UNIVERSITY OF
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MATHEMATICS TRIPOS

Part IV

Non-archimedean Geometry

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1 Introduction

What is non-archimedean geometry?

Definition (non-archimedean field)**.** A *non-archimedean field K* is a field that is complete with respect to a non-archimedean absolute value (multi ... that satisfies the strict triangle inequality.

Notation. We define the valuation ring and its unique maximal ideal to be

$$R = K^0 = \{x \in K : |x| \le 1\}$$
$$\mathfrak{m} = K^{00} = \{x \in K : |x| < 1\}$$

and we let $k = \tilde{K} = R/\mathfrak{m} = K^0/K^{00}$.

Example.

1. A field K with trivial absolute value $|x| = \begin{cases} 0 & x = 0 \\ 1 & x \ne 0 \end{cases}$. Then $K = R = K, \mathfrak{m} = 0$.

2. Let F be a field. Let $K = F((t))$ be the field of Laurent series with t-adic valuation, i.e. let $\varepsilon \in (0,1)$ and define $|x| = \varepsilon^{\mathrm{ord}_t(x)}$. Then $R = F[[t]], \mathfrak{m} = (t)$ and $k = F$.

3. Let p be a prime. Then $K = \mathbb{Q}_p$ with p-adic valuation. Usually we take $\varepsilon = \frac{1}{p}$. $R = \mathbb{Z}_p, \mathfrak{m} = (p)$ and $k = \mathbb{F}_p$.

   More generally let K'/K be an algebraic extension. Then $|\cdot|_K$ extends uniquely to $|\cdot|_{K'}$ which is non-archimedean and also a valuation on K'. If K' is finite then K' is complete with respect to $|\cdot|_{K'}$, but in general it is not.

   We take the algebraic closure $\overline{K}$ and then its completion $\hat{\overline{K}}$. It is a theorem of Krasner that $\hat{\overline{K}}$ is still algebraically closre. In the case $K = \mathbb{Q}_p, \mathbb{C}_p = \hat{\overline{\mathbb{Q}}}_p$ is the field of p-adic complex numbers.

Why do we care about non-archimedean fields?

We want to solve Diophantine problems, which are in general very hard. Hasse proposed in 1920s the *local-global principle*: trivially if we have a (nontrivial) rational solution then we have a solution over $\mathbb{R}$ and $\mathbb{Q}_p$ for all p. The local-global principle asks for the converse: if we have (nontrivial) solutions over $\mathbb{R}$ and $\mathbb{Q}_p$ for all p, can we "patch" together to get a solution over $\mathbb{Q}$?

Theorem 1.1 (Hasse-Minsowski)**.** *If $Q(x_1, \ldots, x_n)$ is a quadratic form over $\mathbb{Q}$, then $Q(x_1, \ldots, x_n)$ has a nontrivial $\mathbb{Q}$-solution if and only if and only if it has nontrivial solutions over $\mathbb{R}$ and $\mathbb{Q}_p$ for all p.*

Counterexample (Selmer): $3x^3 + 4y^3 + 5y^3 = 0$ have $\mathbb{R}$ and $\mathbb{Q}_p$-solutions for all p, but not $\mathbb{Q}$-solutions.

Now we use the algebraic geometric language of schemes. Let X be an algebraic variety over $\mathbb{Q}$ (i.e. a scheme of finite typer). We are interested in

$X(\mathbb{Q})$. The local-global principle then says that we can extend it to $X_{\mathbb{R}}$ (or $X_{\mathbb{C}}$)) and $X_{\mathbb{Q}_p}$.

Elliptic curves over $\mathbb{C}$: we know that if $\Lambda = \tau_1 \mathbb{Z} + \tau_2 \mathbb{Z}$ is a lattice, there is a $\mathbb{C}$-analytic isomorphism $\phi : \mathbb{C}/\Lambda \to E_\Lambda(\mathbb{C})$ for some elliptic curve E_Λ over $\mathbb{C}$ which is also a group homomorphism. The converse is

Theorem 1.2 (uniformisation over $\mathbb{C}$)**.** *Let $E/\mathbb{C}$ be an elliptic curve. Then there exists a lattice Λ unique up to homothety (Λ_1, Λ_2 are homothetic if exists $\lambda \in \mathbb{C}^*$ such that $\lambda \cdot \Lambda_1 = \Lambda_2$) and $\mathbb{C}$-analytic isomorphism $\phi : \mathbb{C}/\Lambda \to E(\mathbb{C})$ which is a group homomorphism.*

We can rescale Λ to get normalised lattice $\frac{1}{\tau_1}\Lambda = \mathbb{Z} + \tau\mathbb{Z}$ where $\tau = \frac{\tau_1}{\tau_2}$. We can further assume that $\tau \in \mathbb{H} = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$. Set $u = e^{2\pi i z}, q = e^{2\pi i \tau}$. There is a $\mathbb{C}$-analytic isomorphism (or isomorphism of complex Lie groups)

$$\mathbb{C}/\Lambda \to \mathbb{C}^*/q^{\mathbb{Z}}$$
$$z \mapsto e^{2\pi i z}$$

Hence we have $\mathbb{C}$-analytic an isomorphism $E(\mathbb{C}) \cong \mathbb{C}^*/q^{\mathbb{Z}}$ (multiplicative parameterisation). Conversely for each $q \in \mathbb{C}, |q| < 1$, there exists an elliptic curve E_q such that $E_q(\mathbb{C}) \cong \mathbb{C}^*/q^{\mathbb{Z}}$.

Tate curves: let $E/\mathbb{Q}_p$ be an elliptic curve. Then the "additive uniformisation" cannot work since $\mathbb{Q}_p$ has no non-trivial discrete subgroups (exercise). But if we consider $\mathbb{Q}_p^*$, there are a lot of discrete subgroups, e.g. $q \in K^*$ with $|q| < 1$ gives a discrete subgroup $q^{\mathbb{Z}}$.

Theorem 1.3 (Tate)**.** *Let K be a non-archimedean field and $q \in \mathbb{Q}_p^*, |q| < 1$. Then*

$$s_k(q) = \sum_{n=1}^{\infty} \frac{n^k q^n}{1 - q^n}, a_4(q) = -5s_3(q), a_6(q) = \frac{-5s_3(q) + 7s_5(q)}{12}$$

then these series converge in K and the elliptic curve $E_q : y^2 + xy = x^3 + a_4(q)x + a_6(q)$ has j-invariant $|j(E_q)| > 1$. Moreover we have a p-adic analytic isomorphism $\phi : \overline{K}^*/q^{\mathbb{Z}} \to E_q(\overline{K})$. ϕ is Galois invariant: $\phi(u^g) = \phi(u)^g$ for all $u \in \overline{K}^*, g \in \operatorname{Gal}(\overline{K}/K)$. In particular we have $L^*/q^{\mathbb{Z}} \cong E_q(L)$ for all L/K algebraic. The elliptic curve E_q is called the Tate curve.

Exercise (Nagell-Lutz)**.** If $E/\mathbb{Q}$ has split multiplicative reduction at p ($|j(E_q)| > 1$) then $E(\mathbb{Q})_{\text{tors}}$ is finite.

Theorem 1.4 (Tate uniformisation)**.** *Let E/K be an elliptic curve with $|j(E)| > 1$. Then there exists a unique $q \in K^*, |q| < 1$ such that $E_q \cong E$.*

What should be the "geometry"?

Naïve approach: define p-adic manifolds, analytic functions locally given by power converging power series. Book: J-P. Serre: p-adic Lie groups.

Problem with this approach: if K is non-archimedean then it is totally disconnected!

Exercise. Easy facts:

1. A series $\sum_{n=0}^{\infty} a_n$ is Cauchy if and only if $\lim |a_n| = 0$ (for if use strict triangle inequality).

2. $|a + b| = \max\{|a|, |b|\}$ if $|a| \neq |b|$.

3. $B(a, r)$, the closed ball of radius r, is both closed and open.

4. $\mathring{B}(a, r)$, the open ball of radius r, is both closed and open.

5. $\partial B(a, r)$ is also both open and closed.

It follows that the closed balls are not connected

Another more series problem: identity theorem does not hold. For example

$$f : K \to K$$
$$x \mapsto \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$$

Solution:

1. Tate's "rigidity" analytic notion: analytic functions should have globally a power series expansion if they are defined on a ball. In particular on $B = B(0, 1)$, $f = \sum a_I x^I$ is analytic if and only if $\lim |a_I| = 0$. They are called *restricted power series*, or *strictly convergent power series*. We then obtain T_n, the *Tate algebra*. The is the analogue of $k[x_1, \ldots, x_n]$ in algebraic geometry.

2. Berkovich spaces/adic spaces (Huber). More "schematic" approach. We are going to focus mostly on the latter after rigid analytic varieties and not so much on Berkovich spaces. Adic spaces are also the framework for perfectoid spaces.

2 Tate algebras

Notation. We use multiindex notation: $f = \sum c_I x^I$ where $I = (i_1, \ldots, i_n)$, and $|I| = \sum_{j=1}^{n} i_j$.

We define $\mathcal{O}_{T_n} = R\langle x_1, \ldots, x_n \rangle$ to be the set of restricted power series with coefficients in R.

Lemma 2.1. *A formal power series $f = \sum c_I x^I \in K[[x_1, \ldots, x_n]]$ converges in $B(\overline{K})$ if and only if $\lim |c_I| = 0$.*

Proof. For only if use $x = (1, \ldots, 1)$. For if, given $x \in B(\overline{K})$, exists K'/K finite such that $x_i \in K'$ for all i. If $\lim |c_I| = 0$ then $\lim |c_I||x^I| = 0$, so $f(x)$ converges in K' as K' is complete. $\qquad\square$

Define the *Gauss norm* on T_n

$$\left\| \sum c_I x^I \right\| = \sup_I \{|c_I|\} = \max_I \{|c_I|\}.$$

It is clear that $\|f\| = 0$ if and only if $f = 0$, $\|cf\| = |c|\|f\|$, and $\|f + g\| \leq \max\{\|f\|, \|g\|\}$. What is not so obvious is that

Lemma 2.2. *The Gauss norm is multiplicative.*

Proof. It is clear that $\|fg\| \leq \|f\|\|g\|$. wlog $\|f\| = \|g\| = 1$, so $f, g \in \mathcal{O}_{T_n}$. We may reduce mod $\mathfrak{m}$ to get $k[x_1, \ldots, x_n]$. $\overline{f}, \overline{g} \neq 0$ implies $\overline{fg} \neq 0$. Thus $\|fg\| = 1$. $\qquad\square$

Therefore $\|\cdot\|$ on T_n is a K-algebra norm.

Lemma 2.3. *T_n is complete with respect to $\|\cdot\|$.*

Proof. Suppose we have a Cauchy sequence $\sum f_i$ with $f_i \in T_n$, $\lim \|f_i\| = 0$. Let $f_i = \sum c_{i,I} x^I$. Then $|c_{i,I}| \leq \|f_i\|$ so $\lim_i |c_{i,I}| = 0$, thus we have $\sum_i c_{i,I} = c_I$. Let $f = \sum c_I x^I$. It is an exercise to show $f \in T_n$. $\qquad\square$

Thus the Tate algebra T_n is a K-Banach algebra.

Lemma 2.4. *For f normalised, i.e. $\|f\| = 1$, TFAE:*

1. *f is a unit in $\mathcal{O}_{T_n}$.*

2. *f is a unit in T_n.*

3. *$\overline{f}$ is constant.*

4. *$|f(0)| = 1$ and $\|f - f(0)\| < 1$.*

Proof. $1 \iff 2$ by multiplicativity of norm. $1 \implies 3$ by using units in a polynomial ring are precisely the non-zero constants. $3 \implies 4$ is clear. $4 \implies 1$: we may assume $f(0) = 1$ so $f = 1 - g$, $\|g\| < 1$ and $\sum g^i$ is an inverse. $\qquad\square$

We would like to construct rigid analytic space using T_n as local models. In complex geometry, we used Weierstrass preparation and division to derive basic properties of the local ring of analytic functions, such as being locally factorial and satisfying Nullstellensatz. We are going to do the same for Tate algebras.

Definition (distinguished power series). A power series $f = \sum g_i x_n^i$ where $g_i \in T_{n-1}$ is *distinguished* in x_n of degree d if

1. $\|f\| = \|g_d\|$ and $\|g_s\| < \|f\|$ for $s > d$.

2. g_d is a unit in T_{n-1}.

Thus if f is normalised then it is distinguished in x_n of degree d if and only if $\overline{f} = \overline{g}_0 + \cdots + \overline{g}_d x_n^d$ with $\overline{g}_d \in k^*$.

Theorem 2.5. *Suppose f is normalised.*

1. *Division: suppose $f \in \mathcal{O}_{T_n}$ is distinguished in x_n of degree d. Then any $g \in T_n$ can be uniquely written as $g = qf + r$ where $q \in T_n, r \in T_{n-1}[x_n]$ with $\deg_{x_n}(r) < d$. Moreover $\|g\| = \max\{\|q\|, \|r\|\}$.*

2. *Preparation: suppose $f \in \mathcal{O}_{T_n}$ is distinguished in x_n of degree d. Then have a unique expression $f = gk$ where $g \in \mathcal{O}_{T_{n-1}}[x_n]$ monic in x_n of degree d, $h \in \mathcal{O}_{T_n}^*$.*

3. *Distinction: if $f_1, \ldots, f_m \in \mathcal{O}_{T_n}$ are all normalised then exists an automorphism τ of T_n (preserving $\|\cdot\|$) such that $f_1^\tau, \ldots, f_n^\tau$ are distinguished in x_n.*

Proof.

1. For uniqueness, suppose $qf + r = q'f + r'$, then $(q - q')f = r' - r$. Let c be such that $|c| = \|q - q'\|^{-1}$. Then reduce $c(q - q')f = c(r' - r)$ modulo $\mathfrak{m}$ and inspect the x_n^d term to derive a contradiction.

   For the statement on norms, suppose for contradiction suppose $\|q\| = \|r\| > \|g\|$. Choose c such that $\|c\| = \|q\|^{-1}$, reduce to get $0 = \overline{cq}\,\overline{f} + \overline{r}$. Again contradiction.

   For existence, suppose first $f = f_0 = c_0 + c_1 x_n + \cdots + c_d x_n^d$ where $c_i \in \mathcal{O}_{T_{n-1}}$. Let $g = \sum d_I x^I$. Decompose $x^I = q_I f + r_I$ (ordinary long division) in $\mathcal{O}_{T_{n-1}}[x_n]$ so $\deg_{x_n}(r_I) < d$. By the result on norms $\max\{\|q_I\|, \|r_I\|\} = \|x^I\| = 1$. Then $q = \sum d_I q_I, r = \sum d_I r_I$ converge in T_n and $g = qf + r$. Have $r \in \mathcal{O}_{T_{n-1}}[x_n]$ with $\deg_{x_n}(r) < d$. For the general case, let $f = f_0 + D$ where $\|D\| < 1$. Let

   $$g_0 = g, g_i = q_i f_0 + r_i = q_i f + r_i - q_i D, g_{i+1} = q_i D.$$

   Then $q = \sum q_i, r = \sum r_i$ converges and $g = qf + r$.

