University of
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Mathematics Tripos

Part IB

Methods

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Contents

[1] Don't panic: it's always OK.
[2] Do panic, it doesn't always work!

1 Fourier Series

1.1 Periodic functions

A function $f(t)$ is *periodic* with period T if $f(t+T) = f(t)$. Consider $A \sin \omega t$, where A is the amplitude, ω is the frequency and $2\pi/\omega = T$ is the period. Sines and cosines have an orthogonality property:

$$\begin{aligned}
\cos(A \pm B) &= \cos A \cos B \mp \sin A \sin B \\
\cos A \cos B &= \frac{1}{2}[\cos(A-B) + \cos(A+B)] \\
\sin A \sin B &= \frac{1}{2}[\cos(A-B) - \cos(A+B)]
\end{aligned}$$

Consider $\sin \frac{n\pi x}{L}, \sin \frac{m\pi x}{L}$ where n, m are non-negative integers. These functions are periodic with period $2L$.

$$\begin{aligned}
ss_{m,n} &:= \int_0^{2L} \sin \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx \\
&= \frac{1}{2} \int_0^{2L} \cos \frac{(m-n)\pi x}{L} - \cos \frac{(m+n)\pi x}{L} dx \\
&= \frac{L}{2\pi} \left[\frac{\sin \frac{(m-n)\pi x}{L}}{m-n} - \frac{\sin \frac{(m+n)\pi x}{L}}{m+n} \right]_0^{2L} \\
&= \begin{cases} 0 & \text{if } m \neq n \\ L & \text{if } m = n \neq 0 \\ 0 & \text{if } m = n = 0 \end{cases}
\end{aligned}$$

Thus

$$ss_{m,n} = \begin{cases} L\delta_{mn} & \text{if } n \neq 0 \\ 0 & \text{if } m = 0 \text{ or } n = 0 \end{cases}$$

Similarly

$$cc_{m,n} := \int_0^{2L} \cos \frac{m\pi x}{L} \cos \frac{n\pi x}{L} dx = \begin{cases} 2L & \text{if } m = n = 0 \\ L\delta_{mn} & \text{otherwise} \end{cases}$$

Finally,

$$\begin{aligned}
cs_{mn} &:= \int_0^{2L} \cos \frac{m\pi x}{L} \sin \frac{n\pi x}{L} dx \\
&= \frac{1}{2} \int_0^{2L} \frac{\sin(m+n)\pi x}{L} - \frac{\sin(m-n)\pi x}{L} dx \\
&= 0
\end{aligned}$$

By analogy with vectors (these integrals are *inner products*), $\sin \frac{n\pi x}{L}$ and $\cos \frac{n\pi x}{L}$ are said to be *orthogonal* on the interval $[0, 2L]$. Actually they constitute an *orthogonal basis*, i.e. it is possible to represent an arbitray (but sufficiently well-behaved) function in terms of an infinite series (Fourier series) formed as a sum of sines and cosines.

1.2 Definition of Fouries Series

Any "well-behaved" (to be defined later) periodic function $f(x)$ with period $2L$ can be written as a Fourier series:

$$\frac{f(x_+) + f(x_-)}{2} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right) \qquad (*)$$

where a_n and b_n are the *Fourier coefficients* and $f(x_+)$ and $f(x_-)$ are the right limit approaching from above and the left limit approaching from below respectively.

1. If $f(x)$ is continuous at x_c, the LHS is just $f(x)$.
2. if $f(x)$ has a *bounded* discontinuity at x_d, i.e., $|f(x_d^-) - f(x_d^+)|$ is non-zero but finite, the Fourier series tends to the mean value of the two limits.

1.3 Coefficient Construction

Multiply RHS of equation $(*)$ by $\sin\frac{m\pi x}{L}$, integrate over $[0, 2\pi]$. Assume we can exchange summation and integration,

$$\begin{aligned}
&\int_0^{2L} f(x)\sin\frac{m\pi x}{L}dx \\
=&\int_0^{2L}\left[\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)\right]\sin\frac{m\pi x}{L}dx \\
=&0 + \sum_{n=1}^{\infty}a_n\int_0^{2L}\cos\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx + \sum_{n=1}^{\infty}b_n\int_0^{2L}\sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L}dx \\
=&0 + 0 + Lb_m
\end{aligned}$$

