j v_j$ then
   $$\begin{aligned}\lambda v \otimes w &= \sum_{i,j} (\lambda \alpha_i) \beta_j v_i \otimes w_j \\ \lambda(v \otimes w) &= \sum_{i,j} (\lambda \alpha_i) \beta_j v_i \otimes w_j \\ v \otimes \lambda w &= \sum_{i,j} (\lambda \alpha_i) \beta_j v_i \otimes w_j\end{aligned}$$
2. exercise.

□

It follows that the map

$$\begin{aligned} V \times W &\to V \otimes W \\ (v, w) &\mapsto v \otimes w \end{aligned}$$

is bilinear.

Lemma 9.3. *If $\{e_1, \dots, e_m\}$ is any basis of V, $\{f_1, \dots, f_n\}$ any basis of W then $\{e_i \otimes f_j : 1 \le i \le m, 1 \le j \le n\}$ is a basis of $V \otimes W$.*

Proof. Writing $v_k = \sum_i \alpha_{ik} e_i, w_\ell = \sum_j \beta_{j\ell} f_j$, we have

$$v_k \otimes w_\ell = \sum_{i,j} \alpha_{ik}\beta_{j\ell} e_i \otimes f_j,$$

hence $\{e_i \otimes f_j\}$ spans $V \otimes W$. And since there are nm of them, they are a basis. ☐

Remark. One can define $V \otimes W$ in a basis independent way in the first place. See Teleman §6.

Proposition 9.4. *Let $\rho : G \to \mathrm{GL}(V), \rho' : G \to \mathrm{GL}(V')$ be complex representations of G. Define*

$$(\rho \otimes \rho')(g) : \sum \lambda_{ij} v_i \otimes w_j \mapsto \sum \lambda_{ij} \rho(g) v_i \otimes \rho'(g) w_j.$$

Then $\rho \otimes \rho'$ is a representation with character

$$\chi_{\rho \otimes \chi'}(g) = \chi_\rho(g) \chi_{\rho'}(g)$$

for all g. Hence product of two characters of G is also a character.

Remark. On example sheet 1, we saw ρ irreducible, ρ' of degree 1 implies that $\rho \otimes \rho'$ is irreducible. If ρ' is not of degree 1 this is usually false.

Proof. It is clear that $(\rho \otimes \rho')(g) \in \mathrm{GL}(V \otimes V')$ for all $g \in G$ and so $\rho \otimes \rho'$ is a homomorphism $G \to \mathrm{GL}(V \otimes V')$. Let $g \in G$. Let $v_1, \dots, v_m$ be a basis of V of eigenvectors of $\rho(g)$, $w_1, \dots, w_n$ be a basis of V' of eigenvectors of $\rho'(g)$, say

$$\rho(g) v_j = \lambda_j v_j, \rho'(g) w_j = \mu_j w_j.$$

Then

$$(\rho \otimes \rho')(g)(v_i \otimes w_j) = \rho(g) v_i \otimes \rho'(g) w_j = \lambda_i v_i \otimes \mu_j w_j = (\lambda_i \mu_j)(v_i \otimes w_j)$$

so

$$\chi_{\rho \otimes \rho'}(g) = \sum_{i,j} \lambda_i \mu_j = \sum \lambda_i \sum \mu_j = \chi_\rho(g) \chi_{\rho'}(g).$$

☐

Let $\mathbb{F} = \mathbb{C}$. Take $V = V'$ and define

$$V^{\otimes 2} = V \otimes V.$$

Let $\tau : \sum \lambda_{ij} v_i \otimes v_j \mapsto \sum \lambda_{ij} v_j \otimes v_i$, a linear G-endomorphism of $V^{\otimes 2}$ such that $\tau^2 = 1$, so has eigenvalues ± 1.

Definition (symmetric/exterior square)**.** Define

$$S^2V = \{x \in V^{\otimes 2} : \tau(x) = x\}$$
$$\Lambda^2 V = \{x \in V^{\otimes 2} : \tau(x) = -x\}$$

the *symmetric* and *exterior square* of V.

Lemma 9.5. *$S^2V, \Lambda^2 V$ are G-subspaces of $V^{\otimes 2}$ and*

$$V^{\otimes 2} = S^2V \oplus \Lambda^2 V.$$

S^2V has basis

$$\{v_i v_j = v_i \otimes v_j + v_j \otimes v_i : 1 \leq i \leq j \leq n\}$$

$\Lambda^2 V$ has basis

$$\{v_i \wedge v_j = v_i \otimes v_j - v_j \otimes v_i : 1 \leq i < j \leq n\}.$$

(Note that in some conventions the definition of $v_i v_j$ and $v_i \wedge v_j$ is half what we defined here.) Hence

$$\dim S^2V = \frac{n(n+1)}{2}$$
$$\dim \Lambda^2 V = \frac{n(n-1)}{2}$$

Proof. Easy exercise by noting that for any $x \in V^{\otimes 2}$,

$$x = \frac{1}{2}(x + \tau(x)) + \frac{1}{2}(x - \tau(x)).$$

□

Lemma 9.6. *If $\rho : G \to \mathrm{GL}(V)$ is a representation affording character χ, then*

$$\chi^2 = \chi_S + \chi_\Lambda$$

where χ_S, χ_Λ are the characters of G in the subrepresentations S^2V and $\Lambda^2 V$. Moreover

$$\chi_S(g) = \frac{1}{2}(\chi^2(g) + \chi(g^2))$$
$$\chi_\Lambda(g) = \frac{1}{2}(\chi^2(g) - \chi(g^2))$$

Proof. Compute the characters χ_S, χ_Λ in the usual way: fix an element and choose an eigenbasis. □

Example. $G = S_4$: We have worked out the character table before

	1	3	8	6	6
	1	(12)(34)	(123)	(1234)	(12)
χ_1	1	1	1	1	1
$\text{sgn} = \chi_2$	1	1	1	-1	-1
$\pi_X - 1 = \chi_3$	3	-1	0	-1	1
$\chi_3\chi_2 = \chi_4$	3	-1	0	1	-1
χ_5	2	2	-1	0	0

Take χ_3, we can work out its symmetric and exterior square

	1	3	8	6	6
	1	(12)(34)	(123)	(1234)	(12)
χ_3^2	9	1	0	1	1
$\chi_3(g^2)$	3	3	0	3	-1
$S^2\chi_3$	6	2	0	2	0
$\Lambda^2\chi_3$	3	-1	0	-1	1

By simply calculating the inner prodcuct, we see that $\chi_4 = \Lambda^2\chi_3$ is irreducible.

We also see

$$S^2\chi_3 = 1 + \chi_3 + \chi_5.$$

9.3 Characters of product groups

Proposition 9.7. *If G, H are finite groups, with their irreducible characters $\chi_1, \dots, \chi_k$ and $\psi_1, \dots, \psi_r$ respectively, then the irreducible characters of their direct product $G \times H$ are precisely $\{\chi_i\psi_j : 1 \le i \le k, 1 \le j \le r\}$, where*

$$\chi_i\psi_j(g,h) = \chi_i(g)\psi_j(h).$$

Proof. If $\rho : G \to \mathrm{GL}(V)$ affords χ, $\rho' : H \to \mathrm{GL}(W)$ affords ψ then

$$\begin{aligned} \rho \otimes \rho' : G \times H &\to \mathrm{GL}(V \otimes W) \\ (g,h) &\mapsto \rho(g) \otimes \rho'(h) \end{aligned}$$

is a representation of $G \times H$ on $V \otimes W$ and $\chi_{\rho\otimes\rho'} = \chi\psi$.

Claim that $\chi_i\psi_j$'s are distinct and irreducible:

$$\begin{aligned} \langle \chi_i\psi_j, \chi_r\psi_s \rangle_{G\times H} &= \frac{1}{|G \times H|} \sum_{(g,h)} \overline{\chi_i\psi_j(g,h)}\chi_r\psi_s(g,h) \\ &= \left(\frac{1}{|G|} \sum_g \overline{\chi_i(g)}\chi_r(g) \right) \left(\frac{1}{|H|} \sum_h \overline{\psi_j(h)}\psi_s(h) \right) \\ &= \delta_{ir}\delta_{js} \end{aligned}$$

To show that they are complete, we take their squares at identity:

$$\sum_{i,j} \chi_i\psi_j(1)^2 = \sum_i \chi_i^2(1) \sum_j \psi_j^2(1) = |G||H| = |G \times H|.$$

$\square$

9.4 Symmetric and exterior powers

Let V be an $\mathbb{F}$-space with $\dim V = d$. Choose a basis $\{v_1, \dots, v_d\}$. Let

$$V^{\otimes n} = \underbrace{V \otimes \cdots \otimes V}_{n}$$

which has a basis $\{v_{i_1} \otimes \cdots \otimes v_{i_n} : i_1, \dots, i_n \in \{1, \dots, d\}\}$ so $\dim V^{\otimes n} = d^n$.

There is an S_n-action on the space V: for each $\sigma \in S_n$, we can define a linear map

$$\begin{aligned} \sigma : V^{\otimes n} &\to V^{\otimes n} \\ v_1 \otimes \cdots \otimes v_n &\mapsto v_{\sigma(1)} \otimes \cdots \otimes v_{\sigma(n)} \end{aligned}$$

for $v_1, \dots, v_n \in V$, which induces a (right) action of S_n on $V^{\otimes n}$.

Given a representation $\rho : G \to \mathrm{GL}(V)$, define a (left) action of G on $V^{\otimes n}$ by

$$\rho^{\otimes n} : v_1 \otimes \cdots \otimes v_n \mapsto \rho(g)v_1 \otimes \cdots \otimes \rho(g)v_n$$

which commutes with the S_n-action. So we can decompose $V^{\otimes n}$ as S_n-spaces, and each isotypical component is a G-invariant subspace of $V^{\otimes n}$. In particular

Definition (symmetric/exterior power)**.** For G-space V, define

1. the *nth symmetric power* of V

$$S^n V = \{x \in V^{\otimes n} : \sigma(x) = x \text{ for all } \sigma \in S_n\},$$

2. the *nth exterior power* of V

$$\Lambda^n V = \{x \in V^{\otimes n} : \sigma(x) = (\operatorname{sgn} \sigma)x \text{ for all } \sigma \in S_n\}.$$

Both are G-subspaces of $V^{\otimes n}$, but for $n > 2$, $S^n V \oplus \Lambda^n V$ is a proper subspace of $V^{\otimes n}$. See example sheet 3 Q7 for bases of $S^n V, \Lambda^n V$.