2. For existence, apply division $x_n^d = q'f + r', q = x_n^d - r'$. $q \in \mathcal{O}_{T_{n-1}}[x_n], \deg_{x_n}(q) = d, q = q'f$. Reduce to get $\overline{q} = \overline{q'f}$. $\overline{q}$ and $\overline{f}$ have the same degree so $\overline{q'}$ is a unit. Thus q' is a unit in T_n. Thus $f = gh$ where $g = q, h = (q')^{-1}$. For uniqueness, suppose $f = gh$ then $x_n^d = h^{-1}f + (x_n^d - g)$. Apply uniqueness of division.

3. Let

$$\tau : T_n \to T_n$$

$$x_i \mapsto \begin{cases} x_i + x_n^{\alpha_i} & i < n \\ x_n & i = n \end{cases}$$

where α_i is to be chosen later. It is clear that τ is a K-algebra homomorphism and τ preserves the Gauss norm. It has an inverse so τ is a K-algebra automorphism that preservse the norm.

To choose α_i, suppose $f_j = \sum c_{j,I} x^I$. $\overline{f}_j$ are polynomials and consider those I such that $|c_{j,I}| = 1$ and define t to be strictly larger than any of the index in $I = (i_1, \ldots, i_n)$ for all I. Let $\alpha_1 = t^{n-1}, \ldots, \alpha_{n-1} = t$. Then

$$\overline{\tau(f_j)} = \sum \overline{c_{j,I}}(x_1 + x_n^{\alpha_1})^{i_1} \cdots (x_{n-1} + x_n^{\alpha_{n-1}})^{i_{n-1}} x_n^{i_n}$$

$$= \sum \overline{c}_{j,I} x_n^{\alpha_1 i_1 + \cdots + \alpha_{n-1} i_{n-1} + i_n} + \overline{g}$$

where $\deg_{x_n} \overline{g} < \alpha_1 i_1 + \cdots + \alpha_{n-1} i_{n-1} + i_n$ for all such indices. Exercise: if $I \neq J$ then $\alpha_1 i_1 + \cdots + \alpha_{n-1} i_{n-1} + i_n \neq \alpha_1 j_1 + \cdots + \alpha_{n-1} j_{n-1} + j_n$. Thus there exists a maximum I_0. Then

$$\overline{\tau(f_j)} = c_{j,I} x_n^{d_j} + \text{ poly of degree } < d_j.$$

Thus $\tau(f_j)$ is distinguished of degree d_j for all $j = 1, \ldots, m$.

$\square$

Proposition 2.6 (Hilbert basis theorem)**.** *The K-algebra T_n is Noetherian.*

Proof. Induction on n. The base case $n = 0$ is obvious. Assume the statement is true for $n - 1$. Let $I \subseteq T_n$ be a nonzero ideal and $0 \neq f \in I$. By Weierstrass distinction exists τ such that f^τ is distinguished in x_n of some degree. By division I^τ is generated by f^τ and $I^\tau \cap T_{n-1}[x_n]$. By induction hypothesis T_{n-1} is Noetherian, and therefore so is $T_{n-1}[x_n]$. Thus $I^\tau \cap T_{n-1}[x_n]$ is finitely generated so I^τ is finitely generated. I is finitely generated. $\square$

Proposition 2.7. *The K-algebra T_n is a UFD.*

Proof. Induction on n. Assume true for $n - 1$. Take $f \in T_n$. f^τ is distinguished in x_n. By preparation $f^\tau = gh$ where g is monic in $\mathcal{O}_{T_{n-1}}[x_n]$ and h is a unit. As $T_{n-1}[x_n]$ is a UFD f^τ can be decomposed into irreducibles. Uniqueness is similar. $\square$

Proposition 2.8. *The Krull dimension on T_n is n.*

Proof. $0 \subseteq (x_1) \subseteq \cdots \subseteq (x_1, \ldots, x_n)$ is a chain of prime ideals of length n. If $f \in T_n$ irreducible, use distinction and preparation to get $T_n/(g)$ ($f^\tau = gh$) which is finite over T_{n-1} so $\dim T_n \leq n$. $\square$

3 Affinoid algebra

Definition (affinoid algebra). An *affinoid algebra* is a K-algebra of the form T_n/I for some $I \subseteq T_n$.

Theorem 3.1 (Noether normalisation). *Let I be an ideal of T_n, $A = T_n/I$. Then there exists a finite injective map $T_d \hookrightarrow A$ of K-algebras for some d. Moreover $\dim A = d$.*

Proof. Induction on n. Assume the result is true for $n-1$. We may asume that $I \subseteq T_n$ is proper. Take $0 \neq f \in I$. Apply Weierstrass distinction and preparation to get a monic polynomial $f \in T_{n-1}[x_n]$ in I in x_n (we write I for I^τ). $T_n/(f)$ is a free T_{n-1}-module of finite rank. Let φ the composition $T_{n-1} \to T_n/(f) \to T_n/I$. If $\ker \varphi = 0$ then done. If not then we may apply the induction hypothesis for $T_{n-1}/\ker \varphi$ to get $T_d \hookrightarrow T_{n-1}/\ker \varphi$, and $T_{n-1}/\ker \varphi \hookrightarrow T_n/I$ is finite and projective. The statement about dimension follows from integrality. $\square$

Remark. Warning: unlike in the affine case, it can happen that an affinoid subalgebra has greater dimension than the affinoid algebra that contains it. See example 2.3.3 in Fresnel & van der Put, Rigid Analytic Geometry and its Applications.

Corollary 3.2 (Hilbert (weak) Nullstellensatz). *Any maximal ideal $\mathfrak{m} \subseteq T_n$ has the property that $T_n/\mathfrak{m}$ is finite over K.*

Proof. $T_n/\mathfrak{m}$ is a field so has dimension 0 so by Noether normalisation it is finite over K. $\square$

3.1 Interlude on p-adic analysis

Facts that carry over from $\mathbb{R}$ and $\mathbb{C}$:

1. For a finite dimensional Banach space over K, any two norms are equivalent.

2. A linear map $f : V \to W$ of K-Banach spaces is continuous if and only if f is bounded.

3. Open mapping theorem: if $f : V \to W$ is a bounded surjective linear map between K-Banach spaces then f is open and the quotient topology on W conincides with the original topology on W. More precisely, exists $c > 0$ such that for any $w \in W$, there is some $v \in V$, $w = f(v)$ satisfying $\|v\| \leq c\|w\|$. As a corollary, any bounded bijective linear map $f : V \to W$ is an isomorphism of Banach spaces.

4. Closed graph theorem: a linear map $f : V \to W$ between K-Banach spaces is bounded if and only if its graph in $V \times W$ with product topology is closed.

Unique to p-adic analysis we have

> **Lemma 3.3.** *Let A be a K-Banach algebra which is Noetherian. Let M be a Banach A-module (i.e. M is Banach space with a norm compatible with the A-module structure: $\|a \cdot m\|_M \le \|a\|_A \|m\|_M$) such that M is finitely generated over A. Then any A-submodule of M is closed.*

Later we'll use this to put a norm on affinoid algebras: T_n is Noetherian so any ideal I is closed. Then we can equip T_n/I with the quotient norm.

Proof. Let N be a submodule of M. Let $\overline{N}$ be the closure of N. A Noetherian, M finitely generated implies that $\overline{N}$ is finitely generated. Take generators $e_1, \ldots, e_n$. Consider $A^n \twoheadrightarrow \overline{N}$. Equip A^n with the maximum norm. By open mapping theorem, exists $c \in (0,1)$ such that every $x \in \overline{N}$ can be written as $\sum a_i e_i$ with $c \cdot \max\|a_i\| \le \|x\|$. N is dense in $\overline{N}$ so choose $f_1, \ldots, f_n \in N$ such that $\sum \|e_i - f_i\| \le c^2$. We show that $f_1, \ldots f_n$ generate $\overline{N}$.

Given $x \in \overline{N}$, write $x_0 = x = \sum a_{0,i} e_i$ where $c \cdot \max\|a_i\| \le \|x_0\|$. Then $x_0 = \sum a_{0,i} f_i + x_1$ where $x_1 = \sum a_{0,i}(e_i - f_i)$ and $\|x_1\| \le c \cdot \|x_0\|$. Then write $x_m = \sum a_{m,i} e_i$ with $c \cdot \max\|a_{m,i}\| \le \|x_m\|$. Take the infinite sum

$$x = \sum_{m=0}^{\infty} x_m - x_{m+1} = \sum (\sum a_{m,i}) f_i = \sum a_i f_i.$$

$\square$

3.2 Completed tensor product

Let A be a K-Banach algebra, M, N Banach A-modules. $M \otimes_A N$ is not complete in general, but there is a seminorm

$$\|x\| = \inf\{\max\|m_i\|\|n_i\| : x = \sum m_i \otimes n_i\}.$$

Define the *completed tensor product* $M \hat{\otimes}_A N$ to be the completion of $M \otimes_A N$ with respect to this seminorm.

Exercise. Let A, M, N be as above. If M, N are finitely generated then $\|\cdot\|$ is a norm and $M \otimes_A N$ is complete. (Hint: choose some finite presentation of M, N and reduce to the case of free modules of finite rank)

The completed tensor product has the following universal property: there exists a continuous A-bilinear map $\alpha : M \times N \to M \hat{\otimes}_A N$ such that for any continuous bilinear map $\beta : M \times N \to T$ into some complete normed A-module, there exists a continuous linear map φ such that the following diagram commutes:

$$
\begin{array}{ccc}
M \times N & \xrightarrow{\beta} & T \\
\downarrow{\alpha} & \nearrow{\varphi} & \\
M \hat{\otimes}_A N & &
\end{array}
$$

Example. Let A be an affinoid algebra. Define the restricted power series to be

$$A\langle x_1, \ldots, x_n \rangle = \{\sum a_I x^I \in A[[x_1, \ldots, x_n]] : a_I \in A, \lim |a_I| = 0\}.$$

It is complete with respect to the Gauss norm and is isomorphic to $A \hat{\otimes}_K T_n$. This is a consequence of the following general observation.

Proposition 3.4. *Let $A_1 \to A_2$ be a homomorphism of affinoid algebras. Then*

$$A_2 \hat{\otimes}_{A_1} A_1 \langle x_1, \ldots, x_n \rangle \cong A_2 \langle x_1, \ldots, x_n \rangle.$$

Proof. Follows from universal property. □

Theorem 3.5. *Let $A \to A_1, A \to A_2$ be homomorphisms of affinoid algebras. Then $A_1 \hat{\otimes}_A A_2$ is also affinoid.*

Sketch proof. We know $T_{n+m} \cong T_n \hat{\otimes}_K T_m$. Prove first the statement over $A = K$. Let $\alpha_1 : T_n \to A_1, \alpha_2 : T_m \to A_2$. Then $T_{n+m}/(\ker \alpha_1, \ker \alpha_2) \cong A_1 \hat{\otimes}_K A_2$. By base change $A_1 \hat{\otimes}_K A_2 \twoheadrightarrow A_1 \hat{\otimes}_R A_2$. Then show for $R \to A, A \to A_1, A \to A_2$, we can exhibit an epimorphism $A_1 \hat{\otimes}_R A_2 \twoheadrightarrow A_1 \hat{\otimes}_A A_2$. □

As hinted in the previous section, every ideal $I \subseteq T_n$ is closed, and hence any affinoid $A = T_n/I$ admits a residue norm

$$\left\| q\left(\sum c_I x^I \right) \right\|_A = \inf_{a \in I} \left\| \sum c_I x^I + a \right\|.$$

where $q : T_n \to A$ is the natural quotient. It then follows that A is a K-Banach algebra with respect to $\|\cdot\|_A$.

Proposition 3.6. *Let B be a Noetherian K-Banach algebra. Suppose exists $\mathcal{S}$ a collection of ideals of B such that*

1. for any $I \in \mathcal{S}$, $\dim_K B/I < \infty$;

2. $\bigcap_{I \in \mathcal{S}} I = (0)$,

then any K-algebra homomorphism $A \to B$, where A is a Noetherian K-Banach algebra, is continuous.

Proof. Applying the closed graph theorem, we need to show if $x_n \to 0$ and $f(x_n) \to y$ then $y = 0$. Note that if $I \subseteq B$ is closed then so is $J = f^{-1}(A) \subseteq A$. The induced map $A/J \to B/I$ is injective. If $I \in \mathcal{S}$ then since $\dim_K B/I < \infty$, have $\dim_K A/J < \infty$. Any linear map between finite dimensional Banach space is continuous, and all norms on finite dimensional spaces are equivalent, so the map is bounded with respect to maximum norm. Thus $y \in I$. As this holds for all $I \in \mathcal{S}$, $y \in \bigcap I = (0)$ so $y = 0$. □

Corollary 3.7. *Any K-algebra homomorphism between affinoids is continuous. In particular for a fixed affinoid A, all the residue norms are equivalent. Thus the Gauss norm on T_n is canonical in the sense that every K-algebra homomorphism $T_n \to T_n$ preserves the Gauss norm.*

Proof. Take all ideals I in A such that $\dim_K A/I < \infty$, in particular all maximal ideals and their powers by Noether normalisation. Suppose $y \neq 0$. Then $\operatorname{Ann}(y) \subseteq \mathfrak{m}$ for some $\mathfrak{m}$ maximal. By Krull intersection $\bigcap \mathfrak{m}^n = 0$ in $A_{\mathfrak{m}}$. Thus $y \notin \bigcap \mathfrak{m}^n$. □

3.3   Newton polygon

Lemma 3.8. *Let $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ over K with roots $\alpha_1, \ldots, \alpha_n$ in $\overline{K}$. Sort the roots by $|\alpha_1| \geq |\alpha_2| \geq \cdots \geq |\alpha_n|$. Then $|a_{n-i}| \leq |\alpha_1 \cdots \alpha_i|$ with equality whenever $|\alpha_i| > |\alpha_{i+1}|$ or $i = n$.*

Proof. Write $P(z) = \prod(z - \alpha_i)$. Then a_{n-i} is up to sign the same as i-folds of α_j's. The first statement then follows from the strict triangle inequality. If $|\alpha_i| > |\alpha_{i+1}|$ then $\alpha_1 \cdots \alpha_i$ is strictly greater than any other i-fold so equality. $\square$

Corollary 3.9. *The maximum absolute value of the roots of P is equal to $\max_j \|a_{n-j}\|^{1/j}$.*

3.4   Maximum spectrum and spectral norm

Let A be an affinoid algebra. Let $\operatorname{Max} A = \operatorname{Sp} A$ be the set of maximal ideals of A. We refer to $\operatorname{Sp} A$ together with its ring of "functions" A as an *affinoid space*, although we haven't defined a topology on $\operatorname{Sp} A$ yet.