Thus

$$b_m = \frac{1}{L}\int_0^{2L} f(x)\sin\frac{m\pi x}{L}dx$$

Similarly, multiply by $\cos\frac{m\pi x}{L}$ and integrate (include $m = 0$), we get

$$\int_0^{2L} f(x)\cos\frac{m\pi x}{L}dx = \int_0^{2L}\left[\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi x}{L} + b_n\sin\frac{n\pi x}{L}\right)\right]\cos\frac{m\pi x}{L}dx$$

The first term is non-zero only when $m = 0$. Therefore

$$\frac{a_0}{2}\cdot 2L = \int_0^{2L} f(x)dx$$

so

$$a_0 = \frac{1}{L}\int_0^{2L} f(x)dx.$$

The second term gives

$$a_m = \frac{1}{L}\int_0^{2L} f(x)\cos\frac{m\pi x}{L}dx.$$

The range of integration is one period so it is also permissible to choose $-L$ and L as the limit of integration. A particularly neat case is when $L = \pi$:

$$a_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\cos mx dx, m \geq 0$$
$$b_m = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x)\sin mx dx, m \geq 1$$

1.4 Dirichlet Conditions

If $f(x)$ is a periodic function with period $2L$ such that

1. it is absolutely integrable, i.e. $\int_0^{2L} |f(x)|dx$ is well-defined,
2. it has a finite number of extrema (i.e. maxima and minima) in $[0, 2L]$,
3. it has a finite number of *bounded* discontinuities in $[0, 2L]$,

then the Fourier series converges to $f(x)$ for all points where $f(x)$ is continuous and at points x_d where $f(x)$ is discontinuous, the series converges to the average value of the left and right limit, i.e. $(f(x_d^+) + f(x_d^-))/2$.

These conditions are satisfied if the function is of *bounded variation.*

Remark. The Fourier series converges but not necessarily uniformly converges. This gives rises to some weird behaviour.

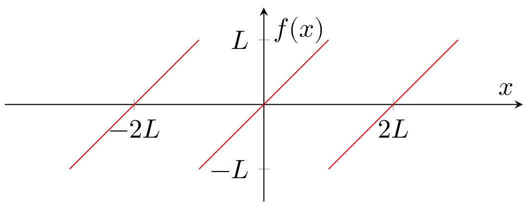
1.4.1 Smoothness and Order of Fourier Coefficients

If the pth derivative is the lowest derivative which is discontinuous somewhere (including the endpoints), then the Fourier coefficients are $O(n^{-(p+1)})$ as $n \to \infty$.

For example, if a function has a bouned discontinuity, the 0th derivative is discontinuous: coefficients are of order $\frac{1}{n}$ as $n \to \infty$.

Example.

1. The sawtooth function $f(x) = x$ for $-L \leq x < L$.



The function is odd so

$$\begin{aligned}
a_m &= \frac{1}{L}\int_{-L}^{L} x\cos\frac{m\pi x}{L}dx = 0\\
b_m &= \frac{1}{L}\int_{-L}^{L} x\sin\frac{m\pi x}{L}dx\\
&= \frac{1}{L}\Big[x\frac{L}{m\pi}(-\cos\frac{m\pi x}{L})\Big]_{-L}^{L} - \frac{1}{L}\frac{L}{m\pi}\int_{-L}^{L} -\cos\frac{m\pi x}{L}dx\\
&= \frac{1}{m\pi}\Big(L(-\cos m\pi) - (-L)(-\cos m\pi)\Big) + \frac{1}{m\pi}\Big[\frac{L}{m\pi}\sin\frac{m\pi x}{L}\Big]_{-L}^{L}\\
&= \frac{2L}{m\pi}(-1)^{m+1}
\end{aligned}$$

So

$$\frac{f(x_+) + f(x_-)}{2} = \frac{2L}{\pi}\Big[\sin\frac{\pi x}{L} - \frac{1}{2}\sin\frac{2\pi x}{L} + \cdots\Big].$$

and we observe that

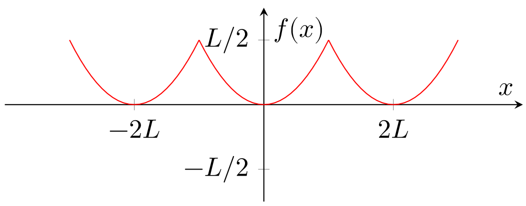
(a) $f_N(x) = \sum_{n=1}^{N} b_n \sin\frac{n\pi x}{L} \to f(x)$ almost everywhere but the convergence is *not* uniform.