9.5 Tensor algebra

Take $\operatorname{ch} \mathbb{F} = 0$.

Definition (tensor algebra)**.** Let $T^n V = V^{\otimes n}$. The *tensor algebra* of V is

$$T(V) = \bigoplus_{n \geq 0} T^n V$$

with $T^0(V) = \mathbb{F}$. This is an $\mathbb{F}$-algebra. $T(V)$ is a graded ring with product

$$x \in T^n(V), y \in T^m(V) \implies x \cdot y = x \otimes y \in T^{n+m}(V).$$

There are two graded quotient rings

$$\begin{aligned} S(V) &= T(V)/(u \otimes v - v \otimes u) \\ \Lambda(V) &= T(V)/(v \otimes v) \end{aligned}$$

the *symmetric* and *exterior algebra* respectively. Have

$$\begin{aligned} S(V) &= \bigoplus_{n\geq 0} S^n V \\ \Lambda(V) &= \bigoplus_{n\geq 0} \Lambda^n V \end{aligned}$$

9.6 Character ring

$\mathcal{C}(G)$ is a commutative ring.

Definition (character ring, virtual character)**.** The $\mathbb{Z}$-submodule of $\mathcal{C}(G)$ spanned by irreducible characters of G is called the *character ring* of G, sometimes also known as *Grothendieck ring*, denoted $R(G)$. Elements of $R(G)$ are called *generalised characters* or *virtual characters*.

$R(G)$ is a ring. Any generalised character is a difference of two ordinary characters. $\{\chi_i\}$ form a $\mathbb{Z}$-basis for $R(G)$ as a free $\mathbb{Z}$-module.

10 Induction and restriction

Throughout the chapter let $\mathbb{F} = \mathbb{C}$ and $H \leq G$.

Definition (restriction)**.** Let $\rho : G \to \mathrm{GL}(V)$ be a representation affording χ. We can think of V as a H-space by restricting attention to $h \in H$. We get $\mathrm{Res}_H^G \rho = \rho\downarrow_H = r_H$, the *restriction* of ρ to H. It affords the character $\mathrm{Res}_H^G \chi = \chi\downarrow_H = \chi_H$.

Lemma 10.1. *If ψ is any nonzero character of H then there exists an irreducible character χ of G such that ψ is a* constituent *of $\mathrm{Res}_H^G \chi$, i.e.*

$$\langle \mathrm{Res}_H^G \chi, \psi \rangle \neq 0.$$

Proof. List the irreducible characters of G as $\chi_1, \dots, \chi_k$. Recall π_{reg}. Have

$$\sum_{i=1}^k \deg \chi_i \langle \mathrm{Res}_H^G \chi_i, \psi \rangle = \langle \mathrm{Res}_H^G \pi_{\mathrm{reg}}, \psi \rangle = \frac{|G|}{|H|} \psi(1) \neq 0$$

so $\langle \mathrm{Res}_H^G \chi_i, \psi \rangle \neq 0$ for some i. ∎

Lemma 10.2. *Let χ be an irreducible character of G and write*

$$\mathrm{Res}_H^G \chi = \sum_i c_i \chi_i$$

where χ_i's are irreducible characters of H. Then

$$\sum_i c_i^2 \leq |G : H|$$

with equality if and only if $\chi(g) = 0$ for all $g \in G \setminus H$.

Proof. We have

$$\begin{aligned}
1 &= \langle \chi, \chi \rangle \\
&= \frac{1}{|G|} \sum_{g \in G} |\chi(g)|^2 \\
&= \frac{1}{|G|} \left(\sum_{g \in H} |\chi(g)|^2 + \sum_{g \in G \setminus H} |\chi(g)|^2 \right) \\
&= \frac{|H|}{|G|} \langle \mathrm{Res}_H^G \chi, \mathrm{Res}_H^G \chi \rangle + \frac{1}{|G|} \sum_{g \in G \setminus H} |\chi(g)|^2 \\
&\geq \frac{1}{|G : H|} \sum_i c_i^2
\end{aligned}$$

with equality if and only if $\chi(g) = 0$ for all $g \in G \setminus H$. ∎

Definition (induction)**.** If ψ is a class function of H, define the *induced class function* $\operatorname{Ind}_H^G \psi = \psi \uparrow^G = \psi^G$ by

$$\operatorname{Ind}_H^G \psi(g) = \frac{1}{|H|} \sum_{x \in G} \mathring{\psi}(x^{-1}gx)$$

where

$$\mathring{\psi}(y) = \begin{cases} \psi(y) & y \in H \\ 0 & \text{otherwise} \end{cases}$$

Lemma 10.3. *If $\psi \in \mathcal{C}(H)$ then $\operatorname{Ind}_H^G \psi \in \mathcal{C}(G)$ is a class function of G and*

$$\operatorname{Ind}_H^G \psi(1) = |G : H| \psi(1).$$

Proof. Obvious. ☐

Let $n = |G : H|$. Let $t_1 = 1, t_2, \dots, t_n$ be a *left transversal* of H in G, i.e. $t_1 H = H, t_2 H, \dots, t_n H$ are precisely the left cosets of H in G.

Lemma 10.4. *Given $\psi \in \mathcal{C}(H)$ and a left transversal $t_1, \dots, t_n$, have*

$$\operatorname{Ind}_H^G \psi(g) = \sum_{i=1}^n \mathring{\psi}(t_i^{-1} g t_i).$$

Proof. Note that every $x \in G$ can be written as $t_i h$ where $h \in H$ and

$$\mathring{\psi}(x^{-1}gx) = \mathring{\psi}(h^{-1}(t_i^{-1}gt_i)h) = \mathring{\psi}(t_i^{-1}gt_i)$$

as ψ is a class function of H. ☐

Theorem 10.5 (Frobenius reciprocity)**.** *Let $\psi \in \mathcal{C}(H), \varphi \in \mathcal{C}(G)$. Then*

$$\langle \operatorname{Res}_H^G \varphi, \psi \rangle = \langle \varphi, \operatorname{Ind}_H^G \psi \rangle.$$

Proof.

$$\begin{aligned}
\langle \varphi, \operatorname{Ind}_H^G \psi \rangle &= \frac{1}{|G|} \sum_{g \in G} \overline{\varphi(g)} \operatorname{Ind}_H^G \psi(g) \\
&= \frac{1}{|G||H|} \sum_{x, g \in G} \overline{\varphi(g)} \mathring{\psi}(x^{-1}gx) \\
&= \frac{1}{|G||H|} \sum_{x, y \in G} \overline{\varphi(y)} \mathring{\psi}(y) \quad \text{set } x^{-1}gx = y \\
&= \frac{1}{|H|} \sum_{y \in G} \overline{\varphi(y)} \mathring{\psi}(y) \\
&= \frac{1}{|H|} \sum_{y \in H} \overline{\varphi(y)} \psi(y) \\
&= \langle \operatorname{Res}_H^G \varphi, \psi \rangle
\end{aligned}$$

☐

Corollary 10.6. *If ψ is a character of H then $\operatorname{Ind}_H^G \psi$ is a character of G.*

Proof. If χ is an irreducible character of G then by Frobenius reciprocity

$$\langle \chi, \operatorname{Ind}_H^G \psi \rangle = \langle \operatorname{Res}_H^G \chi, \psi \rangle \in \mathbb{Z}_{\geq 0}$$

since $\psi, \operatorname{Res}_H^G \chi$ are characters. Hence $\operatorname{Ind}_H^G \psi$ is a linear combination of irreducible characters with nonnegative coefficients, hence a character. □

Proposition 10.7. *Let ψ be a character of $H \leq G$ and let $g \in G$. Let*

$$\mathcal{C}_G(g) \cap H = \bigcup_{i=1}^{m} \mathcal{C}_H(x_i)$$

where x_i's are representatives of the m H-conjugacy classes of elements of H conjugate to g. Then if $m = 0$ then $\operatorname{Ind}_H^G \psi(g) = 0$. Otherwise

$$\operatorname{Ind}_H^G \psi(g) = |C_G(g)| \sum_{i=1}^{m} \frac{\psi(x_i)}{|C_H(x_i)|}.$$

Proof. If $m = 0$ then $\{x \in G : x^{-1}gx \in H\} = \emptyset$ and so $\operatorname{Ind}_H^G \psi(g) = 0$. Assume that $m > 0$ and let

$$X_i = \{x \in G : x^{-1}gx \in H \text{ and conjugate in } H \text{ to } x_i\}.$$

The X_i's are pairwise disjoint and their union is $\{x \in G : x^{-1}gx \in H\}$. By definition

$$\begin{aligned}
\operatorname{Ind}_H^G \psi(g) &= \frac{1}{|H|} \sum_{x \in G} \mathring{\psi}(x^{-1}gx) \\
&= \frac{1}{|H|} \sum_{\substack{x \in G \\ x^{-1}gx \in H}} \psi(x^{-1}gx) \\
&= \frac{1}{|H|} \sum_{i=1}^{m} \sum_{x \in X_i} \psi(x^{-1}gx) \\
&= \frac{1}{|H|} \sum_{i=1}^{m} \sum_{x \in X_i} \psi(x_i) \\
&= \sum_{i=1}^{m} \frac{|X_i|}{|H|} \psi(x_i)
\end{aligned}$$

Need to understand the quotient $\frac{|X_i|}{|H|}$. Fix some $1 \leq i \leq m$ and choose some $g_i \in G$ such that $g_i^{-1} g g_i = x_i$. So for all $c \in C_G(g)$ and $h \in H$,

$$(cg_ih)^{-1} g (cg_ih) = h^{-1} g_i^{-1} c^{-1} g c g_i h = h^{-1} g_i^{-1} g g_i h = h^{-1} x_i h \in H$$

i.e. $cg_ih \in X_i$, hence $C_G(g) g_i H \subseteq X_i$.