Nullstellensatz implies that for any $x \in \operatorname{Sp} A$, $A/\mathfrak{m}_x$ is finite over K. In particular it admits a unique extension of $|\cdot|_K$. Write $f(x)$ for the image of $f \in A$ in $A/\mathfrak{m}_x$. Define the *supremum/spectral norm* to be

$$\|f\|_{\mathrm{spec}} = \sup_{x \in \operatorname{Sp} A} |f(x)|.$$

It is clear that $\|\cdot\|_{\mathrm{spec}}$ is a seminorm and is submultiplicative. It is a norm if and only if $\bigcap_{x \in \operatorname{Sp} A} \mathfrak{m}_x = 0$, and we will see that it is equivalent to A being reduced.

Given $\varphi : A \to B$ a homomorphism of affinoids, we can define

$$\operatorname{Sp} \varphi : \operatorname{Sp} B \to \operatorname{Sp} A$$
$$\mathfrak{m} \mapsto \varphi^{-1}(\mathfrak{m})$$

It is well-defined: by Noether normalisation the composition $K \hookrightarrow A/\varphi^{-1}(\mathfrak{m}) \to B/\mathfrak{m}$ is a finite extension so $\varphi^{-1}(\mathfrak{m})$ is maximal.

Lemma 3.10 (maximum modulus principle for Tate algebra)**.** *Let $f \in T_n$. Then for all $x \in \operatorname{Sp} T_n$, $|f(x)| \leq \|f\|$ and exists x such that $|f(x)| = \|f\|$, i.e. $\|f\|_{\mathrm{spec}} = \|f\|$.*

Proof. Let $f = \sum c_I x^I$. Let $P(x_1, \ldots, x_n)$ be the polynomial of terms in f such that $|c_I| = \|f\|$. Then $\|P - f\| < \|P\|$ so it is enough to show the statement for P. Normalise P so $P \in \mathcal{O}_K[x_1, \ldots, x_n]$ and not all coefficients are in $\mathfrak{m}_K$. Consider $\overline{K}, \overline{k}$. $\overline{P}$ cannot vanish everywhere on $\overline{k}^n$ so exists $(\alpha_1, \ldots, \alpha_n) \in \overline{k}^n$ such that $\overline{P}(\alpha_1, \ldots, \alpha_n) \neq 0$. Any lift $(r_1, \ldots, r_n)$ of $(\alpha_1, \ldots, \alpha_n)$ to $R_{\overline{K}}^n$ will satisfy $|P(r_1, \ldots, r_n)| = 1$.

In general, evaluating functions $f \in T_n$ at $x = (\alpha_1, \ldots, \alpha_n) \in R_{\overline{K}}^n$ gives a surjective map $\varphi_x : T_n \to K(\alpha_1, \ldots, \alpha_n)$, and $\ker \varphi_x$ is a maximal ideal of T_n. Claim $B(\overline{K}) \to \operatorname{Sp} T_n$ is surjective: let $\mathfrak{m} \subseteq T_n$ be maximal and consider the induced map $\varphi : T_n \to \overline{K}$. Claim φ is contractive, i.e. $|\varphi(f)| \leq \|f\|$ for

all f. Assume $|\varphi(f)| > \|f\|$ for some f normalised. Write $\alpha = \varphi(f)$. Let $q(y) = y^k + c_{k-1}y^{k-1} + \cdots + c_0$ be its minimal polynomial over K. $|\alpha_i| = |\alpha|$ for any root α_i of q. Since $|\alpha| > 1$, $|c_{n-i}| \leq |\alpha|^i < |\alpha|^k = |c_0|$ for all $i < k$. Thus $q(f) \in T_n$ is a unit by Lemma 2.4 so its image under φ is also a unit, contradicting $\varphi(q(f)) = q(a) = 0$. Then $(\varphi(x_1), \ldots, \varphi(x_n)) \in B(\overline{K})$ so φ coincides with φ_x. $\qquad\square$

> **Proposition 3.11** (maximum modulus principle for affinoid). *For any affinoid algebra A and $f \in A$, there exists $x \in \operatorname{Sp} A$ such that $\|f\|_{\mathrm{spec}} = |f(x)|$. Moreover $\|f\|_{\mathrm{spec}} = 0$ if and only if f is nilpotent.*

Proof. Noether normalisation says there is a finite map $T_d \hookrightarrow A$, so exists some irreducible polynomial $P(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_0$ such that $P(f) = 0$. Use corollary in Newton polygon subsection, for any $x \in \operatorname{Sp} A$ and any $y \in \operatorname{Sp} T_d$ such that x lies over y, we have $|f(x)| = \max\{\|a_{n-i}(y)\|^{1/i}\}$. Thus $\|f\|_{\mathrm{spec}} = \max\{\|a_{n-i}\|_{\mathrm{spec}}^{1/i}\}$. By maximum modulus principle for Tate algebra, exists y such that the maximum is achieved at y.

If $\|f\|_{\mathrm{spec}} = 0$ then $\|a_{n-i}\|_{\mathrm{spec}} = 0$ so $\|a_{n-i}\| = 0$ so $a_{n-i} = 0$. Thus f is nilpotent. $\qquad\square$

> **Corollary 3.12.** *For any affinoid algebra A, the Jacobson radical equals to the nilradical. In particular $\|\cdot\|_{\mathrm{spec}}$ is a norm if and only if $\mathcal{N}(A) = 0$, i.e. A is reduced.*

Proof. Suppose $f \in \mathcal{J}(A)$, then $\|f\|_{\mathrm{spec}} = 0$ so by maximum modulus principle f is nilpotnent. $\qquad\square$

Remark. In fact more is true. Let A be a reduced affinoid algebra. Then A is complete with respect to $\|\cdot\|_{\mathrm{spec}}$ and $\|\cdot\|_{\mathrm{spec}}$ is equivalent to any residual norm. The first statement is Theorem 3.4.9 in Fresnel & van der Put.

4 Topology on maximum spectrum

4.1 Affinoid subdomain

Let A be an affinoid algebra.

Definition (affinoid subspace)**.** An *affinoid subspace (or affinoid subdomain)* of $X = \operatorname{Sp} A$ is a subset $Y \subseteq X$ for which there exists a morphism $\phi : A \to B$ of affinoid algebra's with $\operatorname{Sp}(\phi)(\operatorname{Sp} B) \subseteq Y$ with the following universal property: given any K-algebra homomorphism $\psi : A \to C$ where C affinoid with $\operatorname{Sp}(\psi)(\operatorname{Sp} C) \subseteq Y$, there exists a unique homomorphism $\tau : B \to C$ making the following diagram commute

$$
\begin{array}{ccc}
A & \xrightarrow{\;\psi\;} & C \\
{\scriptstyle \phi}\downarrow & {\scriptstyle \tau}\nearrow & \\
B & &
\end{array}
$$

Proposition 4.1. *Let $Y \subseteq X$ be an affinoid subdomain.*

1. *The map $\phi : A \to B$ is unique up to isomorphism.*

2. *The induced map $\operatorname{Sp}\phi : \operatorname{Sp} B \to Y$ is a bijection.*

3. *For any $y \in Y$, let $\mathfrak{m}_y$ by the corresponding maximal ideal of A. Then $A/\mathfrak{m}_y^n \to B/\mathfrak{m}_y^n B$ is an isomorphism for all $n \geq 1$.*

4. *If Y is an affinoid subdomain of X and Z is an affinoid subdomain of Y then Z is an affinoid subdomain of X.*

5. *If $\psi : A \to C$ is a homomorphism of affinoids and $X' = \operatorname{Sp} B$ is an affinoid subdomain of X then $((\operatorname{Sp}\psi)^{-1}(Y), B\hat{\otimes}_A C)$ is an affinoid subdomain of $\operatorname{Sp} C$.*

Proof.

1. Universal property.

2. See 3.

3. By universal property $A \to A/\mathfrak{m}_y^n$ factors through B, so the square and the upper triangle in the diagram commute

$$
\begin{array}{ccc}
A & \xrightarrow{\hspace{2cm}} & B \\
{\scriptstyle \pi}\downarrow & {\scriptstyle \alpha}\swarrow & \downarrow{\scriptstyle \pi'} \\
A/\mathfrak{m}_y^n & \xrightarrow{\;\sigma\;} & B/\mathfrak{m}_y^n B
\end{array}
$$

Both π' and $\sigma \circ \alpha$ satisfy the universal property with respect to $\sigma \circ \pi : A \to B/(\mathfrak{m}_y')^n$, so by uniqueness the lower triangle also commutes. As π' is surjective, σ is surjective. As α is surjective and $\ker \pi' = \mathfrak{m}_y^n B \subseteq \ker \alpha$, σ must be injective. Set $n = 1$ we get 2.

4. Definition.

5. Let $X' = \operatorname{Sp} C$. The pushout of rings

$$
\begin{array}{ccc}
A & \xrightarrow{\ \phi\ } & B \\
\downarrow{\scriptstyle\psi} & & \downarrow \\
C & \longrightarrow & B\hat{\otimes}_A C
\end{array}
$$
with α and β to D

gives

$$
\begin{array}{ccc}
Z & \xrightarrow{\ \operatorname{Sp}\beta\ } & \\
 & Y\times_X X' & \xrightarrow{\ p\ } X' \\
\operatorname{Sp}\alpha & \downarrow{\scriptstyle q} & \downarrow{\scriptstyle \operatorname{Sp}\psi} \\
 & Y & \xrightarrow{\ \operatorname{Sp}\phi\ } X
\end{array}
$$

Using commutativity the image of projection p must be contained in $\operatorname{Sp}(\psi)^{-1}(Y)$. Now let $Z = \operatorname{Sp} D$ be affinoid. Assume that $\operatorname{Sp}\beta(Z) \subseteq \operatorname{Sp}(\psi)^{-1}(Y)$, then $\operatorname{Sp}\psi \circ \operatorname{Sp}\beta : Z \to X$ so universal property of Y, we get a unique morphism $\alpha : B \to D$ such that $\operatorname{Sp}\alpha : Z \to X$. Then we use the universal property of tensor product to get a morphism $B\hat{\otimes}_A C \to D$.

$\square$

Corollary 4.2. *If $Y_1,\dots,Y_k$ are affinoid subdomains with coordinate rings $B_1,\dots,B_k$, then $Y_1\cap\cdots\cap Y_k$ is an affinoid subdomain with coordinate ring $B_1\hat{\otimes}_A B_2\hat{\otimes}_A\cdots\hat{\otimes}_A B_k$.*

Definition (mild G-topology)**.** We define the *mild G-topology* on $\operatorname{Sp} A$ on $\operatorname{Sp} A$ to be the Grothendieck topology whose admissible opens (objects) are affinoid subdomains and admissible coverings (coverings) are coverings that have a finite subcovering.

We define the *structure presheaf* on mild G-topology by setting $\mathcal{O}_X(A) = A$ if $U = \operatorname{Sp} A$.

Definition ((locally) closed/open immersion)**.** If $X = \operatorname{Sp} B, Y = \operatorname{Sp} A$, $\phi : X \to Y$ a morphism of affinoid spaces, we say that ϕ is a *closed immersion* if the corresponding homomorphism $A \to B$ is surjective. It is a *locally closed immersion (resp. open immersion)* if the induced homomorphism $\mathcal{O}_{Y,y} \to \mathcal{O}_{X,x}$ where $x \in \operatorname{Sp} B, y = \phi(x)$, is surjective (resp. bijective).

Exercise. A locally closed immersion $\operatorname{Sp} B \to \operatorname{Sp} A$ of affinoids such that the induced map $A \to B$ is finite is in fact a closed immersion.

4.2   Rational subdomain

Rational subdomains are analogous to distinguished open sets in affine schemes.

> **Definition** (rational subdomain)**.** Let A be an affinoid algebra, $X = \operatorname{Sp} A$. We say that a subset $Y \subseteq X$ is a *rational subdomain* if there exist $f_1, \ldots, f_n, g \in A$ such that they generate the unit ideal and
> $$Y = \{x \in X : |f_i(x)| \leq |g(x)| \text{ for all } i\} = X(\tfrac{f_1, \ldots, f_n}{g}).$$

> **Proposition 4.3.** *Any rational subdomain $X(\tfrac{f_1, \ldots, f_n}{g})$ is an affinoid subdomain with coordinate ring*
> $$A\langle \tfrac{f_1, \ldots, f_n}{g} \rangle = A\langle y_1, \ldots, y_n \rangle / (f_1 - gy_1, \ldots, f_n - gy_n).$$

Proof. Let $B = A\langle \tfrac{f_1, \ldots, f_n}{g} \rangle$. Then g is automatically a unit in B. A homomorphism $\psi : A \to C$ of affinoids such that $\operatorname{Sp} \psi(\operatorname{Sp} C) \subseteq X(\tfrac{f_1, \ldots, f_n}{g})$, i.e. $|\psi(f_i)(y)| \leq |\psi(g)(y)|$ for all $y \in \operatorname{Sp} C$ for all i can be characterised as

$$\|\psi(f_i)\psi(g)^{-1}\|_{\mathrm{spec}} \leq 1.$$

Note $\psi(g)$ is indeed a unit in C, for if not $\psi(g) \in \mathfrak{m}$ for some maximal ideal $\mathfrak{m}$. Then not all $\psi(f_i)$ can be in $\mathfrak{m}$ so $|\psi(f_i)(y_{\mathfrak{m}})| > |g(y_{\mathfrak{m}})|$, absurd.

Now let ψ be the canonical map $A \to A\langle \tfrac{f_1, \ldots, f_n}{g} \rangle$. Then

$$\left\| \frac{f_i}{g} \right\|_{\mathrm{spec}} \leq \|y_i\|_{\mathrm{spec}} \leq 1$$

so $\operatorname{im} \operatorname{Sp} \psi \subseteq X(\tfrac{f_1, \ldots, f_n}{g})$. If $\varphi : A \to C$ a map of affinoids satisfying the hypothesis then define

$$\tilde{\varphi} : A\langle \tfrac{f_1, \ldots, f_n}{g} \rangle \to C$$
$$y \mapsto \varphi(f_i)\varphi(g)^{-1}$$

so we have a factorisation of $\varphi : A \to C$. It is unique because of universal property of localisation and density of $A[\tfrac{1}{g}]$. $\square$

Remark.

1. If $f_1, \ldots, f_n, g$ do not generate the unit ideal then we will not get the subset of X we want. For example take $T_2 = K\langle x_1, x_2 \rangle$ and consider $\{y \in \operatorname{Sp} T_2 : |x_1(y)| \leq |x_2(y)|\}$.

2. Consider special kind of rational subspaces

$$X(f, \varepsilon) = \{x \in X : |f(x)| \leq \varepsilon\}$$

for some $f \in A, \varepsilon \in \mathbb{R}_{>0}$. The topology generated by $X(f, \varepsilon)$ is called the *canonical topology*.