(b) Persistent overshooting at $x = L$: Gibbs phenomenon.

(c) $f(L) = 0$, the average of right and left hand limit.

(d) Coefficients are $O(\frac{1}{n})$ as $n \to \infty$.

2. The integral of the sawtooth function, $f(x) = \frac{x^2}{2}$ for $-L \leq x < L$.



Exercise: $f(x) = L^2\big[\frac{1}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{(n\pi)^2}\cos\frac{n\pi x}{L}\big]$.

Note at $x = 0$,

$$\begin{aligned}
0 &= L^2\Big[\frac{1}{6} + 2\sum_{n=1}^{\infty}\frac{(-1)^n}{(n\pi)^2}\Big]\\
\Rightarrow \frac{\pi^2}{12} &= \sum_{n=1}^{\infty}\frac{(-1)^{n+1}}{n^2}
\end{aligned}$$

2 Properties of Fourier Series

2.1 Integration & Differentiation

2.1.1 Integration[1]

Fourier series *can* be integrated term by term. Suppose $f(x)$ with period $2L$ has a Fourier series (so it satisfies the Direichlet condition):

$$\frac{f(x_+) + f(x_-)}{2} = \frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L}\right).$$

Consider

$$\begin{aligned}F(x) &= \int_{-L}^{x} f(u)du \\ &= \frac{a_0(x+L)}{2} + \sum_{n=1}^{\infty}\frac{a_n L}{n\pi}\sin\frac{n\pi x}{L} + \sum_{n=1}^{\infty}\frac{b_n L}{n\pi}\left[(-1)^n - \cos\frac{n\pi x}{L}\right] \\ &= \frac{a_0 L}{2} + L\sum_{n=1}^{\infty}(-1)^n\frac{b_n}{n\pi} - L\sum_{n=1}^{\infty}\frac{b_n}{n\pi}\cos\frac{n\pi x}{L} \\ &+ L\sum_{n=1}^{\infty}\frac{a_n - (-1)^n a_0}{n\pi}\sin\frac{n\pi x}{L}\end{aligned}$$

If a_n, b_n are Fourier coefficients, then series involving $\frac{a_n}{n}, \frac{b_n}{n}$ (multiplied by sine or cosine) must also converge, so they are part of a Fourier series. Fourier series of $f(x)$ exists so b_n is at least of $O(\frac{1}{n})$ as $n \to \infty$. Thus $\frac{b_n}{n}$ is at least of order $O(\frac{1}{n^2})$ as $n \to \infty$ and so by comparison test with $\sum \frac{M}{n^2}$ the second term converges, so $F(x)$ has a Fourier series.

Note. Integration smoothes. The proof relies on discontinuity being bounded ($f(x)$ satisfies the Dirichlet condition).

2.1.2 Differentiation[1]

Let $f(x)$ be a periodic function with period 2 such that

$$f(x) = \begin{cases} 1 & 0 < x < 1 \\ -1 & -1 < x < 0 \end{cases}$$

[1] Don't panic: it's always OK.
[1] Do panic, it doesn't always work!

This is an odd function so

$$\begin{aligned}
a_m &= 0 \\
b_m &= -\int_{-1}^{0} \sin m\pi x dx + \int_{-1}^{0} \sin m\pi x dx \\
&= \frac{\cos m\pi x}{m\pi}\Big|_{-1}^{0} - \frac{\cos m\pi x}{m\pi}\Big|_{0}^{-1} \\
&= \frac{1}{m\pi}(1 - (-1)^m - (-1)^m + 1) \\
&= \begin{cases} \frac{4}{m\pi} & m \text{ is odd} \\ 0 & m \text{ is even} \end{cases}
\end{aligned}$$

Thus

$$\frac{f(x_+) + f(x_-)}{2} = \frac{4}{\pi}\sum_{n=1}^{\infty} \frac{\sin(2n-1)\pi x}{2n-1}.$$

Apply differentiation rules

$$f'(x) = 4\sum_{n=1}^{\infty} \cos(2n-1)\pi x$$

This is clearly divergent even though $f'(x) = 0$ for all $x \neq 0$. The extra factor of $2n - 1$ is the problem, which is related to the discontinuity. $f'(x)$ *does not* satisfy the Dirichlet condition.