Conversely, if $x \in X_i$ then

$$x^{-1}gx = h^{-1}x_ih = h^{-1}(g_i^{-1}gg_i)h$$

for some $h \in H$. Thus $xh^{-1}g_i^{-1} \in C_G(g)$ and

$$x \in C_G(g)g_i h \subseteq C_G(g)g_i H$$

so we have equality

$$X_i = C_G(g)g_i H.$$

Thus

$$\begin{aligned} |X_i| &= |C_G(g)g_i H| \\ &= \frac{|C_G(g)||H|}{|H \cap g_i^{-1}C_G(g)g_i|} \end{aligned}$$

Note that $g_i^{-1}C_G(g)g_i = C_G(g_i^{-1}gg_i) = C_G(x_i)$,

$$\begin{aligned} &= |H : H \cap C_G(x_i)||C_G(g)| \\ &= |H : C_H(x_i)||C_G(g)| \end{aligned}$$

where we used a formula for double coset size. The result thus follows. □

Remark.

1. If $H, K \leq G$, an (H, K)-*double coset* of H and K in G is a set

$$HgK = \{hgk : h \in H, k \in K\}$$

   for some $g \in G$. Facts:

   (a) two double cosets are either disjoint or equal.

   (b) for finite $|HK|$,

$$|HgK| = \frac{|H||K|}{|H \cap gKg^{-1}|} = \frac{|H||K|}{|g^{-1}Hg \cap K|}$$

   See chapter 12 for more on double cosets.

2. An alternative proof can be found in James and Liebeck, chapter 21, 23.

Example. $H = C_4 = \langle (1234) \rangle \leq G = S_4$ with index 6. We calculate the character of induced representations $\mathrm{Ind}_H^G(\alpha)$, where α is a 1 dimensional faithful representation of C_4.

If $\alpha(1234) = i$ then character of α is

	1	(1234)	(13)(24)	(1432)
χ_α	1	i	-1	$-i$

The induced representation of S_4 is

	1	6	8	3	6
	1	(12)	(123)	(12)(24)	(1234)
$\mathrm{Ind}_H^G \chi_\alpha$	6	0	0	-2	0

The first three entries are easy. For (12)(34), only one of the three elements in C_4 it's conjugate to lies in H, namely (13)(24) so

$$\mathrm{Ind}_H^G \chi_\alpha((12)(34)) = 8 \cdot \frac{-1}{4} = -2.$$

For (1234) its conjugate to six elements of S_4, of which two are in C_4: (1234) and (1432). So

$$\operatorname{Ind}_H^G \chi_\alpha(1234) = 4 \cdot \left(\frac{i}{4} - \frac{i}{4}\right) = 0.$$

Lemma 10.8. *If $\psi = 1_H$ then*

$$\operatorname{Ind}_H^G 1_H = \pi_X,$$

the permutation character of G on the set X of left cosets of H in G.

Proof.

$$\begin{aligned}\operatorname{Ind}_H^G 1_H(g) &= \sum_{i=1}^n \mathring{1}_H(t_i^{-1} g t_i) \\ &= |\{i : t_i^{-1} g t_i \in H\}| \\ &= |\{i : g \in t_i H t_i^{-1}\}| \\ &= |\operatorname{fix}_X(g)| \\ &= \pi_X(g)\end{aligned}$$

□

Remark. It follows from Frobenius reciprocity

$$\langle \pi_X, 1_G \rangle_G = \langle \operatorname{Ind}_H^G 1_H, 1_G \rangle_G = \langle 1_H, 1_H \rangle_H = 1$$

as predicted in chapter 7.

10.1 Induced representations

What are the representations affording induced characters? Let $H \leq G$ with index n. Let $1 = t_1, \dots, t_n$ be transversals. Let W be an H-space.

Definition. Let

$$V = \operatorname{Ind}_H^G W = \bigoplus_i t_i \otimes W$$

where $t_i \otimes W = \{t_i \otimes w : w \in W\}$.

Have $\dim V = n \dim W$.

We can define a G-action on V. If $g \in G$ then for all i there exists a unique j with $t_j^{-1} g t_i \in H$ (namely $t_j H$ is the coset containing gt_i). Define

$$g(t_i \otimes w) = t_j \otimes (\underbrace{(t_j^{-1} g t_i)}_{\in H} w) = t_j((t_j^{-1} g t_i) w).$$

where we omit the tensor symbol in the last expression. Check this is a G-action:

$$\begin{aligned}g_1(g_2 t_i w) &= g_1(t_j(t_j^{-1} g_2 t_i) w) \\ &= t_\ell((t_\ell^{-1} g_1 t_j)(t_j^{-1} g_2 t_i) w) \\ &= t_\ell(t_\ell^{-1}(g_1 g_2) t_i) w \\ &= (g_1 g_2)(t_i w)\end{aligned}$$

where j, ℓ is unique such that $g_2 t_i H = t_j H$ and $g_1 t_j H = t_\ell H$. It follows that ℓ is unique such that $(g_1 g_2) t_i H = t_\ell H$. Note that g permutes the cosets as

$$g : t_i w \mapsto t_j (t_j^{-1} g t_i) w$$

so the contribution to the character is 0 unless $j = i$, i.e. $t_i^{-1} g t_i \in H$, then it contributes $\psi(t_i^{-1} g t_i)$ so

$$\operatorname{Ind}_H^G \psi(g) = \sum_{i=1}^n \mathring{\psi}(t_i^{-1} g t_i).$$

Proposition 10.9 (properties of induced modules)**.**

1. $\operatorname{Ind}_H^G(A \oplus B) = \operatorname{Ind}_H^G A \oplus \operatorname{Ind}_H^G B$ *where* A, B *are* H*-spaces.*
2. $\dim \operatorname{Ind}_H^G W = |G : H| \dim W$.
3. $\operatorname{Ind}_{\{1\}}^G 1 = \rho_{reg}$.
4. *If* $H \leq K \leq G$ *then*
$$\operatorname{Ind}_K^G \operatorname{Ind}_H^K W \cong \operatorname{Ind}_H^G W.$$
5. *(Frobenius reciprocity)*
$$\operatorname{Hom}_H(W, \operatorname{Res}_H^G V) \cong \operatorname{Hom}_G(\operatorname{Ind}_H^G W, V)$$
naturally.

Proof. Exercises. For 4 see exmaple sheet 3. For 5 see Teleman 15.6. □

11 Frobenius groups

Theorem 11.1. *Let G be a transitive permutation group on finite set X, say $|X| = n$. Assume that each non-identity element fixes at most one element of X. Then*

$$K = \{1\} \cup \{g \in G : g\alpha \neq \alpha \text{ for all } \alpha \in X\}$$

is a normal subgroup of G of order n.

Note that G is necessarily finite, being isomorphic to a subgroup of Σ_X.

Proof Due to I. M. Issacs. Required to show that $K \trianglelefteq G$. Let $H = G_\alpha$, the stabiliser of α for some $\alpha \in X$, so conjugates of H are the stabilisers of single elements of X, i.e.

$$G_{g\alpha} = gG_\alpha g^{-1}.$$

No two conjugates can share a non-identity element by hypothesis so H has n distinct conjugates and G has $n(|H| - 1)$ elements that fix exactly one element of X. Now

$$|G| = |X||H| = n|H|$$

because X and G/H are isomorphic as G-sets by transitivity. Hence

$$|K| = |G| - n(|H| - 1) = n$$

If $1 \neq h \in H$ and suppose $h = gh'g^{-1}$ for some $g \in G, h' \in H$, then h lies in both $H = G_\alpha$ and $gHg^{-1} = G_{g\alpha}$, by hypothesis $g\alpha = \alpha$, hence $g \in H$. Therefore the intersection of conjugacy class in G of h with H is precisely the conjugacy class in H of h.

Similarly if $g \in C_G(h)$ then

$$h = ghg^{-1} \in G_{g\alpha}$$

and hence $g \in H$, which implies

$$C_G(h) = C_H(h).$$

Every element of G is either an element of K or lies in one of the n stabilisers, each of which is conjugate to H. Thus every element of $G \setminus K$ is conjugate to a non-identity element of H. Hence

$$\{1, h_2, \dots, h_t, y_1, \dots, y_u\}$$

is a set of conjugacy class representatives for G, with $1, \dots, h_t$ representatives of H-conjugacy classes and $y_1, \dots, y_u$ representatives of conjugacy classes of G comprises $K \setminus \{1\}$.

Let $1_H = \psi_1, \psi_2, \dots, \psi_t$ be irreducible characters of H. Fix $1 \leq i \leq t$. Then

$$\operatorname{Ind}_H^G \psi_i(g) = \begin{cases} |G : H|\psi_i(1) = n\psi_i(1) & g = 1 \\ \psi_i(h_j) & g = h_j, 2 \leq j \leq t \\ 0 & g = y_k, 1 \leq k \leq u \end{cases}$$

Let $\theta_1 = 1_G$. Fix some $2 \le i \le t$ and define virtual characters

$$\theta_i = \psi_i^G - \psi_i(1)\psi_1^G + \psi_i(1)\theta_1 \in R(G)$$

Write down a table

	1	h_j	y_k
ψ_i^G	$n\psi_i(1)$	$\psi_i(h_j)$	0
$\psi_i(1)\psi_1^G$	$n\psi_i(1)$	$\psi_i(1)$	0
$\psi_i(1)\theta_1$	$\psi_i(1)$	$\psi_i(1)$	$\psi_i(1)$
θ_i	$\psi_i(1)$	$\psi_i(h_j)$	$\psi_i(1)$

Check the inner product:

$$\begin{aligned}
\langle \theta_i, \theta_i \rangle &= \frac{1}{|G|} \sum_{g \in G} |\theta_i(g)|^2 \\
&= \frac{1}{|G|} \left(\sum_{g \in K} |\theta_i(g)|^2 + \sum_{\alpha \in X} \sum_{1 \ne g \in G_\alpha} |\theta_i(g)|^2 \right) \\
&= \frac{1}{|G|} \left(n\psi_i(1)^2 + n \sum_{1 \ne h \in H} |\theta_i(h)|^2 \right) \\
&= \frac{1}{|H|} \sum_{h \in H} |\psi_i(h)|^2 \\
&= \langle \psi_i, \psi_i \rangle \\
&= 1
\end{aligned}$$

Thus either θ_i or $-\theta_i$ is an irreducible character of G. But since $\theta_i(1) > 0$, it must be that θ_i is an actual character.