Example. Consider $B^n(\overline{K})$. We have seen that $B^n(\overline{K}) \twoheadrightarrow \operatorname{Sp} T_n$. It is a bijection if we take $B^n(\overline{K})/\operatorname{Aut}_K(\overline{K})$ (in the proof we choose an embedding $K \hookrightarrow K' \hookrightarrow \overline{K}$, and the norm is Galois-invariant). Then the quotient topology is the canonical topology. In particular if $K = \overline{K}$ then $\operatorname{Sp} T_n$ is homeomorphic to the closed ball. Note that this topology is not suitable for geometry, as for example it is totally disconntected.

Upon convincing ourselves that the intersection of two rational subdomains is rational, we define

Definition ((very) weak G-topology)**.** We define the *very weak G-topology* on $X = \operatorname{Sp} A$ to be the Grothendieck topology whose admissible opens are rational subdomains, admissible coverings are coverings that contain a finite subcovering.

The *weak G-topology* on X is where admissible opens are finite unions of rational subdomains and admissible coverings are coverings that have a finite subcovering.

Among the three G-topologies, mild topology is finer than very weak topology as we have shown above, and weak topology is finer than mild topology — this is the famous theorem of Gerritzen-Grauert.

5 Tate's acyclicity theorem

Theorem 5.1 (Gerritzen-Grauert theorem)**.** *Let A be an affinoid algebra and $Y = \operatorname{Sp} B$ an affinoid subdomain of $\operatorname{Sp} A = X$. Then Y is a finite union of rational subdomains.*

Note. The converse is not true. A counterexample is given in Fresnel & van der Put.

Detailed sketch. We have $\phi : A \to B$ corresponding to $Y \subseteq X$. Then we may choose a surjection $\varphi : A\langle x_1, \ldots, x_n \rangle \twoheadrightarrow B$. This is possible since B affinoid implies that there is a surjection $T_n \twoheadrightarrow B$. Since $K \hookrightarrow A$, this extends to $\varphi : A\langle x_1, \ldots, x_n \rangle \twoheadrightarrow B$. Choose a minimal such n. Proceed by induction on n.

We need a relative version of Weierstrass preparation, division and distinction. Recall that in the abolute version, an element $f = \sum c_i x_n^i \in T_n$ where $c_i \in T_{n-1}$ is said to be distinguished in x_n of degree d if

- c_d is a unit in T_{n-1},

- $\|c_d\| = \|f\|$,

- $\|c_d\| > \|c_i\|$ for $i > d$.

Definition. We say $f \in A\langle x_1, \ldots, x_n \rangle$ is *distinguished in x_n of degree d_x at $x \in X$* if $f \in A/\mathfrak{m}_x \langle x_1, \ldots, x_n \rangle$ is distinguished in x_n of degree d_x. We say f is *distinguished in x_n of degree $\leq d$ (resp. $= d$)* if f is distinguished in x_n of degree $\leq d$ (resp. $= d$) at all $x \in X$.

Note that f is distinguished of degree 0 if and only if f is a unit.

Lemma 5.2. *Assume $f \in A\langle x_1, \ldots, x_n \rangle$ is distinguished in x_n of degree $\leq d$. Then the set*

$$U_f = \{x \in X : f \text{ distinguished of degree } d \text{ at } x\}$$

is a rational subdomain.

Proof. Write $f = \sum f_i x_n^i$. f distinguished in x_n of degree $\leq d$ means that for all $x \in X$, exists $d_x \leq d$ such that

- $\|\overline{f}_i\| \leq \|\overline{f}_{d_x}\|$ if $i \leq d_x$ and

- $\|\overline{f}_i\| < \|\overline{f}_{d_x}\|$ if $i > d_x$ in $A/\mathfrak{m}_x \langle x_1, \ldots, x_n \rangle$,

- $\overline{f}_{d_x} \in A/\mathfrak{m}_x \langle x_1, \ldots, x_{n-1} \rangle$ is a unit.

Let $c_i \in A$ be the constant term of f_i, which satisfies $|c_i(x)| \leq \|\overline{f}_i\|$ and by Lemma 2.4, for $i = d_x$, $|c_{d_x}(x)| = \|f_{d_x}\|_x$. In particular $|c_{d_x}| \neq 0$. Since f is distinguished of degree $\leq d$ at every point, we see that $c_0, \ldots, c_d$ cannot have a common zero in X. Thus they generate the unit ideal in A so

$$U = \{x \in X : |c_i(x)| \leq |c_d(x)|\}$$

is a rational subdomain. It is the same as U_f by definition. $\qquad\square$

The next two lemmas set up Weierstrass preparation in the relative case. The proofs are very similar to those over K and are left as exercises.

Lemma 5.3 (relative Weierstrass distinction)**.** *Suppose $f \in A\langle x_1, \ldots, x_n \rangle$ is such that its coefficients have no common zero in X. Then exists a K-algebra automorphism τ of $A\langle x_1, \ldots, x_n \rangle$ such that $\tau(f)$ is distinguished in x_n of degree $\leq d$ for some $d \geq 0$.*

Lemma 5.4. *Assume $f \in A\langle x_1, \ldots, x_n \rangle$ is distinguished of degree d. Then the natural map $A\langle x_1, \ldots, x_{n-1} \rangle \to A\langle x_1, \ldots, x_n \rangle /(f)$ is finite.*

Claim that we may find $f \in \ker(\varphi : A\langle x_1, \ldots, x_n \rangle \twoheadrightarrow B)$ with coefficients having no common zero in X.

Proof. Fix $x \in X$ and consider $\varphi_x : A/\mathfrak{m}_x \langle x_1, \ldots, x_n \rangle \twoheadrightarrow B/\mathfrak{m}_x B$. $B/\mathfrak{m}_x B = K'$ for some finite extension K' of K. As the domain of φ_x is not a field, $\ker \varphi_x \neq 0$. Because $\ker \varphi$ is mapped surjectively onto $\ker \varphi_x$, we can find $g_x \in \ker \varphi$ which is nonzero modulo $\mathfrak{m}_x$. Consider the coefficients of one of these, say g_1. Since A is Noetherian, we can find an integer d_1 such that the zero set of the coefficients of g_1 is already defined by coefficients of total degree $< d_1$. Choose some $\alpha_1, \ldots, \alpha_n$ such that $\alpha_1 + \cdots + \alpha_n = d_1$. Then the zero set of g_1: $X(g_1)$ (the zero set of the coefficients of g_1 in X) $\implies$ there must be some g_2 such that $g_2(x) \neq 0$. $g_2' = g_1 + x_1^{\alpha_1} \cdots x_n^{\alpha_n} g_1$ so $X(g_2') \subseteq X(g_1)$. Repeat this step, we get a descending chain

$$X(g_1) \supseteq X(g_2') \supseteq \cdots$$

The chain stabilises after finitely many steps, so say $X(g_r') = \emptyset$. Then $f = g_1 + x_1^{\alpha_1} \cdots x_n^{\alpha_n} g_2 + \cdots \in \ker \varphi$ and the coefficients must have no common zero. $\square$

Use relative distinction, we may assume that $f \in \ker \varphi$ is distinguished in x_n of degree $\leq d$. Then U_f is a rational subdomain. Let $A^{(f)}$ be its coordinate ring. We show that $Y \cap U_f$ is a finite union of rational subdomains of Y, hence of X.

φ induces a surjection $\varphi^{(f)} : A^{(f)}\langle x_1, \ldots, x_n \rangle \twoheadrightarrow B \hat{\otimes}_A A^{(f)}$. Note that the image of f in $A^{(f)}\langle x_1, \ldots, x_n \rangle$ is distinguished in x_n of degree d. We show earlier $f \in \ker \varphi$ so its image is in $\ker \varphi^{(f)}$. Thus by lemma 5.4 $\varphi' : A^{(f)}\langle x_1, \ldots, x_{n-1} \rangle \to B \hat{\otimes}_A A^{(f)}$ is finite homomorphism of K-algebras. Use the exercise (locally closed finite immersion is closed) (to show the exercise, we need that the homomorphism does induce non-trivial extensions on the local rings (prop 4.1 (3) plus $M = 0$ if and only if its localisation at maximal ideals are zero)), we see φ' is surjective. Use induction hypothesis $Y \cap U_f$ is a union of rational subdomains.

Lemma 5.5. *Let Y be an affinoid subdomain of $X = \operatorname{Sp} A$. Let U be a rational subdomain of X given by*

$$U = \{x \in X : |f_i(x)| \leq |f_0(x)|\}$$

where $f_0, \ldots, f_n$ generates the unit ideal. Then exists some $\varepsilon > 1$ such that

> *the rational subspace*
>
> $$U_\varepsilon = \{x \in X : |f_i(x)| \leq \varepsilon \cdot |f_0(x)|\}$$
>
> *has the property that $Y \cap U_\varepsilon$ is also rational.*

Proof. See Lemma 4.2.9 (extension lemma) in Bosch, Lectures on Formal and Rigid Geometry. $\square$

Let us assume that $U \cap U_f = \bigcup_{i=1}^k U_i$. Claim that it is enough to find a cover

$$X = (\bigcup_{\lambda=1}^s V_\lambda) \cup (\bigcup_{i=1}^k U_{i,\varepsilon_i})$$

of rationals such that $V_\lambda \subseteq X - U_f$ for all λ.

Proof. If we have $U_\lambda = \operatorname{Sp} B_\lambda$, tensor φ by B_λ to get $\varphi_\lambda : B_\lambda\langle x_1, \ldots, x_n\rangle \to B \hat{\otimes}_A B_\lambda$. Then $f \in A\langle x_1, \ldots, x_n\rangle$ has image in $\ker \varphi_\lambda$ and it is distinguished in x_n of dgree $\leq d - 1$ (since $V_\lambda \cap X - U_f$). Apply induction on the degree of f. $\square$

How to find this cover? If $k = 1$, denote U_1 by U, $U_{1,\varepsilon_1} = U'$. We know

$$U' = X(\varepsilon^{-1}\tfrac{f_1}{f_0}, \cdots, \varepsilon^{-1}\tfrac{f_n}{f_0}).$$

Then define $V_\lambda = X(\varepsilon\tfrac{f_0}{f_\lambda}, \tfrac{f_j}{f_\lambda} : j \neq \lambda, 0)$. Then V_λ's are rational subcomains and cover $X - U_f$.

For $k > 1$, for each i we find $V_{i,\lambda}$'s that cover $X - U_i$. Then

$$X - \bigcup U_i \subseteq \bigcup V_{1,\lambda_1} \cap \cdots V_{k,\lambda_k}$$

for all k tuples $(\lambda_1, \ldots, \lambda_k)$. This is rational so we get the desired cover. This completes the proof. $\square$

We will consider $X = \operatorname{Sp} A$ with the mild G-topology. We may associate to any finite generated A-module M the presheaf $\mathcal{M}$ with sections $\mathcal{M}(U) = M \otimes B$ for $U = \operatorname{Sp} B$. Note that we do not have to pass to completed tensor product since M is finitely generated. Once we prove that it is in fact a sheaf, we call such $\mathcal{M}$ *coherent sheaf*.

> **Theorem 5.6** (Tate's acyclicity theorem). *Let $X = \operatorname{Sp} A$ be an affinoid space. Then $\mathcal{M}$ is an acyclic sheaf, i.e. it is a sheaf and all higher Čech cohomology groups vanish.*

In particular the structure presheaf is a sheaf and is acyclic, so we may call it structure presheaf.

We need a technical result for Grothendieck topologies.

> **Lemma 5.7** (reduction). *Let X be a G-topological space. Let $\mathcal{F}$ be a presheaf on X, $\{U_i\}_{i \in I}, \{V_j\}_{j \in J}$ covers such that $\{V_j\}$ refines $\{U_i\}$. Assume that $\mathcal{F}$ is an acyclic sheaf on the restriction of $\{V_j\}_{j \in J}$ to each intersection $U_{i_0} \cap \cdots \cap U_{i_n}$. Then $\mathcal{F}$ is an acyclic sheaf for $\{U_i\}_{i \in I}$ if and only if $\mathcal{F}$ is*

an acyclic sheaf for $\{V_j\}_{j\in J}$.

Proof. A standard Leray spectral sequence argument. See Corollary 8.1.4/3 in Bosch, Güntzer, Remmert, Non-Archimedean Analysis, or refer to Stacks project for a more general statement. $\square$

Definition (Laurent domain)**.** A *Laurent domain* of X is a subset defined by

$$X(f_1,\ldots,f_n,g_1^{-1},\ldots,g_n^{-1}) = \{x \in X : |f_i(x)| \leq 1, |g_j(x)| \geq 1\}$$

Lemma 5.8. *Laurent domains are affinoid subdomains.*

Proof. Denote a Laurent domain by $X(f,g^{-1})$ where $f = (f_1,\ldots,f_n), g^{-1} = (g_1^{-1},\ldots,g_n^{-1})$. Consider

$$A\langle f,g^{-1}\rangle = A\langle x_1,\ldots,x_n,y_1,\ldots y_n\rangle/(x_i - f_i, 1 - g_j y_j).$$

We have a canonical map $A \to A\langle f,g^{-1}\rangle$. Similar as before, for any $Y \to B$ affinoids corresponding to $\varphi : A \to B$, $\operatorname{im} Y \subseteq X(f,g^{-1})$ if and only if

$$|\varphi(f_i)(y)| \leq 1, |\varphi(g_j)(y)| \geq 1$$

for all $y \in \operatorname{Sp} B$. The is equivalent to (check $\varphi(g_j)$ are units)

$$|\varphi(f_i)(y)| \leq 1, |\varphi(g_j)^{-1}(y)| \leq 1.$$

Thus $\operatorname{Sp} A\langle f,g^{-1}\rangle \subseteq X(f,g^{-1})$. Check B satisfies the universal property. $\square$

Definition (standard rational covering)**.** Let $X = \operatorname{Sp} A$ be affinoid and $f_1,\ldots,f_n \in A$ have no common zero. Let

$$U_i = \{x \in X : |f_j(x)| \leq |f_i(x)| \text{ for all } j \neq i\}.$$

$U_1,\ldots,U_n$ form a covering of X, called the *standard rational covering* of X.

Lemma 5.9. *Every finite covering of X by affinoids can be refined by a standard rational covering.*

Proof. By Gerritzen-Grauert we may start with a finite rational covering $V_1,\ldots,V_n$

$$V_i = \{x \in X : |f_{j,i}(x)| \leq |f_{0,i}(x)| : 0 \leq j \leq n_i\}$$

with $\{f_{j,i}\}_j$ generates the unit ideal. Consider the product over a multiindex $(v_1,\ldots,v_n), 0 \leq v_i \leq n_i$

$$f_{v_1,\ldots,v_n} = \prod_{i=1}^{n} f_{v_i,i}$$

Choose those $f_{v_1,\ldots,v_n}$ such that at least one of the v_i's is zero. Claim these functions generate the unit ideal and thus give a standard covering of X.