Note. Intuitively, this behaviour can be explained by noticing that Dirac delta function is the derivative of Heaviside function.

2.1.3 Differentiation Under Certain Circumstances

Fourier series can be differentiated under certain circumstances:

Example. Assume $f(x)$ is continuous and is extended as a $2L$-periodic function, piecewise continuously differentiable on $(-L, L)$. Let $g(x) = \frac{df}{dx}$. $g(x)$ satisfies the Dirichlet condition as it has at worst a finite number of bounded discontinuities.

$$\begin{aligned}
f(x) &= \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \\
\frac{g(x_+) + g(x_-)}{2} &= \frac{A_0}{2} + \sum_{n=1}^{\infty} A_n \cos\frac{n\pi x}{L} + B_n \sin\frac{n\pi x}{L}
\end{aligned}$$

Then

$$\begin{aligned}
A_0 &= \frac{1}{L}\int_0^{2L} g(x)dx \\
&= \frac{f(2L) - f(0)}{L} \\
&= 0
\end{aligned}$$

by periodicity.

$$\begin{aligned} A_n &= \frac{1}{L}\int_0^{2L} \frac{df}{dx}\cos\frac{n\pi x}{L}dx \\ &= \frac{1}{L}\Big[f(x)\cos\frac{n\pi x}{L}\Big]\Big|_0^{2L} + \frac{n\pi}{L^2}\int_0^{2L} f(x)\sin\frac{n\pi x}{L}dx \\ &= 0 + \frac{n\pi b_n}{L} \end{aligned}$$

Similarly, $B_n = -\frac{n\pi a_n}{L}$.

This reduces differentiation to multiplication by $\pm\frac{n\pi}{L}$.

2.2 Alternate Representation: Complex Form

Recall

$$\begin{aligned} \cos\frac{n\pi x}{L} &= \frac{1}{2}(e^{i\frac{n\pi x}{L}} + e^{-i\frac{n\pi x}{L}}) \\ \sin\frac{n\pi x}{L} &= \frac{1}{2i}(e^{i\frac{n\pi x}{L}} - e^{-i\frac{n\pi x}{L}}) \end{aligned}$$

so

$$\begin{aligned} \frac{f(x_+)+f(x_-)}{2} &= \frac{a_0}{2} + \sum_{n=1}^{\infty}\frac{a_n}{2}(e^{i\frac{n\pi x}{L}} + e^{-i\frac{n\pi x}{L}}) - \sum_{n=1}^{\infty}\frac{b_n}{2}(e^{i\frac{n\pi x}{L}} - e^{-i\frac{n\pi x}{L}}) \\ &= \frac{a_0}{2} + \sum_{n=1}^{\infty}(\frac{a_n - ib_n}{2}e^{i\frac{n\pi x}{L}}) + \sum_{n=1}^{\infty}(\frac{a_n+ib_n}{2}e^{-i\frac{n\pi x}{L}}) \\ &= \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}} \end{aligned}$$

with

$$\begin{aligned} c_0 &= \frac{a_0}{2}, \\ c_n &= \frac{a_n - ib_n}{2}, n > 0, \\ c_n &= \frac{a_n + ib_n}{2}, n < 0. \end{aligned}$$

Note that $c_n^* = c_{-n}$. It can be easily shown that complex exponentials are orthogonal:

$$\begin{aligned} \int_0^{2L} e^{i\frac{n\pi x}{L}}e^{-i\frac{m\pi x}{L}}dx &= \int_0^{2L}\cos\frac{(n-m)\pi x}{L}dx + i\int_0^{2L}\sin\frac{(n-m)\pi x}{L}dx \\ &= 2L\delta_{n,m} + 0 \end{aligned}$$

so

$$c_m = \frac{1}{2L}\int_0^{2L} f(x)e^{-i\frac{m\pi x}{L}}dx = \frac{1}{2L}\int_0^{2L}\Big(\sum_{n=-\infty}^{\infty}c_n e^{i\frac{n\pi x}{L}}\Big)e^{i\frac{-m\pi x}{L}}dx.$$

Now assume $g(x) = \frac{df}{dx} = \sum_{n=-\infty}^{\infty} c_n e^{i\frac{n\pi x}{L}}$. Then

$$\begin{aligned} c_n &= \frac{1}{2L}\int_0^{2L} \frac{df}{dx} e^{-i\frac{n\pi x}{L}}\, dx \\ &= \frac{1}{2L}\left[f(x)e^{-i\frac{n\pi x}{L}}\right]_0^{2L} + \frac{in\pi}{2L^2}\int_0^{2L} f(x)e^{-i\frac{n\pi x}{L}}\, dx \\ &= \frac{in\pi}{L} c_n \end{aligned}$$

by periodicity.