Now define $\theta = \sum_{i=1}^t \theta_i(1)\theta_i$. By column orthogonality, for $1 \ne h \in H$

$$\theta(h) = \sum_{i=1}^t \psi_i(1)\psi_i(h) = 0,$$

and for any $y \in K$,

$$\theta(y) = \sum_{i=1}^t \psi_i(1)^2 = |H|.$$

Hence $\theta(g) = \begin{cases} |H| & g \in K \\ 0 & g \notin K \end{cases}$ so

$$K = \{g \in G : \theta(g) = \theta(1)\} = \ker \theta \trianglelefteq G.$$

□

Definition (Frobenius group). A *Frobenius group* is a group G having a subgroup H such that $H \cap gHg^{-1} = 1$ for all $g \notin H$. H is the *Frobenius complement* of G.

Proposition 11.2. *Any finite Frobenius group satisfies the hypothesis of Theorem 11.1. The normal subgroup K is a* Frobenius kernel *of G.*

Proof. Suppose G is Frobenius with complement H. Then the action of G on G/H is transitive and faithful. Furthermore, if $1 \neq g \in G$ fixes both xH and yH then $g \in xHx^{-1} \cap yHy^{-1}$ and hence

$$H \cap (y^{-1}x)H(y^{-1}x)^{-1} \neq 1$$

so $xH = yH$. □

Example.

1. If p, q are distinct primes and $p = 1 \pmod{q}$, the unique non-abelian group of order pq is a Frobenius group. See JL §25 and Teleman §11.

2. If n is odd, D_{2n} is a Frobenius group with complement C_2. The smallest example is S_3 with $K = C_3, H = C_2$.

Remark.

1. J. Thompson (thesis, 1959) proved that any finite group having a fixed-point-free automorphism of prime power order is nilpotent. This implies that the Frobenius kernel of a Frobenius group is nilpotent (which is equivalent to K being the direct product of its Sylow subgroups).

2. There is no known proof of Theorem 11.1 in which character theory is not used.

12 Mackey theory

Let $\mathbb{F} = \mathbb{C}$. Mackey theory describes restriction to a subgroup $K \leq G$ of an induced representation $\mathrm{Ind}_H^G W$. K and H are unrelated, but usually we take $K = H$, in which case we can characterise when $\mathrm{Ind}_H^G W$ is irreducible.

We'll work with the special case $W = 1_H$ first. Then $\mathrm{Ind}_H^G 1_H$ is the permutation representation of G on G/H. Recall that if G is transitive on a set X and $H = G_\alpha$ for some $\alpha \in X$ then the action of G on X is isomorphic to the action of G on G/H, namely

$$g.\alpha \leftrightarrow gH$$

is a well-defined bijection and commutes with the G-action

$$x(g\alpha) = (xg)\alpha \leftrightarrow x(gH) = (xg)H.$$

Consider the action of G on G/H and let $K \leq G$. Then G/H splits into K-orbits: those correspond to *double cosets*

$$KgH = \{kgh : k \in K, h \in H\},$$

namely the K-orbits containing gH.

Notation. Denote by $K\backslash G/H$ the set of (K,H)-double cosets. They partition G. Let S be a set of representatives. Note

$$\#K\backslash G/H = \langle \pi_{G/K}, \pi_{G/H} \rangle$$

by Lemma 7.3.

Clearly $G_{gH} = gHg^{-1}$. Restricting to K, we get

$$H_g := K_{gH} = gHg^{-1} \cap K.$$

So by above as K-set, $KgH \cong K/K \cap gHg^{-1} = K/H_g$. As

$$\mathrm{Ind}_H^G 1_H = \mathbb{C}X$$

where $X = G/H$, and if $X = \bigcup X_i$ then it decomposes into orbits $\mathbb{C}X = \bigcup \mathbb{C}X_i$ we have

Proposition 12.1. *If G is finite, $H, K \leq G$ then*

$$\mathrm{Res}_K^G \mathrm{Ind}_H^G 1 \cong \bigoplus_{g \in S} \mathrm{Ind}_{K \cap gHg^{-1}}^K 1.$$

Let $S = \{1 = g_1, \dots, g_r\}$ be the such that $G = \bigcup_i Kg_iH$ as a union of disjoint set. Let $H_g = gHg^{-1} \cap K \leq K$. Take a representation (ρ, W) of H. For $g \in G$ define (ρ_g, W_g) to be the representation of H_g with the same underlying vector space W but now the H_g-action is

$$\rho_g(x) = \rho(h) = \rho(g^{-1}xg)$$

where $x = ghg^{-1}$. This is well-defined because $g^{-1}xg \in H$ for $x \in gHg^{-1}$. Since $H_g \leq K$ we obtain an induced representation $\mathrm{Ind}_{H_g}^K W_g$.

Theorem 12.2 (Mackey's restriction formula)**.** *Let W be an H-space. Then*

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G W \cong \bigoplus_{g \in S} \operatorname{Ind}_{H_g}^K W_g$$

as representations of K.

Corollary 12.3 (character version of Mackey's restriction formula)**.** *If ψ is a character of a representation of H then*

$$\operatorname{Res}_K^G \operatorname{Ind}_H^G \psi = \sum_{g \in S} \operatorname{Ind}_{H_g}^K \psi_g$$

where ψ_g is the character of H_g given as $\psi_g(x) = \psi(g^{-1}xg)$.

Corollary 12.4 (Mackey's irreducibility criterion)**.** *Let $H \le G$ and W an H-space. Then $V = \operatorname{Ind}_H^G W$ is irreducible if and only if*

1. *W is irreducible,*
2. *and for each $g \in S \setminus H$ the two H_g-spaces W_g and $\operatorname{Res}_{H_g}^H W$ have no irreducible constituents in common.*

Remark. The set of representatives is arbitrary so we could just as easily demand in 2 that $g \in G \setminus H$. However it suffices to check for $g \in S \setminus H$.

Proof of Mackey's irreducibility criterion. Use characters and recall that W is irreducbile if and only if $\langle \psi, \psi \rangle = 1$ where W affords the character ψ. Take $K = H$ in Mackey's restriction formula. Note $H_g = gHg^{-1} \cap H$. Use Frobenius reciprocity,

$$\begin{aligned} \langle \operatorname{Ind}_H^G \psi, \operatorname{Ind}_H^G \psi \rangle_G &= \langle \psi, \operatorname{Res}_H^G \operatorname{Ind}_H^G \psi \rangle_H \\ &= \sum_{g \in S} \langle \psi, \operatorname{Ind}_{H_g}^H \psi_g \rangle_H \\ &= \sum_{g \in S} \langle \operatorname{Res}_{H_g}^H \psi, \psi_g \rangle_{H_g} \\ &= \langle \psi, \psi \rangle_H + \sum_{\substack{g \in S \\ g \notin H}} d_g \end{aligned}$$

where $d_g = \langle \operatorname{Res}_{H_g}^G \psi, \psi_g \rangle_{H_g}$. For $g \in H$ we have $H_g = H$. Hence this is a sum of nonnegative integers which is ≥ 1, so $\operatorname{Ind}_H^G \psi$ is irreducible if and only if $\langle \psi, \psi \rangle = 1$ and all the other terms are 0. In other words W is irreducible and for all $g \notin H$, W and W_g are disjoint representations (of $H \cap gHg^{-1}$). □

Corollary 12.5. *If $H \trianglelefteq G$ and ψ is an irreducible character of H then $\operatorname{Ind}_H^G \psi$ is irreducible if and only if ψ is distinct from all its conjugates ψ_g for $g \in G \setminus H$.*

Proof. Take $K = H$. Double cosets are left or right cosets and $H_g = gHg^{-1} \cap H = H$ for all g. Moreover W_g is irreducible since W is irreducible. Thus Ind_H^G is irreducible precisely if $W \ncong W_g$ for all $g \in G \setminus H$. This is equivalent to $\psi \neq \psi_g$. (Again could check condition on set of representatives: actually the isomorphism class of W_g, where $g \in G$, depends only on gH) □

Proof of Mackey's restriction formula. Write $V = \mathrm{Ind}_H^G W$. Fix $g \in G$. Now V is direct sum of $x \otimes W$ with x running through set of representatives of left cosets of H in G. Consider a particular double coset $KgH \in K\backslash G/H$. The terms

$$\mathcal{V}(g) = \bigoplus_{\substack{x \text{ rep} \\ x \in KgH}} x \otimes W$$

form a subspace invariant under the action of K (it is the direct sum of an orbit of subspaces permuted by K as $kx \in KgH$ for all $x \in KgH$).

Now viewing V as a K-space, $\mathrm{Res}_K^G V = \bigoplus_{g \in S} \mathcal{V}(g)$. Thus need to show $\mathcal{V}(g) \cong \mathrm{Ind}_{H_g}^K W_g$ as K-spaces for each $g \in S$.

Now

$$\begin{aligned}
\mathrm{Stab}_K(g \otimes W) &= \{k \in K : kg \otimes W = g \otimes W\} \\
&= \{k \in : g^{-1}kg \in \mathrm{Stab}_G(1 \otimes W) = H\} \\
&= K \cap gHg^{-1} \\
&= H_g
\end{aligned}$$

This implies that if $x = kgh, x' = k'gh'$ then $x \otimes W = x' \otimes W$ if and only if k, k' lie in the same coset in K/H_g. Hence $\mathcal{V}(g)$ is the direct sum $\bigoplus_{\text{rep } k \in K/H_g} k \otimes (g \otimes W)$.