Proof. Assume exists $x \in X$ such that all such functions vanish. Suppose $x \in V_k$. If $f_{0,k}(x) \neq 0$ then $\prod_{i \neq k} f_{v_i, i}$ must vanish. Thus we can find at least one index i_0 such that $f_{0,i_0}, \ldots, f_{n,i_0}$ vanish at x. But $f_{0,i_0}, \ldots, f_{n,i_0}$ generate the unit ideal, contradiction. $\qquad\square$

Claim that the above standard rational covering is a refinement of $\{V_i\}$.

Proof. Let
$$X_{v_1, \ldots, v_n} = X\left(\frac{f_{\lambda_1, \ldots, \lambda_n}}{f_{v_1, \ldots, v_n}}\right)$$
where $\lambda_1, \ldots, \lambda_n$ are as above. Suffice to show if $v_n = 0$ then $X_{v_1, \ldots, v_n} \subseteq V_n$ as other cases follow similarly. Need
$$|f_{\lambda_n, n}(x)| \leq |f_{0,n}(x)| = |f_{v_n, n}(x)|$$
for all $x \in X_{v_1, \ldots, v_n}$. Fix x. Then exists some index j such that $x \in V_j$. If $j = n$ then done. Otherwise wlog $j = 1$. We know
$$|f_{\lambda_1, 1}(x)| \leq |f_{0,1}(x)|$$
so
$$|\prod_{i \neq n} f_{v_i, i}(x)||f_{\lambda_n, n}(x)| \leq |f_{0,1}(x)| \prod_{i \neq n} |f_{\lambda_i, i}(x)||f_{\lambda_n, n}(x)| \leq \prod |f_{v_i, i}(x)|$$
RHS is non-zero so we can divide through by $\prod_{i \neq n} |f_{v_i, i}(x)|$ and the claim follows. $\qquad\square$

$\hfill\square$

Choose elements $f_1, \ldots, f_n \in A$. Consider the sets
$$X_i^+ = X(f_i), X_i^- = X(f_i^{-1}).$$
Then $X_1^* \cap \cdots \cap X_n^*$ where $* = \pm$ is a Laurent domain and they give a covering of X, called *Laurent covering*.

Lemma 5.10. *Every finite covering by affinoids can be refined by a Laurent covering.*

Proof. We may start with a standard rational covering given by $f_1, \ldots, f_n$. Since f_i is invertible in $\mathcal{O}_X(U_i)$ and f_i^{-1} assumes its maximum on U_i by maximum modulus principle, we can find $c \in K^*$ such that
$$|c|^{-1} < \inf_{x \in X} \max_{i=0, \ldots, n} |f_i(x)|.$$
Let us consider the Laurent covering $V_1, \ldots, V_n$ defined by $cf_1, \ldots, cf_n$. Claim if we restrict the standard rational covering to each of the V_j then we get a standard rational covering on V_j generated by units in $\mathcal{O}_X(V_j)$.

Proof. Consider $V = V_j$ for some fixed j.

$$V = X((cf_0)^{\alpha_1}, \ldots, (cf_n)^{\alpha_n}), \alpha_i = \pm 1.$$

We may assume that $\alpha_1 = \cdots = \alpha_s = 1, \alpha_{s+1} = \cdots = \alpha_n = -1$ for some s. Then $X(\frac{f_1,\ldots,f_n}{f_i}) \cap V = \emptyset$ for all $i = 0,\dot{,}s$. Since

$$\max_{i=0,\ldots,s} |f_i(x)| \le |c|^{-1} \le \max_{i=0,\ldots,n} |f_i(x)|$$

for all $x \in V$. In particular

$$\max_{i=0,\ldots,n} |f_i(x)| = \max_{i=s+1,\ldots,n} |f_i(x)|$$

Hence $\{U_i|_V\}$ is a rational covering generated by $f_0|_V, \ldots, f_n|_V$ and all these are units in $\mathcal{O}_X(V)$. $\qquad\square$

Claim if we have a rational covering generated by units $f_1, \ldots, f_n$, then it can be refined to a Laurent covering. From this the lemma follows.

Proof. Consider the Laurent covering generated by $\frac{f_j}{f_i}, 0 \le i < j \le n$. Call this covering $\{V_i\}$. Pick one $V \in \{V_i\}$. Write $i <' j$ if $|f_j(x)| \le |f_i(x)|$ for all $x \in V$. $<$ is transitive and a total ordering so exists a maxumum i_0. Then $V \subseteq X(\frac{f_0,\ldots,f_n}{f_{i_0}})$. $\qquad\square$

$\qquad\square$

Proof of Tate's acyclicity theorem. Using the reduction lemma and induction on the number of functions generating the Laurent covering. We only need to consider a Laurent covering generated by one element $f \in A, X(f), X(f^{-1})$.

First prove for $M = A$. It suffices to prove

$$0 \longrightarrow A \longrightarrow A\langle f \rangle \oplus A\langle f^{-1} \rangle \longrightarrow A\langle f, f^{-1} \rangle \longrightarrow 0$$
$$\parallel \qquad\qquad\qquad\qquad \parallel$$
$$A\langle x\rangle/(x-f) \oplus A\langle y\rangle/(1-yf) \qquad A\langle x,y\rangle/(x-f,1-yf)$$

is a short exact sequence. It is part of the following commutative diagram which can be checked by diagram chasing

$$
\begin{array}{ccccccccc}
 & & 0 & & & & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
 & & (x-f)A\langle x\rangle \oplus (1-yf)A\langle y\rangle & \xrightarrow{\delta''} & (x-f)A\langle x, x^{-1}\rangle & \longrightarrow & 0 & & \\
 & & \downarrow & & & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{\varepsilon'} & A\langle x\rangle \oplus A\langle y\rangle & \xrightarrow{\delta'} & A\langle x, x^{-1}\rangle & \longrightarrow & 0 \\
 & & \parallel & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & A & \xrightarrow{\varepsilon} & A\langle f\rangle \oplus A\langle f^{-1}\rangle & \xrightarrow{\delta} & A\langle f, f^{-1}\rangle & \longrightarrow & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}
$$

Now for the general case, suffices to show after tensoring with a finitely generated module preserves exactness, as $- \otimes M$ gives complete modules for M finitely generated. This follows as the short exact sequence splits.

Proof. We have

$$A[x, y] \to A\langle x\rangle \oplus A\langle y\rangle$$

$$x^i y^j \mapsto \begin{cases} x^{i-j} & i \geq j \\ y^{j-i} & i < y \end{cases}$$

Compose with the natural map to $A\langle f\rangle \oplus A\langle f^{-1}\rangle$ and extend continuously to $A\langle x, y\rangle \to A\langle f\rangle \oplus A\langle f^{-1}\rangle$. Observe that $(x - f), (yf - 1)$ are in the kernel so it induces $A\langle f, f^{-1}\rangle \to A\langle f\rangle \oplus A\langle f^{-1}\rangle$ which splits δ. $\square$

$\square$

> **Definition** (strong G-topology). Let X be an affinoid space. The *strong G-topology* on X is given by
>
> - $U \subseteq X$ admissible open if exists a covering $U = \bigcup_{i \in I} U_i$ by affinoids such that for any morphism $\varphi : Z \to X$ of affinoids with $\varphi(Z) \subseteq U$, $\{\varphi^{-1}(U_i)\}$ can be refined to a finite cover of affinoids of Z.
>
> - an admissible covering $\{V_j\}$ of an admissible open V is a covering such that for any $\varphi : Z \to X$ of affinoids with $\varphi(Z) \subseteq V$, $\{\varphi^{-1}(V_i)\}$ admits a refinement by a finite covering of affinoids of Z.

We need another technical lemma to extend the structure sheaf from the weak G-topology to the strong G-topology.

> **Lemma 5.11** (extension lemma). *Let X be a set with G-topologies T, T' such that*
>
> *1. T' is finer than T,*
>
> *2. each T'-open $U \subseteq X$ admits a T'-covering by T-open subsets,*
>
> *3. each T'-covering of a T-open admits a T-covering as a refinement.*
>
> *Then any T-acyclic sheaf $\mathcal{F}$ on X admits a unique (up to isomorphism) extension $\mathcal{F}'$ as a T'-acyclic sheaf on X.*

We omit the proof but outline the construction briefly. For any admissible open $U \subseteq X$ with respect to T', define the presheaf

$$U \mapsto \varinjlim_{\{U_i \to U\}} H^0(\{U_i\}, \mathcal{F})$$

where $\{U_i \to U\}$ runs through the set of T'-coverings by T-opens. By condition 3 $\mathcal{F}'$ is an extension and using that $\mathcal{F}$ is an acyclic sheaf, it's easy to prove that $\mathcal{F}'$ is a T'-acyclic sheaf.

Corollary 5.12. $\mathcal{O}_X$ *extends from the mild topology to the strong topology.*

Note that every affinoid space X is a ringed G-topological space with the strong G-topology and sheaf $\mathcal{O}_X$. To show this is a locally G-ringed space we show

Proposition 5.13. *Let* X *be as above. Then* $\mathcal{O}_{X,x}$ *is a local ring with maximal ideal* $\mathfrak{m}_x\mathcal{O}_{X,x}$ *for every* $x \in X$.

Proof. For any affinoid subdomain $U \ni x$, we know $\mathcal{O}_X(X)/\mathfrak{m}_x \cong \mathcal{O}_X(U)/\mathfrak{m}_x\mathcal{O}_X(U)$. By taking direct limit we get a surjection $\varphi : \mathcal{O}_{X,x} \to \mathcal{O}_X(X)/\mathfrak{m}_x \cong K'$. Thus $\mathfrak{n} = \ker \varphi$ is maximal. Obviously $\mathfrak{m}_x\mathcal{O}_{X,x} \subseteq \mathfrak{n}$. Conversely, let $f_x \in \mathfrak{n}$. Consider the short exact sequence

$$0 \longrightarrow \mathfrak{m}_x\mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(U) \longrightarrow \mathcal{O}_X(U)/\mathfrak{m}_x\mathcal{O}_X(U) \longrightarrow 0$$

Since direct limit is exact and commutes with tensor product, have an exact sequence

$$0 \longrightarrow \mathfrak{m}_x\mathcal{O}_{X,x} \longrightarrow \mathcal{O}_{X,x} \longrightarrow K' \longrightarrow 0$$

$\mathfrak{m}_x\mathcal{O}_{X,x}$ is the only maximal ideal so if $f_x \in \mathcal{O}_{X,x} - \mathfrak{m}_x\mathcal{O}_{X,x}$ and f_x is represented by (f, U) then $f(x) \neq 0$ so we may assume that $|f(x)| \geq 1$. Then $x \in X(f^{-1})$ and $f|_{X(f^{-1})}$ is a unit so f_x is a unit in $\mathcal{O}_{X,x}$. $\qquad\square$

6   Rigid analytic spaces

Proposition 6.1. *The contravariant functor $A \mapsto (\operatorname{Sp} A, \mathcal{O}_X)$ (viewed as a locally G-ringed space with the strong topology) is an equivalence of categories. The quasi-inverse is given by the global section functors.*

Sketch proof. Fully faithfulness follows from the universal property of affinoid subdomains. $\qquad\qquad\square$

Definition (rigid analytic space)**.** A *rigid analytic K-space X* is a G-ringed space for which there exists an admissible covering $\{U_i\}_{i \in I}$ of X such that

- U_i's are affinoid spaces,

- $U \subseteq X$ is admissible if and only if $U \cap U_i$ is admisible.

Definition (coherent sheaf)**.** A sheaf on X is *coherent* if there exists an admissible covering $\{U_i\}$ of affinoids such that $\mathcal{F}|_{U_i}$ is induced by a finitely generated $\mathcal{O}_X(U_i)$-module.

Definition (closed analytic subspace)**.** A *closed analytic subspace* of X is a subspace Y such that $Y \cap U_i$ is isomorphic to the zero locus of some ideal in $\mathcal{O}_X(U_i)$.

Remark. Rigid analytic spaces are formally very similar to schemes.

- We can glue spaces: let X_i be rigid K-spaces and $X_{ij} \subseteq X_i$ open subspaces and isomorphisms $\varphi_{ij} : X_{ij} \to X_{ji}$ such that

  - $\varphi_{ij} \circ \varphi_{ji} = \operatorname{id}$,
  - $X_{ii} = X_i$ and $\varphi_{ii} = \operatorname{id}$,
  - φ_{ij} induces isomorphisms $\varphi_{ijk} : X_{ij} \cap X_{ik} \to X_{ji} \cap X_{jk}$ such that $\varphi_{ijk} = \varphi_{kji} \circ \varphi_{ikj}$.

  Then there exists a rigid analytic space X with an admissible covering $\{X_i'\}$ such that $X_i' \cong X_i, X_i' \cap X_j' \cong X_{ij}$ and the diagram commutes:

$$
\begin{array}{ccc}
X_{ij} & \longrightarrow & X_i' \cap X_j' \\
\downarrow{\scriptstyle\varphi_{ij}} & & \| \\
X_{ji} & \longrightarrow & X_i' \cap X_j'
\end{array}
$$

- We can also glue morphisms: let X, Y be rigid analytic spaces, $\{X_i\}_{i \in I}$ an admissible cover of X and $\varphi_i : X_i \to Y$ such that $\varphi_i|_{X_i \cap X_j} : X_i \cap X_j \to Y$ coincides with $\varphi_j|_{X_i \cap X_j}$. Then there exists a unique map $\varphi : X \to Y$ such that $\varphi|_{X_i} = \varphi_i$.

- If X, Y are rigid analytic spaces and Y is affinoid then the natural map

$$\mathrm{Hom}(X, Y) \to \mathrm{Hom}(\mathcal{O}_Y(Y), \mathcal{O}_X(X))$$

  is bijective.

- The fibre product of X and Y exists (because of completed tensor product).

6.1 Rigid analytification

Analogous to the complex analytic case, we want to construct a functor that assigns to every K-scheme Z locally of finite type a rigid analytic space Z^{rig} called the *rigid analytification*. We begin by giving two hands-on examples.

Rigid analytic affinespace $\mathbb{A}_K^{n,\mathrm{rig}}$ Denote by

$$T_n(r) = \{\sum a_I x^I \subseteq K[[x]] : \lim |a_I r^{|I|}| = 0\} = K\langle r^{-1} x\rangle$$

For $r > 0$, $\mathrm{Sp}\, T_n(r) = B^n(r)$. For any $c \in K, |c| > 1$ we may consider

$$T_n = T_n(c^0) \supseteq T_n(c) \supseteq \cdots \supseteq K[x]$$

which corresponds to

$$B^n \hookrightarrow \mathrm{Sp}\, T_n(c) \hookrightarrow \cdots$$

By gluing, the union of these balls is a rigid analytic space $\mathbb{A}_K^{n,\mathrm{rig}} = \bigcup_{i=0}^{\infty} \mathrm{Sp}\, T_n(c^i)$. Claim $\mathrm{Max}\, K[x] = \bigcup_{i=0}^{\infty} \mathrm{Max}\, T_n(c^i)$.