2.3 Half-range Series

Consider a function defined *only* on $0 \leq x \leq L$. There are two possible ways to extend this function to a $2L$-periodic function that can be represented as a Fourier series.

2.3.1 Odd function: Fourier Sine Series

$f(x)$ can be extended as an *odd* function $f(x) = -f(-x)$ on $-L \leq x \leq L$ and then extended as a $2L$-periodic function. In this case $a_n = 0$ we can define the *Fourier sine series*:

$$\frac{f(x_+) + f(x_-)}{2} = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}$$

where

$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi x}{L} dx.$$

Note the range of integration.

Example (Sawtooth function)**.**

2.3.2 Even function: Fourier Cosine Series

$f(x)$ can also be extended as an even function on $-L \leq x \leq L$, i.e. $f(x) = f(-x)$ and then extended as a $2L$-periodic function. $b_n = 0$ for all n. The Fourier cosine series is

$$\frac{f(x) + f(x)}{2} = \frac{a_0}{2} + \sum_{n-1}^{\infty} a_n \cos\frac{n\pi x}{L}$$

where

$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi x}{L}.$$

2.4 Parseval's Theorem

"Energy" of a periodic signal is often of interest, i.e. $E = \int_0^{2L} f^2(x)dx$. Consider the general case

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}, g(x) = \sum_{m=-\infty}^{\infty} d_m e^{\frac{im\pi x}{L}};$$

$$\begin{aligned}
\int_0^{2L} f(x)g(x)dx &= \sum_{n=-\infty}^{\infty}\sum_{m=-\infty}^{\infty} c_n d_m \int_0^{2L} \exp\Big[\frac{i\pi x}{L}(n+m)\Big]dx \\
&= \sum_{n=-\infty}^{\infty}\sum_{m=\infty}^{\infty} c_n d_m (2L\delta_{n,-m}) \\
&= 2L \sum_{n=-\infty}^{\infty} c_n d_{-n} \\
&= 2L \sum_{n=-\infty}^{\infty} c_n d_n^*
\end{aligned}$$

so if $g = f$,

$$\int_0^{2L} f^2(x)dx = 2L\sum_{n=-\infty}^{\infty}|c_n|^2 = L\Big[\frac{a_0^2}{2} + \sum_{n=1}^{\infty}(a_n^2 + b_n^2)\Big]$$

Example (Sawtooth function)**.** Remember $f(x) = x$ for $-L \le x \le L$.

$$b_n = \frac{2L}{m\pi}(-1)^{m+1}$$

so

$$\int_{-L}^{L} x^2 dx = \frac{2L^3}{3} = L\sum_{m=1}^{\infty}\frac{4L^2}{m^2\pi^2}$$

so

$$\sum_{m=1}^{\infty}\frac{1}{m^2} = \frac{\pi^2}{6}.$$

Exercise. From the Fourier series of $x^2/2$ show that

$$\sum_{m=1}^{\infty}\frac{1}{m^4} = \frac{\pi^4}{90}.$$

3 Sturm-Liouville Theory

3.1 Second Order ODEs

Consider a general second order ordinary partical differential equation

$$\mathcal{L}y(x) = \alpha(x)\frac{d^2y}{dx^2} + \beta(x)\frac{dy}{dx} + \gamma(x)y = f(x).$$

α, β, γ are continuous, with α non-zero except perhaps at a finite number of isolated points. $f(x)$ is bounded, defined on $a \leq x \leq b$ (a or b may be infinity). tary function is

$$y_c = Ay_1 + By_2.$$

Inhomogeneous or *forced* equation $\mathcal{L}y = f(x)$ where f is the forcing, has a particular integral $y_p(x)$. The general solution is $y = y_c(x) + y_p(x)$ where A, B are determined in a problem by applying condition.