Therefore as a representation of K, this space is

$$\mathcal{V}(g) \cong \mathrm{Ind}_{H_g}^K (g \otimes W).$$

But $W_g \cong g \otimes W$ as representations of H_g using linear isomorphism $w \mapsto g \otimes w$. Putting all these expressions together gives the result. □

13 Integrality and group algebra

Definition (algebraic integer)**.** $a \in \mathbb{C}$ is an *algebraic integer* if a is a root of a monic polynomial in $\mathbb{Z}[x]$. Equivalently, the subring of $\mathbb{C}$

$$\mathbb{Z}[a] = \{f(a) : f(x) \in \mathbb{Z}[x]\}$$

is a finitely-generated $\mathbb{Z}$-algebra.

Fact.

1. The algebraic integers form a subring of $\mathbb{C}$.
2. If $a \in \mathbb{C}$ is both an algebraic integer and a rational number then $a \in \mathbb{Z}$.
3. Any subring S of $\mathbb{C}$ which is a finitely-generately $\mathbb{Z}$-module consists of algebraic integers. (suppose $s_1, \dots, s_n$ are generators of S as $\mathbb{Z}$-module and $a \in S$. Then for all i exists $a_{ij} \in \mathbb{Z}$ such that $as_i = \sum_j a_{ij} s_j$. Put $A = (a_{ij})$ then $Av = av$ where $v = (s_1, \dots, s_n)$, so a is the root of the characteristic polynomial of A, and is thus an algebraic integer)

Proposition 13.1. *If χ is a character of G and $g \in G$ then $\chi(g)$ is an algebraic integer.*

Proof. $\chi(g)$ is the sum of nth roots of unity, where n is the order of g. Each root of unity is an algebraic integer. □

Corollary 13.2. *There are no entries in the character table of any finite group which are rational but not integers.*

13.1 The centre of CG

Recall that the *group algebra* $\mathbb{C}G$ of a finite group G, the $\mathbb{C}$-space with basis G and dimension $|G|$. It is also a ring and a $\mathbb{C}$-algebra.

Let $\{1\} = \mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_k$ be the G-conjugacy classes. Define the *class sums*

$$C_j = \sum_{g \in \mathcal{C}_j} g \in \mathbb{C}G.$$

Now each $C_j \in Z(\mathbb{C}G)$, the centre of $\mathbb{C}G$. Moreover

Proposition 13.3. *$C_1, \dots, C_k$ is a basis of $Z(\mathbb{C}G)$. There exist non-negative integers $a_{ij\ell}$, $1 \le i, j, \ell \le k$ with*

$$C_i C_j = \sum_\ell a_{ij\ell} C_\ell.$$

These are the class algebra constants *or* structure constants *for $Z(\mathbb{C}G)$.*

Proof. Check $gC_jg^{-1} = C_j$ for all $g \in G$ so $C_j \in Z(\mathbb{C}G)$. Clearly C_j's are linearly independent because $\mathcal{C}_j$'s are disjoint. For spanning, suppose $z = \sum_{g\in G} a_g g \in Z(\mathbb{C}G)$. Then for all $h \in G$, $a_{h^{-1}gh} = \alpha_g$ so the function $g \mapsto a_g$ is constant on G-conjugacy classes. Writing $a_g = \alpha_j$ if $g \in \mathcal{C}_j$. Then

$$z = \sum_{j=1}^{k} \alpha_j C_j.$$

Finally $Z(\mathbb{C}G)$ is a $\mathbb{C}$-algebra so $C_iC_j = \sum_{\ell=1}^{k} a_{ij\ell} C_\ell$ as the C_ℓ's span. We claim that $a_{ij\ell} \in \mathbb{Z}_{\geq 0}$: fix $g_\ell \in \mathcal{C}_\ell$ then

$$a_{ij\ell} = |\{(x, y) \in \mathcal{C}_i \times \mathcal{C}_j : xy = g_\ell\}| \in \mathbb{Z}_{\geq 0}.$$

□

Definition (representation of algebra)**.** Let $\rho : G \to \mathrm{GL}(V)$ be an irreducible representation over $\mathbb{C}$ affording character χ. Extend linearly to a map $\rho : A = \mathbb{C}G \to \mathrm{End}(V)$, an algebra homomorphism. Such a homomorphism of algebra A into $\mathrm{End}(V)$ is called a *representation* of A.

A *central homomorphism* is a ring homomorphism $Z(A) \to \mathbb{C}$.

Let $z \in Z(\mathbb{C}G)$. Then $\rho(z)$ commutes with $\rho(g)$ for all $g \in G$, so by Schur's lemma $\rho(z) = \lambda_z I$ for some $\lambda_z \in \mathbb{C}$. Consider the central homomorphism

$$\begin{aligned} \omega_\chi = \omega : Z(\mathbb{C}G) &\to \mathbb{C} \\ z &\mapsto \lambda_z \end{aligned}$$

Now $\rho(C_i) = \omega_\chi(C_i)I$ so taking traces,

$$\chi(1)\omega_\chi(C_i) = \sum_{g\in\mathcal{C}_i} \chi(g) = |\mathcal{C}_i|\chi(g_i).$$

Thus

$$\omega_\chi(C_i) = \frac{\chi(g_i)}{\chi(1)}|\mathcal{C}_i|.$$

Lemma 13.4. *The values of $\omega_\chi(C_i)$ are algebraic integers.*

Proof. Since ω_χ is a homomorphism

$$\omega_\chi(C_i)\omega_\chi(C_j) = \sum_{\ell=1}^{k} a_{ij\ell}\omega_\chi(C_\ell)$$

where $a_{ij\ell} \in \mathbb{Z}_{\geq 0}$. Thus the span of $\{\omega_\chi(C_j) : 1 \leq j \leq k\}$ is a subring of $\mathbb{C}$, and is a finitely-generated abelian group, so consists of algebraic integers. □

Exercise. Show that $a_{ij\ell}$ can be obtained from the character table. In fact,

$$a_{ij\ell} = \frac{|G|}{|C_G(g_i)||C_G(g_j)|} \sum_{s=1}^{k} \frac{\chi_s(g_i)\chi_s(g_j)\chi_s(g_\ell^{-1})}{\chi_s(1)}.$$

See JL 30.4.

Theorem 13.5. *The degree of any irreducible complex character of G divides $|G|$.*

Proof. Given an irreducible character χ,

$$\begin{aligned}\frac{|G|}{\chi(1)} &= \frac{1}{\chi(1)}\sum_{g\in G}\chi(g)\chi(g^{-1}) \\ &= \frac{1}{\chi(1)}\sum_{i=1}^{k}|\mathcal{C}_i|\chi(g_i)\chi(g_i^{-1}) \\ &= \sum_{i=1}^{k}\underbrace{\frac{|\mathcal{C}_i|\chi(g_i)}{\chi(1)}}_{\text{alg integer}}\chi(g_i^{-1})\end{aligned}$$

which is algebraic integer. LHS is rational. □

Example.

1. If G is a p-group then $\chi(1)$ is a p-power. In particular if $|G| = p^2$ then $\chi(1) = 1$ for all χ, hence G must be abelian.
2. If $G = S_n$ then every prime p dividing the degree of an irreducible character also divides $n!$, so in particular $p \leq n$.
3. No simple group has an irreducible character of degree 2. See James and Liebeck 22.13.

Theorem 13.6. *If χ is irreducible then $\chi(1)$ divides $|G : Z(G)|$.*

Proof. Exercise. □

14 Burnside's theorem

Theorem 14.1 (Burnside)**.** *Let p, q be primes. Let $|G| = p^a q^b$ where $a, b \in \mathbb{Z}_{\geq 0}$, with $a + b \geq 2$. Then $|G|$ is not nonabelian simple.*

Remark.

1. If fact more is true: G is soluble.
2. This is the best possible in the sense that $|A_5| = 2^2 \cdot 3 \cdot 4$ has exactly 3 prime factors.
3. If either a or $b = 0$ then G is p-group, so nilpotent so soluble.
4. Feit and Thompson proved in 1963 that any group of odd order is soluble.
5. H. Bender and D. Goldschmidt independently found the first proof without the use of representation.

The theorem follows from two lemmas, one of which is starred.

Lemma 14.2. *Suppose $0 \neq \alpha = \frac{1}{m} \sum_{j=1}^{m} \lambda_j$ with $\lambda_j^n = 1$ is an algebraic integer. Then $|\alpha| = 1$.*

Proof.* Clearly $0 < |\alpha| \leq 1$. Observe that $\alpha \in F = \mathbb{Q}(\varepsilon)$ where $\varepsilon = e^{\frac{2\pi i}{n}}$. Let $G = \mathrm{Gal}(F/\mathbb{Q})$. We know

$$\{\beta \in F : \sigma(\beta) = \beta \text{ for all } \sigma \in G\} = \mathbb{Q}.$$

Define norm

$$N(\alpha) = \prod_{\sigma \in G} \sigma(\alpha).$$

Then $N(\alpha)$ is fixed by every element of G so $N(\alpha) \in \mathbb{Q}$. Now $N(\alpha)$ is an algebraic integer since Galois conjugates of algebraic integers are algebraic integers. Thus $N(\alpha) \in \mathbb{Z}$. But for $\sigma \in G$,

$$|\sigma(\alpha)| = \left| \frac{1}{m} \sum \sigma(\lambda_j) \right| \leq 1.$$

Thus $N(\alpha) = \pm 1$, which implies that $|\alpha| = 1$. □

Lemma 14.3. *Suppose χ is an irreducible character of G and $\mathcal{C}$ is a conjugacy class in G such that $\chi(1)$ and $|\mathcal{C}|$ are coprime. Then for all $g \in \mathcal{C}$, $|\chi(g)| = \chi(1)$ or 0.*

Proof. By Bézout's theorem exist $a, b \in \mathbb{Z}$ with $a\chi(1) + b|\mathcal{C}| = 1$. Define

$$\alpha = \frac{\chi(g)}{\chi(1)} = a\chi(g) + b\frac{\chi(g)}{\chi(1)}|\mathcal{C}|$$

which is an algebraic integer. Thus α satisfies the conditions of the previous lemma. □

Proposition 14.4. *If in a finite group G, the number of elements in a conjugacy class $\mathcal{C}_i \neq 1$ is of prime power order then G is not nonabelian simple.*

Granted this, we can prove Burnside: if $a, b > 0$ let Q be a Sylow q-subgroup, so $Q \neq 1$ (otherwise G is p-group). Now $1 \neq Z(Q)$ so exists $1 \neq g \in Z(Q)$. Then as $C_G(g) \geq Q$, we have

$$|\mathcal{C}_G(g)| = |G : C_G(g)| = p^r$$

for some $0 \leq r \leq a$.