Proof. Let $\mathfrak{m} \subseteq K\langle x\rangle$ maximal. Claim that $\mathfrak{m}' = \mathfrak{m} \cap K[x]$ is maximal and $\mathfrak{m} = \mathfrak{m}' \cdot K\langle x\rangle$.

Proof. For the first claim use the commutative diagram

$$
\begin{array}{ccc}
K[x] & \longrightarrow & K\langle x\rangle \\
\downarrow & & \downarrow \\
K \hookrightarrow K[x]/\mathfrak{m}' & \longrightarrow & K\langle x\rangle/\mathfrak{m} \cong K'
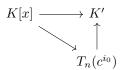\end{array}
$$

Since K' is a finite field extension of K, $K[x]/\mathfrak{m}'$ is a field.

For the second claim, consider

$$
\begin{array}{ccc}
K[x]/\mathfrak{m}' & \hookrightarrow & K\langle x\rangle/\mathfrak{m}' \cdot K\langle x\rangle \\
\| & & \downarrow \\
K[x]/\mathfrak{m}' & \hookrightarrow & K\langle x\rangle/\mathfrak{m}
\end{array}
$$

Since $K[x]$ is dense in $K\langle x\rangle$ and finite dimensional vector spaces are closed, the images of the horizontal maps are closed and dense. Thus the horizontal maps are surjections so isomrophisms. It follows that the vertical map on the right is also an isomorphism $\qquad\qquad\square$

Thus have an injection $\operatorname{Max} K\langle c^{-i}x\rangle \hookrightarrow \operatorname{Max} K[x]$. For the other direction, claim for any $\mathfrak{m}' \subseteq K[x]$, exists some integer i_0 such that $\mathfrak{m}' \cdot K\langle c^{-i_0}x\rangle$ is a maximal ideal.

Sketch proof. By Nullstellensatz for polynomial algebras, $K[x]/\mathfrak{m}' \cong K'$ is a finite extension of K. Extend the valuation uniquely to K' and choose an integer i_0 such that $\overline{x}_i \in K'$ satisfies $|\overline{x}_i| \le |c|^{i_0}$. Have a factorisation

$$
\begin{array}{ccc}
K[x] & \longrightarrow & K' \\
& \searrow & \uparrow \\
& & T_n(c^{i_0})
\end{array}
$$

and it is easy to check the assertion. $\qquad\square$

$\square$

Analytification of affine schemes Similarly as above, take $\mathfrak{a} \subseteq K[x]$ and consider the chain

$$T_n/\mathfrak{a} \leftarrow T_n(c)/\mathfrak{a} \leftarrow \cdots$$

where we also denote by $\mathfrak{a}$ its extension. Then $\bigcup_{i=0}^{\infty} \operatorname{Sp} T_n(c^i)/\mathfrak{a}$ is the analytification of $K[x]/\mathfrak{a}$.

> **Lemma 6.2.** *Let Z be an affine K-scheme of finite type (with Zariski topology), Y a rigid K-space. Then the natural map*
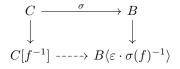>
> $$\operatorname{Hom}(Y, Z) \to \operatorname{Hom}_k(\mathcal{O}_Z(Z), \mathcal{O}_Y(Y)),$$
>
> *where LHS are morphisms in the category of locally G-ringed spaces, is a bijection.*
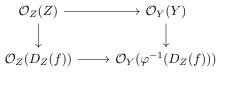
Sketch proof. Suffices to construct an inverse in the case $Y = \operatorname{Sp} B$ affinoid, $Z = \operatorname{Spec} C$ for $\sigma : C \to B$. Pulling back maximal ideals gives

$$\varphi : \operatorname{Max} B \to \operatorname{Max} C \hookrightarrow \operatorname{Spec} C.$$
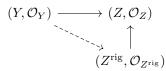
For $\varepsilon \in K, f \in C$, consider

$$
\begin{array}{ccc}
C & \xrightarrow{\ \sigma\ } & B \\
\downarrow & & \downarrow \\
C[f^{-1}] & \dashrightarrow & B\langle \varepsilon \cdot \sigma(f)^{-1}\rangle
\end{array}
$$

As $B\langle \varepsilon \cdot \sigma(f)^{-1}\rangle$ corresponds to the Laurent domain $X(\varepsilon^{-1} \cdot \sigma(f))$, by universal property of localisation we get a unique map completing the square. By varying ε we get

$$
\begin{array}{ccc}
\mathcal{O}_Z(Z) & \longrightarrow & \mathcal{O}_Y(Y) \\
\downarrow & & \downarrow \\
\mathcal{O}_Z(D_Z(f)) & \longrightarrow & \mathcal{O}_Y(\varphi^{-1}(D_Z(f)))
\end{array}
$$

$\square$

Definition (rigid analytification). Let $(Z, \mathcal{O}_Z)$ be a K-scheme locally of finite type. The *rigid analytification* of Z, Z^{rig}, is a rigid K-space $(Z^{\mathrm{rig}}, \mathcal{O}_{Z^{\mathrm{rig}}})$ together with a morphism $(Z^{\mathrm{rig}}, \mathcal{O}_{Z^{\mathrm{rig}}}) \to (Z, \mathcal{O}_Z)$ of locally G-ringed spaces with the following universal property: given any rigid analytic space $(Y, \mathcal{O}_Y)$, any morphism $(Y, \mathcal{O}_Y) \to (Z, \mathcal{O}_Z)$ factorises unqiuely through Z^{rig}.

$$
\begin{array}{ccc}
(Y, \mathcal{O}_Y) & \longrightarrow & (Z, \mathcal{O}_Z) \\
 & \searrow & \uparrow \\
 & & (Z^{\mathrm{rig}}, \mathcal{O}_{Z^{\mathrm{rig}}})
\end{array}
$$

Exercise. $\mathbb{A}_K^{n,\mathrm{rig}} \to \operatorname{Spec} K[x]$ and $\bigcup \operatorname{Spec} T_n(c^i)/\mathfrak{a} \to \operatorname{Spec} K[x]/\mathfrak{a}$ are rigid analytifications.

We conclude this chapter by giving two more examples.

Analytification of projective spaces $\mathbb{P}_K^n$ is covered by $n+1$ affine opens $U_i = \operatorname{Spec} K[\frac{x_0}{x_i}, \ldots, \frac{x_n}{x_i}]$. Take analytifications U_i^{rig} and glue them together to get $\mathbb{P}_K^{n,\mathrm{rig}}$. Check that $\mathbb{P}_K^{n,\mathrm{rig}}$ is already covered by $n+1$ balls $\operatorname{Sp} K\langle \frac{x_0}{x_i}, \ldots, \frac{x_0}{x_i} \rangle \subseteq U_i^{\mathrm{rig}}$: take $x \in \mathbb{P}_k^{n,\mathrm{rig}}$. It has residue field $L = K(x)$ some finite extension of K, so defines an L-valued point of $\mathbb{P}_K^n$. Represent x in homogeneous coordinates $x = (x_0 : \cdots : x_n)$. For some i, $\max_j |x_j| = |x_i|$. Then x factors through $\operatorname{Sp} K\langle \frac{x_0}{x_1}, \ldots, \frac{x_n}{x_i} \rangle$.

Multiplicative group variety The annulus $\{x \in T_1 : |a| \leq |x| \leq |b|\}$ where $a, b \in K$ is the intersection $X(\frac{a}{x}) \cap X(\frac{x}{b})$, which has coordinate ring $K\langle x, y \rangle / (xy - \frac{a}{b})$. The set of K'-rational points for any $K \subseteq K'$ corresponds to the annulus

$$\{z \in K' : |a| \leq |z| \leq |b|\}.$$

Define

$$\mathbb{G}_{m,K}^{\mathrm{rig}} = \bigcup_{n \in \mathbb{Z}} \operatorname{Sp} K\langle \tfrac{a^{n+1}}{x}, \tfrac{x}{a^n} \rangle$$

for some a such that $|a| < 1$.

Let X be a rigid K-space. Assign to X a contravariant functor from the category of affinoid algebras, send R/K to the set of morphisms $\operatorname{Sp} R \to X$ of rigid spaces over K. G has the form $R \mapsto R^*$.

Tate curve Recall that a Tate curve is given by $\mathbb{G}_{m,K}^{\mathrm{rig}}/q^{\mathbb{Z}}$ for some $q \in \overline{K}^*, |q| < 1$ (uniformisation). Pick $a \in K$, $|a|^k = |q|$ for some $k \geq 1$. We have the analytic covering $\mathbb{G}_{m,K}^{\mathrm{rig}} = \bigcup_{n \in \mathbb{Z}} A_n$ where $A_n = \operatorname{Sp} K\langle \frac{a^{n+1}}{x}, \frac{x}{a^n} \rangle$. Multiplication by q maps A_n isomorphically to A_{n+k}. We can glue the boundaries of the annuli together. The Tate curve, being the union of finitely many affinoids, is then quasicompact.

7 Rigid GAGA

Since the category of rigid K spaces has fibre products, we can define

Definition ((quasi)separated)**.** A rigid K-space X is *separated (quasiseparated respectively)* if $X \to X \times_K X$ is a closed immersion (a quasicompact morphism respectively).

$f : X \to Y$ is *separated* if $X \to X \times_Y X$ is closed immersion.

Proposition 7.1. *A rigid K-space X is separated if and only if it admits an admissible affinoid covering $\{U_i\}_{i \in I}$ such that for $i \neq j$ with $U_i \cap U_j \neq \emptyset$, the intersection $U_i \cap U_j$ is affinoid and the canonical map $\mathcal{O}(U_i)\hat{\otimes}\mathcal{O}(U_j) \to \mathcal{O}(U_i \cap U_j)$ is surjective.*

Proof. Same as schemes. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ $\square$

Example. All affinoids are separated.

The definition of properness, on the other hand, is a bit more subtle.

Definition (relatively compact)**.** Let X and Y be affinoid spaces with coordinate rings A and B respectively and let $f : X \to Y$ be a morphism. Let $U \cap X$ be an affinoid subdomain. We say U is *relatively compact in X over Y* if there exists an affinoid generating system $\{f_1, \ldots, f_n\}$ (i.e. a surjection $B\langle f_1, \ldots, f_n \rangle \twoheadrightarrow A$) such that $U \subseteq \{x \in X : |f_1(x)| \leq \varepsilon, \ldots, |f_n(x)| \leq \varepsilon\}$ for some $\varepsilon < 1$.

Definition (proper)**.** Let X and Y be rigid K-spaces and $f : X \to Y$ a morphism. Then f is *proper* if f is separated and satsifies the following condition: exists an affinoid covering $\{Y_i\}_{i \in I}$ of Y and for any $i \in I$, there exist two finite admissible coverings $\{X_{ij}\}_{j=0,\ldots,n_i}, \{X'_{ij}\}_{j=0,\ldots,n_i}$ of $f^{-1}(Y_i)$ such that X_{ij} is relatively compact in X'_{ij} over Y_i.

X is proper if it is proper over $\operatorname{Sp} K$.

For separated varieties, Čech cohomology agrees with sheaf cohomology.

We state here the rigid GAGA correspondence. The proof uses the rigid analogue of Cartan-Serre

Theorem 7.2 (Khiel)**.**

1. *Let X be proper over $\operatorname{Sp} K$. Then $H^i(X, \mathcal{F})$ are finite dimension over K for any coherent sheaf $\mathcal{F}$.*

2. *(Higher) direct image theorem: let X be proper, Y separated, $f : X \to Y$ a morphism and $\mathcal{F}$ a coherent sheaf. Then $f_*\mathcal{F}$ and higher direct images $R^i f_* \mathcal{F}$ are cohernet.*

Theorem 7.3 (rigid GAGA)**.** *Let X be a proper (maybe projective?) scheme over K. Then*

1. *For any coherent sheaf $\mathcal{F}$ over X, the canonical homomorphism be-*

tween coherent algebraic cohomology and analytic cohomology, induced by $f : (X^{\mathrm{rig}}, \mathcal{O}_{X^{\mathrm{rig}}}) \to (X, \mathcal{O}_X)$ is an isomorphism.

2. *The analytification functor from coherent sheaves on X to coherent sheaves on X^{rig} is fully faithful.*

3. *Every coherent analytic sheaf is isomorphic to the pullback of some algebraic coherent sheaf.*

2 and 3 together say analytification induces an equivalence of categories between coherent analytic and algebraic sheaves.

8 Formal models à la Raynaud

Let R be a complete DVR with a uniformiser π. Let $K = R[\frac{1}{\pi}]$. We quote the following facts:

1. An R-algebra A is topologically of finite type (tft) if it is isomorphic to $R\langle x_1, \ldots, x_n \rangle / I$ for some ideal I. We could do the same analytification process as in affinoids. Raynaud's theorem measures how far the functor is from being an equivalence of a category.

   Then $A \cong \varprojlim_n A/\pi^{n+1}A = \varprojlim A_n$

2. Define $\operatorname{Spf} A = \varinjlim \operatorname{Spec} A_n$, the set of open prime ideals of A with the natural π-adic topology. It has a structure sheaf

$$\mathcal{O}_{\operatorname{Spf} A} = \varprojlim \mathcal{O}_{\operatorname{Spec} A_n}.$$

   $(\operatorname{Spf} A, \mathcal{O}_{\operatorname{Spf} A})$ is a locally topologically ringed space, which is called an *affine tft formal scheme*.

3. Given $f \in A$, let $D(f)$ be the set of all open primes avoiding f. We have

$$\mathcal{O}_{\operatorname{Spf} A}(D(f)) = \hat{A}_f,$$

   the π-adic completion.

4. A locally tft formal scheme $\mathcal{X}$ is a locally topologically ringed space such that it has an open covering by affine tft formal schemes.

5. As usual there is an isomorphism

$$\operatorname{Hom}(\operatorname{Spf} A, \operatorname{Spf} B) \cong \operatorname{Hom}_{\operatorname{cont}}(B, A).$$

6. If we have $f : X \to Y$ of tft affine formal schemes where $X = \operatorname{Spf} A, Y = \operatorname{Spf} B$, then passing to the generic fibre gives

$$B \otimes_R K \to A \otimes_R K$$

   of affinoid K-algebras, thus a mophism $\operatorname{Sp} A \otimes K \to \operatorname{Sp} B \otimes K$ of rigid K-spaces. It induces a functor $(-)_{\operatorname{rig}}$ from the cateogry of locally tft formal schemes to the category of rigid K-spaces.

7. An R-algebra tft is *admissible* if it is flat, i.e. having no π-torsion. $\mathcal{X}$ is *admissible* if it is covered by admissible affine formal schemes.

8. Given a rigid K-space X, any admissible formal scheme $\mathcal{X}$ over R such that $\mathcal{X}_{\operatorname{rig}} \cong X$ is called a *formal model* of X.