3.2 Hermitian Matrices: an Analogy

Remember the problem: find $\mathbf{x}$ such that

$$A\mathbf{x} = \mathbf{b}.$$

If A is an Hermitian matrix, A is $N \times N$, i.e. $A^\dagger A$ where the † denotes complex conjugate transpose. It has 4 properties:[1]

1. λ_n are real,
2. if $\lambda_m \neq \lambda_n$ then $\mathbf{y}_m \cdot \mathbf{y}_n = 0$,
3. the eigenvectors form, on scaling, an orthonormal basis and so specially $\mathbf{b} \in \mathbb{C}^N$ can be described by a linear combination of eigenvectors,
4. if A is non-singular, i.e. all eigenvalues are non-zero, the solution to $A\mathbf{x} = \mathbf{b}$ can be written as a sum of eigenvector.

3.2.1 Gaussian Elimination

From property 3 above we can write

$$\mathbf{b} = \sum_{n=1}^{N} b_n y_n$$
$$\mathbf{x} = \sum_{n=1}^{N} C_n y_n$$

Since A is linear,

$$A\mathbf{x} = \sum_{n=1}^{N} c_n A\mathbf{y}_n = \sum_{n=1}^{N} c_n \lambda_n \mathbf{y}_n = \sum_{n=1}^{N} b_n \mathbf{y}_n = \mathbf{b}.$$

[1] Recall an eigenvector and eigenvalue are defined such that $A\mathbf{y}_n = \lambda_n \mathbf{y}_n$ where λ_n is the eigenvalue and y_n is the eigenvector.

For simplicity assuming the λ_n are distinct and non-zero. From property 2,

$$\mathbf{y}_m \cdot (\sum_{n=1}^{N} c_n \lambda_n \mathbf{y}_n) = c_m \lambda_m = \mathbf{y}_m \cdot (\sum_{n=1}^{N} c_n \lambda_n \mathbf{y}_n) = b_m$$

Therefore

$$c_m = \frac{b_m}{\lambda_m}$$
$$x = \sum_{n=1}^{N} \frac{b_n}{\lambda_n} y_n$$

Question. Can this be generalised to differential operators?

3.3 Motivating Example: Fourier Series

For continuous forcing functions $f(x)$, we want to find $y(x)$ on a finite interval such that

$$-\frac{d^2y}{dx^2} = f(x), 0 \leq x \leq L,\ f(0) = f(L) = y(0) = y(L) = 0,$$

satisfies the Dirichlet conditions and so we can write a Fourier sine series if we extend f to be a $2L$-periodic odd function:

$$f(x) = \sum_{n=1}^{\infty} b_n \sin \frac{n\pi x}{L}$$
$$b_n = \frac{2}{L} \int_0^L f(\xi) \sin \frac{n\pi\xi}{L} d\xi$$

Let $\mathcal{L} = -\frac{d^2}{dx^2}$, can we find solutions to

$$\mathcal{L} y_n = \lambda_n y_n,\ y_n(0) = y_n(L) = 0?$$

Indeed, we can solve the equations

$$\frac{d^2}{dx^2} y_n = -\lambda_n y_n,$$

and obtain the solutions

$$y_n = \sin \frac{n\pi x}{L},$$
$$\lambda_n = \frac{n^2\pi^2}{L^2}.$$

In particular, the boundary value quantises the λ_n.

Note. λ_n are real and strictly positive and so y_n is an eigenfunction with associated eigenvalue λ_n.

Note $\lambda_n = \frac{n^2\pi^2}{L^2}$ has property 1 and we have already met the orthogonality property 2, i.e. $\int_0^L y_n y_m dx = \frac{L}{2}\delta_{m,n}$. There is also a generalisation of property 3: sines and cosines form a complete (infinite-dimensional) basis for functions that satisfy Dirichlet condition.

For this problem $y(x)$ must be sufficiently smooth so that its *second derivative* satisfies the Dirichlet condition.