Proof. Suppose G is nonabelian simple, and there exists $1 \neq g \in G$ lying in the conjugacy class $\mathcal{C}$ of order p^r. If $\chi \neq 1_G$ is a non-trivial irreducible character of G then $|\chi(g)| < \chi(1)$ (otherwise G not simple). Then for every non-trivial irreducible character, either $p \mid \chi(1)$ or $|\chi(g)| = 0$. By column orthogonality applied to $\{1\}$ and $\mathcal{C}$,

$$0 = 1 + \sum_{\substack{\chi \neq 1_G \\ p \mid \chi(1)}} \chi(1)\chi(g)$$

so

$$-\frac{1}{p} = \sum_{\chi \neq 1} \frac{\chi(1)}{p} \chi(g)$$

is an algebraic integer in $\mathbb{Q}$. Absurd. $\square$

15 Representations of compact groups

See Teleman §19 - 22 and C. Thomas §6 for more detailed treatment of this chapter.

Definition (topological group)**.** A *topological group* G is a group which is also a topological space and for which multiplication $G \times G \to G$ and inversion $G \to G$ are continuous. It is *compact* if it is so as a topological space.

Example.

1. Any finite group G with discrete topology.
2. $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{GL}_n(\mathbb{R})$ are topological groups (as open subsets of $\mathbb{C}^{n^2}$ or $\mathbb{R}^{n^2}$).
3. Examples of compact groups:
   (a) finite groups,
   (b) $S^1 = \{z \in \mathbb{C} : |z| = 1\}$ under multiplication, the *circle group*,
   (c) torus: finite product $S^1 \times \cdots \times S^1$,
   (d) $\mathrm{O}(n) = \{A \in \mathrm{GL}_n(\mathbb{R}) : AA^t = I_n\}$, orthogonal group,
   (e) $\mathrm{SO}(n) = \{A \in O(n) : \det A = 1\}$, special orthogonal group,
   (f) $\mathrm{U}(n) = \{A \in \mathrm{GL}_n(\mathbb{C}) : A\overline{A}^t = I_n\}$, unitary group,
   (g) $\mathrm{SU}(n) = \{A \in U(n) : \det A = 1\}$, speical unitary group.

Remark.

1. $U(1) \cong \mathrm{SO}(2) \cong_h S^1$ where $\cong_h$ means the homomorphism is also a homeomorphism.
2. $\mathrm{SU}(2) = \{(z_1, z_2) \in \mathbb{C}^2 : z_1\overline{z}_1 + z_2\overline{z}_2 = 1\} \subseteq \mathbb{R}^4 \cong \mathbb{C}^2$ is isomorphic and homeomorphic to S^3.

Definition (representation of topological group)**.** A representation of a topological group G on a finite-dimensional space V is a continuous group homomorphism $\rho : G \to \mathrm{GL}(V)$.

Remark. If X is a topological space then $\rho : X \to \mathrm{GL}(V) \cong \mathrm{GL}_n(\mathbb{C})$ is continuous if and only if $x \mapsto \rho(x)_{ij}$ are continuous for all i, j.

15.1 The compact group U(1)

We prove

Theorem 15.1. *Every 1-dimensional continuous representation of S^1 is of the form $z \mapsto z^n$ for some $n \in \mathbb{Z}$.*

Remark. It can be easily seen that these are representations. Why are they the only ones? If one drops continuity condition, the number of 1-dimensional representations is uncountably infinite. See Teleman §19.8.

To prove the theorem we need two lemmas from real analysis

Lemma 15.2. *If $\psi : (\mathbb{R}, +) \to (\mathbb{R}, +)$ is a continuous group homomorphism then there exists $c \in \mathbb{R}$ such that $\psi(x) = cx$ for all $x \in \mathbb{R}$.*

Proof. Given $\psi : (\mathbb{R}, +) \to (\mathbb{R}, +)$ continuous, let $c = \psi(1)$. As ψ is a homomorphism,

$$\psi(nx) = \psi(x + \cdots + n) = n\psi(x)$$

for $x \in \mathbb{R}, n \in \mathbb{Z}_{\geq 0}$. In particular $\psi(n) = cn$. Also $\psi(-n) = -\psi(n) = -cn$ so $\psi(n) = cn$ for all $n \in \mathbb{Z}$. Put $x = \frac{m}{n} \in \mathbb{Q}$,

$$n\psi(x) = \psi(nx) = \psi(m) = cm$$

so $\psi(x) = cx$ for all $x \in \mathbb{Q}$. As $\mathbb{Q} \subseteq \mathbb{R}$ is dense and ψ is continuous, $\psi(x) = cx$ for all $x \in \mathbb{R}$. □

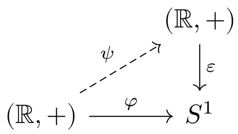
Lemma 15.3. *Continuous homomorphisms $\varphi : (\mathbb{R}, +) \to S^1$ are of the form $\varphi(x) = e^{icx}$ for some $c \in \mathbb{R}$.*

Proof. Define

$$\begin{aligned} \varepsilon : (\mathbb{R}, +) &\to S^1 \\ x &\mapsto e^{ix} \end{aligned}$$

This homomorphism wraps real line around S^1 with period 2π.

Claim given any continuous map $\varphi : (\mathbb{R}, +) \to S^1$ such that $\varphi(0) = 1$, there exists a unique continuous map $\psi : \mathbb{R} \to \mathbb{R}$, called a *lifting*, such that $\psi(0) = 0$, making the diagram



commute. (The lifting is constructed by starting with $\psi(0) = 0$ and then extending a small interval at a time to get a continuous map $(\mathbb{R}, +) \to (\mathbb{R}, +)$)

If φ is a homomorphism then so is its lifting ψ: $\varphi(x + y) = \varphi(x)\varphi(y)$ so $\varepsilon(\psi(x + y) - \psi(x) - \psi(y)) = 1$. Thus $\psi(x + y) - \psi(x) - \psi(y) = 2k\pi$ for some integer k depending continuously on x, y, so must be constant. Setting $x = y = 0$ we get $k = 0$. □

Proof of Theorem 15.1. Let $\rho : S^1 \to \mathbb{C}^\times$ be a continuous 1-dimensional representation. Then $\rho : S^1 \to S^1$: since S^1 is compact and ρ is continous, $\rho(S^1)$ is closed and bounded. As $\rho(z^n) = \rho(z)^n$ for all $n \in \mathbb{Z}$, we must have $\rho(S^1) \subseteq S^1$. We get a continuous homomorphism

$$\begin{aligned} \mathbb{R} &\to S^1 \\ x &\mapsto \rho(e^{ix}) \end{aligned}$$

so exists $c \in \mathbb{R}$ such that $\rho(e^{ix}) = e^{icx}$. But $1 = \rho(e^{2\pi i}) = e^{2\pi ic}$ so $c \in \mathbb{Z}$. Put $n = c$, $\rho(z) = z^n$ as claimed. □

In studying representations of finite groups we "averaged" over the group via the operation $\frac{1}{|G|}\sum$. An analogous operation exists for topological groups, if we replace "sum" by $\int_G \mathrm{d}g$.

Definition (Haar measure)**.** If G is a topological group, let

$$\mathcal{C}(G) = \{f : G \to \mathbb{C} : f \text{ continuous}, f(gxg^{-1}) = f(x) \text{ for all } g, x \in G\}$$

Then a non-trivial functional

$$\int_G : \mathcal{C}(G) \to \mathbb{C}$$

is called a *Haar measure* if

1. normalisation: $\int_G 1\mathrm{d}g = 1$,
2. translation invariance: $\int_G f(xg)\mathrm{d}g = \int_G f(g)\mathrm{d}g = \int_G f(gx)\mathrm{d}g$ for all $x \in G$.

Example.

1. If G is finite then
$$\int_G f = \frac{1}{|G|}\sum_{g\in G} f(g)$$
is a Haar measure.
2. $G = S^1$:
$$\int_G f = \frac{1}{2\pi}\int_0^{2\pi} f(e^{i\theta})d\theta.$$
3. $G = \mathrm{SU}(2)$: see later.

Theorem 15.4. *If G is compact and Hausdorff then there exists a unique Haar measure on G.*

Proof. Omitted. □

We compute Haar measure for SU(2) below. Henceforth "compact" means "compact Hausdorff".

As a general theme, results we proved using "averaging" techniques work for compact groups by replacing averaging by the Haar measure on the topological group.

Corollary 15.5 (Weyl's unitary trick)**.** *Let G be compact. Then every representation (ρ, V) has G-invariant inner product.*

Proof. Take any inner product $(\cdot, \cdot)$ on V. Then

$$\langle v, w\rangle = \int_G (\rho(g)v, \rho(g)w)\mathrm{d}g$$

is a G-invariant inner product. □

Corollary 15.6 (Maschke). *If G is compact then every representation of G is completely reducible.*

We can use the Haar measure to endow $\mathcal{C}(G)$, the space of continuous functions, with an inner product

$$\langle f, f' \rangle = \int_G \overline{f(g)} f'(g) \mathrm{d}g.$$

If $\rho : G \to \mathrm{GL}(V)$ is a representation then $\chi_V = \chi_\rho = \operatorname{tr} \rho$ is a continuous class function since each $\rho(g)_{ii}$ is continuous.