9. A *coherent sheaf* on locally tft $\mathcal{X}$ is an $\mathcal{O}_{\mathcal{X}}$-module $\mathcal{F}$ such that there exists an open affine cover $\{\mathcal{X}_i\}$ such that $\mathcal{F}|_{\mathcal{X}_i} \cong \widehat{M_i}$, where M_i is a finitely generated $\mathcal{O}_{\mathcal{X}_i}(\mathcal{X}_i)$-modules, where $\widehat{M_i}$ is the sheaf induced by $D_f \mapsto M \otimes \hat{A}_f$.

10. A *formal blow-up* associated to a coherent open ideal sheaf $\mathcal{I} \subseteq \mathcal{O}_{\mathcal{X}}$ is

$$\mathcal{X}_{\mathcal{I}} = \varinjlim \operatorname{Proj}(\bigoplus_{d=0}^{\infty} \mathcal{I}^d \otimes_{\mathcal{O}_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}/\pi^n \mathcal{O}_{\mathcal{X}}).$$

We have a canonical projection $\mathcal{X}_{\mathcal{I}} \to \mathcal{X}$ with the universal property that for any $\mathcal{Y} \to \mathcal{X}$ such that $\mathcal{I} \cdot \mathcal{O}_{\mathcal{Y}}$ is invertible, $\mathcal{Y} \to \mathcal{X}$ factors uniquely throught $\mathcal{X}_{\mathcal{I}}$.

Theorem 8.1 (Raynaud). $(-)_{\mathrm{rig}}$ *induces an equivalence of categories between*

- *quasi-paracompact admissible formal R-schemes localised by the class of formal blow-ups and*

- *quasi-separated quasi-paracompact rigid K-spaces.*

Quasi-paracompact means there exists a covering (admissible respectively) $\{U_i\}$ of finite type (i.e. U_i is disjoint from almost all U_j) of quasicompact (admissible respectively) opens.

9 Adic spaces

9.1 Huber pairs

> **Definition** (Huber ring). A *Huber ring* (or *f*-adic ring in old literature) is a topological ring R such that there exists an open subring $R_0 \leq R$ and a finitely generated ideal $I \subseteq R_0$ such that the topology on R_0 is the I-adic topology. R_0 is called a *ring of definition* and I is an *ideal of definition*.

Remark. R_0 may not be complete with respect to the I-adic topology. One can always complete R_0 to get the completion $\hat{R}_0$. Then $\hat{R}$ is Huber with ring of definition $\hat{R}_0$. We will see that in the case of adic spaces it does not make any difference.

Example.

1. Schemes: any ring R with discrete topology, any subring $R_0 \leq R$ and $I = 0$.

2. Formal schemes: any adic ring R is Huber if it has a finitely generated ideal of definition. Then $R_0 = R$.

3. Rigid spaces: R_0 any ring, $g \in R_0$ a non-zero divisor and $R = R_0[\frac{1}{g}]$ with the topology making $\{g^n R_0\}$ the fundamental system of open neighbourhoods. Then R is Huber with ring of definition R_0 and gR_0 an ideal of definition.

   If R is a K-Banach algebra, we can take $R_0 \subseteq R$ to be the coordinate ring of the unit ball, g any non-zero element with norm smaller than 1. For example $T_n = K\langle x_1, \ldots, x_n \rangle, R_0 = \mathcal{O}_K \langle x_1, \ldots, x_n \rangle, g = \pi_K$.

> **Definition.** A subset $S \subseteq R$ of a topological ring R is *bounded* if for any open neighbourhood U of 0, exists open neighbourhood V of 0 such that $S \cdot V \subseteq U$.

Remark. In verifying this condition in Huber rings, we may assume that U is closed under addition as $\{I^n\}$ is a basis of open neighbourhoods of 0.

> **Lemma 9.1.** *A subring $R_0 \subseteq R$ of a Huber ring R is a ring of definition if and only if R_0 is open and bounded.*

Proof. We prove if and leave only if as an exercise. wlog we may assume that $U = I^n$ for some n. Then $V = I^n$ suffices. $\qquad\square$

> **Definition** (Tate ring). A Huber ring R is *Tate* if it contains a topological nilpotent unit $g \in R$, called a *pseudo-uniformiser* (topological nilpotent means $g^n \to 0$).

Proposition 9.2.

1. If $R = R_0[\frac{1}{g}]$ as in example 3 then R is Tate.

2. If R is Tate with topological nilpotent unit g, $R_0 \subseteq R$ any ring of definition, then exists n large enough such that R_0 is $(g^n R_0)$-adic. Moreover $R = R_0[\frac{1}{g^n}]$.

3. Let R, R_0, g be as above. A subset $S \subseteq R$ is bounded if and only if $S \subseteq g^k R_0$ for some k.

Proof.

1. Clear from definition.

2. Let $I \subseteq R_0$ be an ideal of definition. Since $g^n \to 0$, exists n such that $g' = g^n \in I$. $(g')^{-1} : R_0 \to R_0$ is a continuous map. Then $g' R_0$ is the preimage of R_0 under the map so is open and thus contains I^m for some m. Then
$$g^{nm} R_0 = (g')^m R_0 \subseteq I^m \subseteq g^n R_0$$
   so R_0 is g^n-adic.

   Clearly $R_0[\frac{1}{g'}] \subseteq R$. For any $x \in R, g^n x \to 0$ so exists k such that $(g')^k x \in I$, so in R_0. Thus $x \in R[\frac{1}{g'}]$.

3. Exercise.

$\square$

Definition (power-bounded)**.** Let R be a Huber ring. An element $x \in R$ is called *power-bounded* if $\{x^n : n \geq 0\}$ is bounded. Denote by $R^0 \subseteq R$ the set of power bounded elements and by R^{00} the subset of topological nilpotent elements.

Example. If $R = T_n$ then $R^0 = \mathcal{O}_K \langle x_1, \ldots, x_n \rangle$, which is the same as the ring of definition. In particular for $n = 0$ this agrees with the notation we gave in introduction.

Lemma 9.3.

1. R^0 is an integrally closed open subring of R.

2. R^0 is the colimit of all rings of definitions of R.

3. R^{00} forms an open ideal of R^0.

Proof.

1. Let $f, g \in R^0$. Then $f^{\mathbb{N}} \cdot g^{\mathbb{N}}$ is a bounded set, since if S, T are bounded sets then $S \cdot T$ is also bounded. Then since
$$f^{\mathbb{N}} \cdot g^{\mathbb{N}} = \{(f + g)^n\} \cup \{(fg)^n\},$$
   both $f + g$ and fg are power-bounded.

Suppose x is integral over R^0, i.e. $x^n + a_{n-1}x^{n-1} + \cdots + a_0 = 0$ where $a_{n-1}, \ldots, a_0 \in R^0$. Then

$$x^{\mathbb{N}} \subseteq a_0^{\mathbb{N}} \cdots a_{n-1}^{\mathbb{N}} \{1, x, \ldots, x^{n-1}\}.$$

Since finite sets are bounded, it follows that x is power-bounded.

2. Let $f \in R^0$. Then $R_0 f^{\mathbb{N}}$ is an R_0-subring of R and open. Since $R_0 \subseteq R_0 f^{\mathbb{N}}$, $R_0 f^{\mathbb{N}}$ is a ring of definition and $f \in R_0 f^{\mathbb{N}}$.

3. Exercise.

$\square$

Definition (Huber pair)**.** Let R be a Huber ring. A subring R^+ is called a *subring of integral elements* if it is open, integrally closed in R and $R^+ \subseteq R^0$.
 R is called *uniform* if R^0 is a ring of definition.
 A *Huber pair* (R, R^+) is a pair where R is a Huber ring and R^+ is a ring of integral elements.

We will often take $R^+ = R^0$. For example, all the rings that come from rigid geometry are uniform.
 Note $R^{00} \subseteq R^+$: for any $f \in R^{00}$, $f^n \in R^+$ for some $n \geq 1$ since R^+ is open. R^+ integrally closed implies $f \in R^+$.

9.2   Adic spectrum of a Huber pair

Definition (valuation)**.** A *valuation* on a ring R is the data of a totally ordered abelian group Γ (written multiplicatively) and a map $x : R \to \Gamma \cup \{0\}$ (with $0 < \gamma$ and $0 \cdot \gamma = 0$ for all $\gamma \in \Gamma$) such that

- $x(0) = 0, x(1) = 1$,

- $x(fg) = x(f) \cdot x(g)$,

- $x(f + g) \leq \max(x(f), x(g))$.

The *value group* $\Gamma_x \subseteq \Gamma$ is the subgroup generated by $x(R) \setminus \{0\}$.
 The *support* of x is the prime ideal $P_x = \{f \in R : x(f) = 0\}$.
 The *residue field* is $k(x) = \mathrm{Frac}\, R/P_x$. We can extend the valuation to $k(x)$ by

$$x : k(x) \to \Gamma \cup \{0\}$$
$$\frac{f}{g} \mapsto \frac{x(f)}{x(g)}.$$

$\mathcal{O}_x = \{f \in k(x) : x(f) \leq 1\} \supseteq R/P_x$ is the *valuation ring* of R.

Lemma 9.4. *Let* $x : R \to \Gamma_1 \cup \{0\}, y : R \to \Gamma_2 \cup \{0\}$ *be two valuations. then TFAE:*

 1. for any $f, g \in R$, $x(f) \leq x(g)$ *if and only if* $y(f) \leq y(g)$.

2. $\operatorname{supp} x = \operatorname{supp} y$ *and* $\mathcal{O}_x = \mathcal{O}_y$.

3. *there exists an isomorphism of totally ordered abelian groups* $i : \Gamma_{1,x} \to \Gamma_{2,x}$ *such that* $y = i \circ x$.

If any of the above holds then we say x and y are equivalent.

Proof. 3 $\implies$ 1 is obvious. 1 $\implies$ 2 is also immediate as $x(f) = 0$ if and only if $x(f) \leq x(0)$ and $\frac{f}{g} \in \mathcal{O}_x$ if and only if $x(f) \leq x(g)$. For 2 $\implies$ 3, x induces a surjection $k(x)^\times \twoheadrightarrow \Gamma_{1,x}$ with kernel $\mathcal{O}_x^\times$, similar for y. Since $P_x = P_y$, $k(x) = k(y)$ so we can define i via

$$
\begin{array}{ccc}
k(x)^\times / \mathcal{O}_x^\times & \xrightarrow{\cong} & \Gamma_{1,x} \\
\Big\| & & \Big\downarrow i \\
k(y)^\times / \mathcal{O}_y^\times & \xrightarrow{\cong} & \Gamma_{1,y}
\end{array}
$$

$\square$

Definition (adic spectrum of a Huber pair)**.** Let (R, R^+) be a Huber pair. Its asscociated *adic spectrum*, $\operatorname{Spa}(R, R^+)$, is the set of equivalence classes of valuations $x : R \to \Gamma \cup \{0\}$ such that

- $x(f) \leq 1$ for all $f \in R^+$.

- x is continuous with respect to the order topology, i.e. $\{f \in R : x(f) < \gamma\}$ is open for all $\gamma \in \Gamma$.

The topology on $\operatorname{Spa}(R, R^+)$ is generated by

$$
\{x \in \operatorname{Spa}(R, R^+) : x(f) \leq x(g) \neq 0\}
$$

for $f, g \in R$.

Sometimes we also denote the valuation $x(f)$ by $|f(x)|$, to imitate the notation for rigid spaces.

Example. Consider $\operatorname{Spa}(\mathbb{Z}, \mathbb{Z})$, where $\mathbb{Z}$ is given the discrete topology. Its points are

- a point η, which corresponds to the trivial valuation, i.e. taking all non-zero integers to 1.

- a special point s_p for each prime p, which corresponds to the composition $\mathbb{Z} \to \mathbb{F}_p \to \{0, 1\}$.

- a point η_p for each prime p, which corresponds to $\mathbb{Z} \to \mathbb{Z}_p \to p^{\mathbb{Z}} \cup \{0\}$.

These are all the points by Ostrowski (no archimedean valuation). $\{s_p\}$ are closed, $\overline{\{\eta_p\}} = \{\eta_p, s_p\}$ and $\overline{\{\eta\}} = \operatorname{Spa}(\mathbb{Z}, \mathbb{Z})$.

In general, for a discrete ring R, we have a map $\operatorname{Spec} R \to \operatorname{Spa}(R, R)$ which sends a prime ideal $\mathfrak{p}$ to its corresponding special point $s_{\mathfrak{p}}$. It has a left inverse

$$
\operatorname{Spa}(R, R) \to \operatorname{Spec} R
$$

$$
x \mapsto \operatorname{supp} x
$$

Example. Assume K is a non-archimedean field with absolute value $|\cdot|$ and let $R = K\langle T \rangle, R^+ = R^0 = \mathcal{O}_K\langle T \rangle$. In general there are five types of points of $\mathrm{Spa}(R, R^+)$:

- type I points (classical points): they come from rigid geometry and correspond to points of $\mathrm{Sp}\, R$. For any $\lambda \in \mathcal{O}_K = K^0$, i.e. $|\lambda| \leq 1$, for any $f \in R$, evaluating f and λ and taking absolute value gives

$$f = \sum a_i T^i \mapsto \sum a_i \lambda^i \mapsto |f(\lambda)|.$$

  This is obviously a continuous valuation and for any $f \in R^0$, $|f| \leq 1$.

- type II + III points: let $r \in \mathbb{R}, 0 \leq r \leq 1, \lambda \in K^0$, then

$$f = \sum a_n(T - \lambda)^i \mapsto \sup_{x \in \overline{B}(\lambda, r)} |f(x)| = \sup_n |a_n| \cdot r^n$$

  is a continuous valuation and for $f \in R^0$, $|f| \leq 1$. Thus "every closed disc gives a point". If $r \in |K^\times|$ then the corresponding valuation is of type II, otherwise of type III (irrational discs). If $r = 0$ we get the classical points. If $r = 1$ every λ is a centre and the corresponding point, independent of λ, is called the *Gauss point*.

- type IV points: if K is not spherically complete (e.g. $\mathbb{C}_p$) then we have a decreasing sequence

$$D_1 \supseteq D_2 \supseteq D_3 \supseteq \cdots$$

  with $\bigcap D_i = \emptyset$. We get a correponding valuation

$$f \mapsto \inf_i \sup_{x \in D_i} |f(x)|.$$

- type V points: these are only visible in the theory of adic spaces. Let $\lambda \in K^0$ and fix $r \in \mathbb{R}, 0 < r \leq 1$ and define a totally ordered abelian group

$$\Gamma_{<r} = \mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$$

  endowed with the unique total ordering such that $r' < \gamma < r$ for all $r' < r$. Then

$$x_{<r} : f = \sum a_n(T - \lambda)^n \mapsto \sup |a_n|\gamma^n$$

  is a continuous valuation. Similar define

$$\Gamma_{>r} = \mathbb{R}_{>0} \times \gamma^{\mathbb{Z}}$$

  with $r < \gamma < r'$ for all $r < r'$ and get

$$x_{>r} : f = \sum a_n(T - \lambda)^n \mapsto \sup |a_n|\gamma^n$$

  By definition if $r \notin |K^\times|$ then $x_{<r} = x_{>r} = x_r$, corresponding to type III point associated to $\overline{B}(\lambda, r)$. For each type II point we get an additional type V point for each branching.