Theorem 15.7 (row orthogonality). *Suppose G is compact and V, W are irreducible representations of G. Then*

$$\langle \chi_V, \chi_W \rangle = \begin{cases} 1 & V \cong W \\ 0 & V \ncong W \end{cases}$$

Naturally one may wonder if irreducible characters form a basis of $\mathcal{C}(G)$. The answer is not quite. We need some Hilbert space theory and Peter-Weyl theorem. For S^1 see Teleman §19.14, 19.15.

15.2 Representations of SU(2)

Let

$$\begin{aligned} G = \mathrm{SU}(2) &= \{A \in \mathrm{GL}_2(\mathbb{C}) : \overline{A}^t A = I, \det A = 1\} \\ &= \left\{ \begin{pmatrix} a & b \\ -\overline{b} & \overline{a} \end{pmatrix}, a, b \in \mathbb{C}, |a|^2 + |b|^2 = 1 \right\} \end{aligned}$$

Topologically $G \cong_h S^3$. More precisely, let

$$\mathbb{H} = \left\{ \begin{pmatrix} z & w \\ -\overline{w} & \overline{z} \end{pmatrix} \right\}$$

Hamilton's quaternion algebra. $\mathbb{H}$ is a 4 dimensional Euclidean space and $\|A\|^2 = \det A$ defines a norm on $\mathbb{H} \cong \mathbb{R}^4$ with G the unit ball. If $A \in G$ and $X \in \mathbb{H}$ then

$$\|AX\| = \|X\| = \|XA\|.$$

Thus after normalisation (by $\frac{1}{2\pi^2}$), usual integration of functions on S^3 defines Haar measure on G.

We first discuss conjugacy classes in G. Let

$$T = \left\{ \begin{pmatrix} a & 0 \\ 0 & \overline{a} \end{pmatrix} : a \in \mathbb{C}, |a| = 1 \right\} \cong S^1$$

the *maximal torus* in G. Let $s = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \in G$.

Lemma 15.8.

1. *If* $t \in T$ *then* $sts^{-1} = t^{-1}$.
2. $s^2 = -I \in Z(G)$.
3. $N_G(T) = T \cup sT = \left\{ \begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}, \begin{pmatrix} 0 & a \\ -a^{-1} & 0 \end{pmatrix} : a \in \mathbb{C}, |a| = 1 \right\}$
4. *Every conjugacy class* $\mathcal{C}$ *in* G *contains an element of* T*, i.e.* $\mathcal{C} \cap T \neq \emptyset$. *In fact,*
5. $\mathcal{C} \cap T = \{t, t^{-1}\}$ *for some* $t \in T$. *Moreover* $t = t^{-1}$ *if and only if* $t = \pm I$ *when* $\mathcal{C} = \{t\}$.
6. *There exists a bijection* $\{$*conjugacy classes of* $G\} \leftrightarrow [-1, 1]$ *given by* $A \mapsto \frac{1}{2} \operatorname{tr} A$.

Proof.

1. Direct computation.
2. Ditto.
3. Ditto.
4. Every unitary matrix X has an orthonormal basis of eigenvectors, hence is conjugate in $\mathrm{U}(2)$ to one in T, say $QX\overline{Q}^t \in T$. We want Q such that $\det Q = 1$. Put $\delta = \det Q$ so $|\delta| = 1$. If ε is a square root of δ then $Q_1 = \overline{\varepsilon} Q \in \mathrm{SU}(2)$ and $Q_1 X \overline{Q}_1^t \in T$.
5. Let $g \in G$ and suppose $g \in \mathcal{C}$. If $g = \pm I$ then $\mathcal{C} \cap T = \{g\}$. Otherwise g has distinct eigenvalues λ, λ^{-1} and $\mathcal{C} = \{h \left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) h^{-1} : h \in G\}$. Hence $\mathcal{C} \cap T = \{\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right), \left(\begin{smallmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{smallmatrix}\right)\}$, by noting that

   $$s \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} s^{-1} = \begin{pmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{pmatrix}.$$

   Furthermore, if $\left(\begin{smallmatrix} \mu & 0 \\ 0 & \mu^{-1} \end{smallmatrix}\right) \in \mathcal{C}$ then $\{\mu, \mu^{-1}\} = \{\lambda, \lambda^{-1}\}$ (i.e. eigenvalues preserved under conjugation).
6. By 5 matrices are conjugate in G if and only if their eigenvalues agree up to reordering. Now

   $$\frac{1}{2} \operatorname{tr} \begin{pmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{pmatrix} = \frac{1}{2}(\lambda + \overline{\lambda}) = \operatorname{Re} \lambda = \cos\theta$$

   where $\lambda = e^{i\theta}$. Hence the map is surjective. It's also injective: if $\frac{1}{2} \operatorname{tr} g = \frac{1}{2} \operatorname{tr} g'$ then g, g' have the same characteristic polynomial, namely

   $$X^2 - (\operatorname{tr} g) X + 1,$$

   hence the same eigenvalues and are conjugate.

□

Thus we write

$$\mathcal{C}_t = \{g \in G : \frac{1}{2}\operatorname{tr} g = t\}$$

for $t \in [-1,1]$. In particular $\mathcal{C}_1 = \{I\}, \mathcal{C}_{-1} = \{-I\}$. In fact

Proposition 15.9. *For $t \in (-1,1)$, $\mathcal{C}_t \cong_h S^2$.*

Proof. Exercise. □

Now we can study the representations of G. Let V_n be the space of all homogeneous polynomials of degree n in variables x, y, i.e.

$$V_n = \{r_0x^n + r_1x^{n-1}y + \cdots + r_ny^n : r_i \in \mathbb{C}\},$$

an $(n+1)$ dimensional $\mathbb{C}$-space, with standard basis $x^n, x^{n-1}y, \dots, y^n$. Then $\mathrm{GL}_2(\mathbb{C}) = \mathrm{GL}(\mathbb{C}^2)$ acts on V_n via

$$\rho_n : \mathrm{GL}(\mathbb{C}^2) \to \mathrm{GL}(V_n)$$
$$\rho_n\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)f(x,y) = f(ax+cy, bx+dy)$$

Exercise.

1. If $n = 0$ then ρ_0 is trivial.
2. If $n = 1$ then ρ_1 is the natural 2 dimensional representation where $\rho_1\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ has matrix $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ with respect to standard basis of V_1.
3. If $n = 2$ then
$$\rho_2\begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{pmatrix}$$
with repsect to standard basis of V_2.

Now $G \le \mathrm{GL}_2(\mathbb{C})$ so view V_n as a representation of G by restriction.

Lemma 15.10. *A continuous class function $f : G \to \mathbb{C}$ is determined by its restriction to T, and $f|_T$ is even (in the sense that $f\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) = f\left(\begin{smallmatrix} \lambda^{-1} & 0 \\ 0 & \lambda \end{smallmatrix}\right)$).*

Proof. Each conjugacy class in G meets T so a class function is determined by its restriction to T. Evenness follows from $T \cap \mathcal{C} = \{t, t^{-1}\}$. □

Lemma 15.11. *If χ is a character of a representation of G then $\chi|_T$ is a Laurent polynomial, i,e. finite $\mathbb{N}_0$-linear combination of functions $\left(\begin{smallmatrix} \lambda & 0 \\ 0 & \lambda^{-1} \end{smallmatrix}\right) \mapsto \lambda^n$ where $n \in \mathbb{Z}$.*

Proof. If V is a representation of G then $\mathrm{Res}_T^G V$ is a representation of T and its character $\chi_{\mathrm{Res}_T^G V}$ is the restriction of χ_V to T. But every representation of T has character of given form by Theorem 15.1. □

Put

$$\mathbb{N}_0[z, z^{-1}] = \left\{ \sum_{n\in\mathbb{Z}} a_n z^n : a_n \in \mathbb{N}_0, \text{ finitely many } a_n \neq 0 \right\}$$
$$\mathbb{N}_0[z, z^{-1}]_{\text{ev}} = \{f \in \mathbb{N}_0[z, z^{-1}] : f(z) = f(z^{-1})\}$$

By these lemmas for continuous representations of G, the character χ_V is in $\mathbb{N}_0[z, z^{-1}]_{\text{ev}}$ by identifying it with its restriction to T. We calculate the character χ_n of (ρ_n, V_n). Recall

$$\rho_n \begin{pmatrix} a & b \\ c & d \end{pmatrix} : x^{n-j}y^j \mapsto (ax + cy)^{n-j}(bx + dy)^j$$

and extend linearly. To find $\chi_n(g) = \operatorname{tr} \rho_n(g)$, note that $g \sim \left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right) \in T$ and

$$\rho_n(\begin{pmatrix} z & 0 \\ 0 & z \end{pmatrix}) (x^i y^j) = (zx)^i (z^{-1}y)^j = z^{i-j} x^i y^j$$

so $\rho_n\left(\begin{smallmatrix} z & 0 \\ 0 & z^{-1} \end{smallmatrix}\right)$ has matrix

$$\begin{pmatrix} z^n & & & & \\ & z^{n-2} & & & \\ & & \ddots & & \\ & & & z^{2-n} & \\ & & & & z^{-n} \end{pmatrix}$$

with respect to standard basis. Hence

$$\chi_n \begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix} = z^n + z^{n-2} + \cdots + z^{-n}.$$

Exercise. $\chi_0 = 1_G$, $\chi_1 = e^{i\theta} + e^{-i\theta} = 2\cos\theta$, $\chi_3 = 1 + 2\cos 2\theta$. In general it equals to $\frac{z^{n+1} - z^{-(n+1)}}{z - z^{-1}}$ unless $z = \pm 1$.