The subspace of $\mathrm{Spa}(R, R^+)$ of points of type I to IV is the *Berkovich space*. The classification theorem of Berkovich says we get all rank 1 valuations.

Note. If Γ is a totally ordered abelian group, we define a convex subgroup Δ to be a subgroup such that if $\gamma \le \delta \le \gamma'$ and $\gamma, \gamma' \in \Delta$ then $\delta \in \Delta$. The number of non-trivial convex subgroups of Γ for a valuation $x : R \to \Gamma \cup \{0\}$ is called its *rank*. For example Γ has rank 0 if and only if $\Gamma = 1$. $\mathbb{R}_{\ge 0}$ or any subgroup have rank 1.

There is a nice way to visualise $\mathrm{Spa}(K\langle T \rangle, K^0 \langle T \rangle)$. Let $X_r, X_{r'}$ correspond to $\overline{B}(a, r), \overline{B}(b, r')$. There are two possibilities:

1. $\overline{B}(a, r) \subseteq \overline{B}(b, r')$ (or the other way round): there exists a unique path which consists of all closed balls that contain $\overline{B}(a, r)$ and contained in $\overline{B}(b, r')$. For example if $a = b$ then the path just corresponds to $[r, r'] \subseteq \mathbb{R}$.

2. $\overline{B}(a, r) \cap \overline{B}(b, r') = \emptyset$: this implies $r, r' \le |a - b|$. Then the smallest closed ball containing both $\overline{B}(a, r)$ and $\overline{B}(b, r')$ is just $\overline{B}(a, |a-b|) = \overline{B}(b, |a-b|)$. The corresponding point is denoted by $X_r \vee X_{r'}$. So the unique map $X_r \rightsquigarrow X'_r$ is the concatenation $X_r \rightsquigarrow X_r \vee X_{r'} \rightsquigarrow X_{r'}$. In other words, we have all the closed ball $\overline{B}(a, t)$ such that $r \le t \le |a - b|$, or $\overline{B}(b, t')$ such that $r' \le t' \le |a - b|$.

(pic)

Definition (rational subset)**.** Let (R, R^+) be a Huber pair. Let $U \subseteq \mathrm{Spa}(R, R^+)$. We say U is a *rational subset* of $\mathrm{Spa}(R, R^+)$ if there exist $f_1, \ldots, f_n \in R$ which generate an open ideal and $g \in R$ such that

$$U = \mathrm{Spa}(R, R^+)(\tfrac{f_1, \ldots, f_n}{g}) = \{x \in \mathrm{Spa}(R, R^+) : |f_i(x)| \le |g(x)| \ne 0\}.$$

Remark. For Tate-Huber pair, the only open ideal of R is R itself, since for any open ideal I, some power of the pseudo-uniformiser is in I.

Lemma 9.5. *Let (R, R^+) be a Huber pair. Then rational subsets are open in $\mathrm{Spa}(R, R^+)$ and are closed under finite intersection.*

Proof. Let $U_1, \ldots, U_k$ be rational subsets. Almost by definition, $U_1 \cap \cdots \cap U_k$ is open. By the defintion of valuation on (R, R^+), it is clear that if U, U' are rational given by $f_1, \ldots, f_n, g$ and $f'_1, \ldots, f'_k, g'$ respectively then

$$U \cap U' = \mathrm{Spa}(R, R^+)(\tfrac{f_i \cdot f'_j}{gg'}).$$

Note that $\{f_i \cdot f'_j : 1 \le i \le n, 1 \le j \le k\}$ does generate an open ideal as

$$I^n \subseteq (f_1, \ldots, f_n), I^m \subseteq (f'_1, \ldots, f'_k)$$

implies $I^{n+m} \subseteq (f_i \cdot f'_j)$. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \square$

Definition (morphism of Huber pairs)**.** A *morphism* of Huber pairs $(R, R^+) \to (S, S^+)$ is a countinuous ring homomorphism $\varphi : R \to S$ such that $\varphi(R^+) \subseteq S^+$.

It is clear that such a morphism induces a continuous map $\mathrm{Spa}(S, S^+) \to \mathrm{Spa}(R, R^+)$.

Definition (spectral topological space)**.** A topological space is *spectral* it is quasicompact, has a basis of quasicompact open subsets that are closed under finite intersections and is sober (every irreducible subset has a unique generic piont).

It can be shown that it is equivalent to being homeomorphic to the spectrum of a ring.

The upshot is

Theorem 9.6. *Let* (R, R^+) *be a Huber pair.*

1. *The topological space* $\mathrm{Spa}(R, R^+)$ *is spectral.*

2. *The rational subsets form a basis of for the topology on* $\mathrm{Spa}(R, R^+)$.

3. *Any rational subset is quasicompact.*

4. $\mathrm{Spa}(R, R^+) = \emptyset$ *if and only if the topology on* $\mathrm{Spa}(R, R^+)$ *is trivial.*

5.

$$R^+ = \{f \in R : x(f) \leq 1 \text{ for all } x \in \mathrm{Spa}(R, R^+)\}$$
$$R^{00} = \{f \in R : x(f) < 1 \text{ for all } x \in \mathrm{Spa}(R, R^+)\}$$

Proof. Huber *Continuous Valuation* section 2, 3. $\qquad\square$

Note. The defining opens in $\mathrm{Spa}(R, R^+)$ are not quasicompact in general. Consider a Tate-Huber pair with pseudo-uniformiser $t \in R$, then

$$\mathrm{Spa}(R, R^+)(\tfrac{f}{g}) = \bigcup_n \mathrm{Spa}(R, R^+)(\tfrac{f, t^n}{g}),$$

a union of rational subsets since $\{f, t^n\}$ generates an open ideal. $|g(x)| \neq 0$ implies that $|g(x)| > |t^n(x)|$ for $n \gg 0$. Typically it is no a finite union (exercise: look at the Zariski open corresponding to $f = g$).

9.3 Completion of Huber pairs

Recall that for a metric space X, its completion $\hat{X}$ is defined as the quotient of all Cauchy sequences under the equivalence relation $(x_n) \sim (y_n)$ if and only if $\lim_n d(x_n, y_n) = 0$.
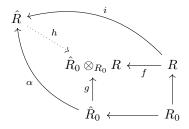
Similarly for a topological ring R with an ideal of definition I, a sequence (x_n) is Cauchy if for all n_0, $x_n - x_m \in I^{n_0}$ for all $n, m \gg 0$. The set of all Cauchy sequences form a ring. Let I_0 be the ideal of all Cauchy sequences such that all but finitely many x_n's are zero. Then define $R \to \hat{R} = \{\text{Cauchy sequences}\}/I_0$ to be the completion of R.

Proposition 9.7. $\hat{R}$ *is a Huber ring. If* $I \subseteq R_0$ *is an ideal (resp. ring) of definition of R then*

1. $I \cdot \hat{R}_0 \subseteq \hat{R}_0$ *is an ideal (resp. ring) of definition in* $\hat{R}$, *where* $\hat{R}_0 = \varprojlim R_0/I^n$.

2. *the canonical map $\hat{R}_0 \otimes_{R_0} R \to \hat{R}$ is an isomorphism.*

3. *$\hat{R}^+$ is a subring of integral elements of $\hat{R}$ and $(\hat{R}, \hat{R}^+)$ is called the completion of (R, R^+).*

Proof. 1 is well-known (see stacks project). To define the map in 2, consider the commutative diagram

$i(R)$ is dense in $\hat{R}$ and $\hat{R}_0$ is open in $\hat{R}$. Thus $\hat{R} = i(R) + \hat{R}_0$ and $i^{-1}(\hat{R}_0) = R_0$. Thus the outer square is also a pushout so exists unique h such that $h \circ i = f, h \circ \alpha = g$. $\square$

Corollary 9.8. $(R, R^+) \to (\hat{R}, \hat{R}^+)$ *is a morphism of Huber pairs.*

Theorem 9.9. $\mathrm{Spa}(\hat{R}, \hat{R}^+) \to \mathrm{Spa}(R, R^+)$ *is a homeomorphism. Moreover, rational subsets are mapped to rational subsets.*

Proof. Theorem 3.9 in Huber, Continuous Valuations. $\square$

Proposition 9.10. *Let (R, R^+) be a complete Huber pair. Then*

1. *$R^\times = \{f \in R : x(f) \neq 0 \text{ for all } x \in \mathrm{Spa}(R, R^+)\}$.*

2. *if $I \subseteq R$ is a proper ideal then exists $x \in \mathrm{Spa}(R, R^+)$ such that $x(f) = 0$ for all $f \in I$.*

Proof. For the first part $\subseteq$ is clear. For the other direction, assume $f \in R$ is a non-unit. Then exists $\mathfrak{m} \subseteq R$ maximal containing f. We show that $R \setminus \mathfrak{m}$ contains $1 + R^{00}$, and therefore $\mathfrak{m}$ is closed. Claim $R^\times$ contains $1 + R^{00}$: let $I \subseteq R_0$ be an ideal (resp. ring) of definition in R. Then if $g \in R^{00}$ then $g^m \in I$ for some $m \in \mathbb{N}$. R_0 is I-adic complete by assumption so $1 - g^m \in R_0^\times \subseteq R^\times$. But

$$(1 - g)(1 + g + \cdots + g^{m-1}) = 1 - g^m$$
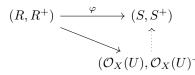
and therefore $1 - g \in R^\times$.

Let $K = R/\mathfrak{m}$ and K^+ be the integral closure of R^+ in K. Consider the pair (K, K^+). Since $\mathfrak{m}$ is closed, the topology on K is separable. Thus by Theorem 9.6 4, $\mathrm{Spa}(K, K^+) \neq \emptyset$. Then the image of an $x \in \mathrm{Spa}(K, K^+)$ satisfies $x(f) = 0$.

For the second part apply the same argument to a maximal $\mathfrak{m}$ containing I. $\square$

Theorem 9.11. *Let $U \subseteq \operatorname{Spa}(R, R^+)$ be a rational subset. Then exists a complete Huber pair $(R, R^+) \to (\mathcal{O}_X(U), \mathcal{O}_X(U)^+)$ over (R, R^+) such that the map*

$$\operatorname{Spa}(\mathcal{O}_X(U), \mathcal{O}_X(U)^+) \to \operatorname{Spa}(R, R^+)$$

factors through U and has the unique property that if we have a morphism $\varphi : (R, R^+) \to (S, S^+)$ to a complete Huber pair such that the image of $\operatorname{Spa}(S, S^+)$ is contained in U then we have a unique map making the following diagram commute:

$$
\begin{array}{ccc}
(R, R^+) & \xrightarrow{\quad \varphi \quad} & (S, S^+) \\
 & \searrow & \uparrow \\
 & & (\mathcal{O}_X(U), \mathcal{O}_X(U)^+)
\end{array}
$$

Here the notation $\mathcal{O}_X(U)$ is purely formal but should prompt one to think about the possibility of attaching a structure sheaf of adic spectrum.

Sketch proof. Let $f_1, \ldots, f_n, g$ be elements that define U. Let $I \subseteq R_0$ be an ideal (resp. ring) of definition in R. Take $\varphi : (R, R^+) \to (S, S^+)$ as in statement. Then by part 1 of the above proposition, g is invertible in S and therefore we have a unique ring homomorphism $R[\frac{1}{g}] \to S$. By definition all $\frac{f_i}{g}$ have image in S such that $x(\frac{\varphi(f_i)}{\varphi(g)}) \leq 1$ for all $x \in \operatorname{Spa}(S, S^+)$, so $\frac{\varphi(f_i)}{\varphi(g)} \in S^+ \subseteq S^0$ by Theorem 9.6 5. Recall S^0 is the colimit of all rings of definitions so we can choose some ring of definition S_0 such that $\frac{\varphi(f_i)}{\varphi(g)} \in S_0$ for all i.

We get a ring homomorphism $R_0[\frac{f_1, \ldots, t_n}{g}] \to S_0$. Endow it with the $I \cdot R_0[\frac{f_1, \ldots, t_n}{g}]$-adic topology. Technical claim: this defines a topology on $R[\frac{1}{g}]$ making $I \cdot R_0[\frac{f_1, \ldots, f_n}{g}] \subseteq R_0[\frac{f_1, \ldots, f_n}{g}]$ an ideal (resp. ideal) of definition, and $R[\frac{1}{g}]$ is a Huber ring.

Define $R[\frac{1}{g}]^+$ to be the integral closure of $R^+[\frac{f_i}{g}]$ in $R[\frac{1}{g}]$ and $(R\langle \frac{f_1, \ldots, f_n}{g} \rangle, R\langle \frac{f_1, \ldots, f_n}{g} \rangle^+)$ be its completion. By construction we have an induced morphism of Huber pairs $(R\langle \frac{f_1, \ldots, f_n}{g} \rangle, R\langle \frac{f_1, \ldots, f_n}{g} \rangle^+) \to (S, S^+)$. $\qquad \square$

Definition. Define the *structure presheaf of topological rings* on $X = \operatorname{Spa}(R, R^+)$ as follow: if $U \subseteq X$ is rational then let

$$\mathcal{O}_X(U) = R\langle \frac{f_1, \ldots, f_n}{g} \rangle$$
$$\mathcal{O}_X(U)^+ = R\langle \frac{f_1, \ldots, f_n}{g} \rangle^+$$

For general open W, define

$$\mathcal{O}_X(W) = \varprojlim_{U \subseteq W} \mathcal{O}_X(U)$$

and $\mathcal{O}_X(W)^+$ similarly.

Observe that since

$$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) : x(f) \leq 1 \text{ for all } x \in U\},$$

$\mathcal{O}_X^+$ is a sheaf if $\mathcal{O}_X$ is a sheaf.

Theorem 9.12 (Tate, Bosch-Günter-Remert, Huber). *Let (R, R^+) be a Huber pair and let $X = \mathrm{Spa}(R, R^+)$. Assume*

1. *either R is Tate-Huber and is strongly noetherian (i.e. $R\langle x_1, \ldots, x_n \rangle$ is noetherian for all $n \geq 0$),*

2. *or R^+ has a subring of definition which is noetherian,*

then $\mathcal{O}_X$ is an acyclic sheaf. X is called sheafy.

The most recent result when this set of notes is taken is

Theorem 9.13 (Buzzard-Verberkmoes). *Let (R, R^+) be a Tate-Huber pair which is stably uniform (i.e. all rational subsets $U \subseteq X$ are uniform). Then $\mathcal{O}_X$ is an acyclic sheaf.*

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