Theorem 15.12. *The representations $\rho_n : G \to \mathrm{GL}(V_n)$ of dimension $n+1$ are irreducible for all $n \in \mathbb{Z}_{\geq 0}$.*

Proof. Assume $0 \neq W \leq V_n$ is a G-invariant subspace. Claim $V_n = W$. Claim if $0 \neq w = \sum r_j x^{n-j} y^j \in W$ with some $r_i \neq 0$ then $x^{n-i}y^i \in W$. Argue by induction on the number of non-zero r_j. If unique $r_i \neq 0$ then result is clear (as w is a non-zero multiple of $x^{n-i}y^i$). So assume more than one and choose i such that $r_i \neq 0$. Pick $z \in S^1$ with $z^n, z^{n-2}, \dots, z^{2-n}, z^{-n}$ distinct in $\mathbb{C}$. Then

$$\rho_n(\begin{pmatrix} z & 0 \\ 0 & z^{-1} \end{pmatrix})w - z^{n-2i}w = \sum_j r_j(z^{n-2j} - z^{n-2i})(x^{n-j}y^j) \in W$$

as W is G-invariant. Now $r_j(z^{n-2j} - z^{n-2i}) \neq 0$ precisely when $r_j \neq 0$ and $j \neq i$. By induction $x^{n-j}y^j \in W$ for all $j \neq i$ with $r_j \neq 0$. Hence also

$$x^{n-i}y^i = \frac{1}{r_i}(w - \sum_j r_j x^{n-j} y^j) \in W$$

as required.

We now know $x^{n-i}y^i \in W$ for some i. We find matrices in G, the action of which will give all $x^{n-i}y^i \in W$. Since

$$\frac{1}{\sqrt{2}}\begin{pmatrix}1 & -1\\ 1 & 1\end{pmatrix} : x^{n-i}y^i \mapsto \frac{1}{\sqrt{2}^n}(x+y)^{n-i}(-x+y)^i \in W$$

and we can use the claim to deduce $x^n \in W$. Similarly if $a, b \neq 0$

$$\begin{pmatrix}a & -\bar{b}\\ b & \bar{a}\end{pmatrix} : x^n \mapsto (ax+by)^n \in W$$

and so by the claim $x^{n-i}y^i \in W$ for all i. Thus $W = V_n$. □

Remark. Alternatively, see Teleman §21.1 we can evaluate $\langle \chi_n, \chi_n \rangle = 1$ using Weyl's integration formula.

Now show all irreducible representations of G are of this form.

Theorem 15.13. *Every finite-dimensional continuous irreducible representation of G is one of the $\rho_n : G \to \mathrm{GL}(V_n)$ above.*

Proof. Assume $\rho_V : G \to \mathrm{GL}(V)$ is irreducible affording character $\chi_V \in \mathbb{N}_0[z, z^{-1}]_{\mathrm{ev}}$. We show $\chi = \chi_n$ for some n. Now $\chi_0, \chi_1, \ldots$ form a basis of $\mathbb{Q}[z, z^{-1}]_{\mathrm{ev}}$. Hence $\chi_V = \sum_n a_n \chi_n$, a finite $\mathbb{Q}$-linear combination. Clearing the denominators and moving all summands with negative coefficients to LHS gives the relation

$$m\chi_V + \sum_{i\in I} m_i \chi_i = \sum_{j\in J} n_j \chi_j$$

with I, J disjoint finite subsets of $\mathbb{N}_0$ and $m, m_i, n_i \in \mathbb{N}_0$. The left and right hand sides are characters of G. Hence

$$mV \oplus \bigoplus_{i\in I} m_i V_i \cong \bigoplus_{j\in J} n_j V_j.$$

Since V is irreducible we must have $V \cong V_n$ for some $n \in J$. □

15.2.1 Tensor products of representations of G

We know from Lemma 15.10 for V, W representation of G, $\mathrm{Res}^G_T V \cong \mathrm{Res}^G_T W$ implies $V \cong W$. We want to understand tensor products of representations for G.

Proposition 15.14. *If $G = \mathrm{SU}(2)$ or S^1, V, W representations of G then*

$$\chi_{V\otimes W} = \chi_V \chi_W.$$

Proof. Suffice to consider $G \cong S^1$. V, W have eigenbases $e_1, \ldots, e_n, f_1, \ldots, f_m$ such that

$$\begin{aligned} ze_i &= z^{n_i} e_i \\ zf_j &= z^{m_j} f_j \end{aligned}$$

respectively. Then

$$z(e_i \otimes f_j) = z^{n_i+m_j}(e_i \otimes f_j)$$

so

$$\chi_{V\otimes W}(z) = \sum_{i,j} z^{n_i+m_j} = \chi_V(z)\chi_W(z).$$

□

Example.

$$\chi_{V_1\otimes V_1}(z) = (z+z^{-1})^2 = z^2+2+z^{-2} = (z^2+1+z^{-1})+1 = \chi_{V_2}+\chi_{V_0}$$
$$\chi_{V_1\otimes V_2}(z) = (z^2+1+z^{-2})(z+z^{-1}) = z^3+2z+2z^{-1}+z^{-3} = \chi_{V_3}+\chi_{V_1}$$

The next result analyses the product structure of representations

Theorem 15.15 (Clebsch-Gordon formula)**.** *For any $n, m \in \mathbb{N}_0$,*

$$V_n \otimes V_m \cong V_{n+m} \oplus V_{n+m-2} \oplus \cdots \oplus V_{|n-m|+2} \oplus V_{|n-m|}.$$

Proof. Use characters. Wlog $n \geq m$ so

$$\begin{aligned}(\chi_n\chi_m)(z) &= \frac{z^{n+1}-z^{-n-1}}{z-z^{-1}}(z^m+z^{m-2}+\cdots+z^{-m}) \\ &= \sum_{j=0}^m \frac{z^{n+m+1-2j}-z^{2j-n-m-1}}{z-z^{-1}} \\ &= \sum_{j=0}^m \chi_{n+m-2j}(z)\end{aligned}$$

□

15.2.2 Representation of some closely related groups

Proposition 15.16.

1. $\mathrm{SO}(3) \cong \mathrm{SU}(2)/\{\pm 1\} = \mathrm{PSU}(2)$.
2. $\mathrm{SO}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)/\{\pm(I,I)\}$.
3. $\mathrm{U}(2) \cong \mathrm{U}(1) \times \mathrm{SU}(2)/\{\pm(I,I)\}$.

In fact, these are not only group isomorphisms but also homeomorphisms.

The homeomorphism bit can be deduced from a continuous bijection from compact space to Hausdorff space being a homeomorphism.

Corollary 15.17. *Every irreducible representation of* SO(3) *is of the form*

$$\rho_{2m} : \mathrm{SO}(3) \to \mathrm{GL}(V_{2m})$$

for some $m \geq 0$.

Proof. Irreducible representations of SO(3) correspond to irreducible representations of SU(2) such that $-I$ acts trivially. But $-I$ acts on V_n as -1 when n is odd, and as 1 when n is even. $\square$

Sketch proof of Proposition 15.16 1. Recall SU(2) can be viewd as the space of unit norm quaternions in $\mathbb{H} \cong \mathbb{R}^4$. Let $\mathbb{H}_0 = \{A \in \mathbb{H} : \operatorname{tr} A = 0\}$, called *pure quaternions*, which are spanned as $\mathbb{R}$-space by

$$\mathbf{i} = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \mathbf{j} = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \mathbf{k} = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix},$$

equipped with norm $\|A\|^2 = \det A$. It is a 3 dimensional Euclidean space and SU(2) acts by isometries on $\mathbb{H}_0$:

$$X \cdot A = XAX^{-1}.$$

This gives a group homomorphism $\varphi : \mathrm{SU}(2) \to \mathrm{O}(3)$ with kernel $Z(\mathrm{SU}(2)) = \{\pm I\}$. Now SU(2) is compact, O(3) is Hausdorff, hence we have a continuous group isomorphism $\overline{\varphi} : \mathrm{SU}(2)/\{\pm I\} \to \operatorname{im} \phi$ which is also a homeomorphism.

Left to show $\operatorname{im} \varphi = \mathrm{SO}(3)$. $\operatorname{im} \varphi \leq \mathrm{SO}(3)$: we know SU(2) is path-connected, so only one of the two possible values ± 1 can be taken by the continuous function $\det \varphi$. But $\varphi(I_2) = I_3$ with determinant 1, so have value 1.

Need to show that all rotations in $(\mathbf{i}, \mathbf{j})$-plane are implemented by elements $a + b\mathbf{k}$, and similarly with any permutations of $\mathbf{i}, \mathbf{j}, \mathbf{k}$. (the rotations generate SO(3)). Now

$$\begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \begin{pmatrix} ai & b \\ -\bar{b} & -ai \end{pmatrix} \begin{pmatrix} e^{-i\theta} & 0 \\ 0 & e^{i\theta} \end{pmatrix} = \begin{pmatrix} ai & e^{2i\theta} b \\ -\bar{b} e^{-2i\theta} & -ai \end{pmatrix}$$

so $\left(\begin{smallmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{smallmatrix}\right)$ acts on $\mathbb{R}\langle \mathbf{i}, \mathbf{j}, \mathbf{k} \rangle = \mathbb{H}_0$ by rotation in $(\mathbf{j}, \mathbf{k})$-plane through an angle 2θ. Check

$$\begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix}, \begin{pmatrix} \sin\theta & i\sin\theta \\ -i\sin\theta & \cos\theta \end{pmatrix}$$

act by rotation of 2θ in $(\mathbf{i}, \mathbf{k})$- and $(\mathbf{i}, \mathbf{j})$-planes respectively. $\square$

Exercise. Mimick this for SO(4) and U(2).

To get the representations in 2 and 3, we need results about products $G \times H$ of two compact groups G and H. Complete list of irreducible representations comprises the tensor products $V \otimes W$, as V, W ranges over the irreudibles of G, H respectively. Compare with the finite case. So complete list of irreducibles of SO(4) is $\rho_m \otimes \rho_n$, $m, n \geq 0, m = n \bmod 2$. Complete list of U(2) is $\det^{\otimes m} \otimes \rho_n$, $m, n \in \mathbb{Z}, n \geq 0$.

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