$y(x)$ has a Fourier sine series:

$$\begin{aligned} y(x) &= \sum_{n=1}^{\infty} c_n \sin\frac{n\pi x}{L} \\ -\frac{d^2}{dx^2}y = \mathcal{L}y &= \sum_{n=1}^{\infty} \frac{n^2\pi^2}{L^2} c_n \sin\frac{n\pi x}{L} = \sum_{n=1}^{\infty} \lambda_n c_n \sin\frac{n\pi x}{L} \\ &= \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L} \end{aligned}$$

The orthogonality property tell us that

$$\lambda_n c_n = b_n.$$

Thus

$$\begin{aligned} y(x) &= \sum_{n=1}^{\infty} \frac{b_n}{\lambda_n} \sin\frac{n\pi x}{L},\ \lambda_n = \frac{n^2\pi^2}{L^2} \\ &= \frac{2}{L}\int_0^L \sum_{n=1}^{\infty} \frac{1}{\lambda_n} \sin\frac{n\pi x}{L} \sin\frac{n\pi\xi}{L} f(\xi)d\xi \\ &= \int_0^L G(x;\xi)f(\xi)d\xi \end{aligned}$$

where $G(x;\xi)$ is the *Green's function.*

Note the loose analogy between Hermitian matrices and self-adjoint operators:

$$\begin{aligned} A\mathbf{x} = \mathbf{b} &\Rightarrow \mathbf{x} = A^{-1}\mathbf{b} \\ \mathcal{L}y = f &\Rightarrow y = \mathcal{L}^{-1}f \end{aligned}$$

4 Self-adjoint Operators

4.1 Definition of Self-adjoint Form

Consider a second order linear differential operator where the eigenvalue problem is to determine eigenfunction y and associated eigenvalue λ such that

$$\alpha(x)\frac{d^2y}{dx^2} + \beta(x)\frac{dy}{dx} + \gamma(x)y = \mathcal{L}y = -\lambda\kappa(x)y, a \leq x \leq b \qquad (*)$$

where κ and α are real and positive on $[a, b]$. (In the previous example, $\kappa = 1 = \alpha, \beta = \gamma = 0$.) This general differential operator can *always* be written in *Sturm-Liouville* or *self-adjoint* form

$$\mathcal{L}y = -\frac{d}{dx}\Big[p(x)\frac{dy}{dx}\Big] + q(x)y = \lambda w(x)y$$

where $w(x)$ is called the *weight* function, wlog real and positive on $[a, b]$ except possibly on isolated points where $w = 0$.

Multiply $(*)$ by $-\phi(x)$ where

$$\phi(x) = \frac{1}{\alpha(x)}\exp\Big(\int^x \frac{\beta(u)}{\alpha(u)}du\Big),$$

the result is

$$\begin{aligned} p(x) &= \exp\Big(\int^x \frac{\beta(u)}{\alpha(u)}du\Big) \\ q(x) &= -\frac{\gamma(x)}{\alpha(x)}\Big(\int^x \frac{\beta(u)}{\alpha(u)}du\Big) \\ w(x) &= -\frac{\kappa(x)}{\alpha(x)}\Big(\int^x \frac{\beta(u)}{\alpha(u)}du\Big) \end{aligned}$$

where p, q, w are all real and positive on $[a, b]$.

4.2 Definition of Self-adjointness

Consider a linear 2nd order differential operator $\mathcal{L}$ defined on $[a, b]$. The *adjoint* of $\mathcal{L}$, denoted by $\mathcal{L}^\dagger$ has the property that for all pairs of functions y_1 and y_2 satisfying appropriate boundary conditions, which shall be defined below,

$$\int_a^b y_1^* \mathcal{L} y_2 dx = \int_a^b y_2 (\mathcal{L}^\dagger y_1)^* dx.$$

If $\mathcal{L} = \mathcal{L}^\dagger$ (with appropriate boundary conditions) then $\mathcal{L}$ is said to be *self-adjoint* or *Hermitian*.

4.2.1 Self-adjointness of Sturm-Liouville Operator

To derive the necessary and sufficient condition for the Sturm-Liouville operator to be self-adjoint, we consider the real case here:

$$\begin{aligned}\int_a^b y_1 \mathcal{L} y_2 dx &= \int_a^b y_1 \Big[-\frac{d}{dx}(p\frac{dy_2}{dx}) + q y_2 \Big] dx \\ &= \Big[-y_1 p \frac{dy_2}{dx} \Big]_a^b + \int_a^b q y_1 y_2 dx + \int_a^b p \frac{dy_2}{dx}\frac{dy_1}{dx} dx \\ &= \Big[-y_1 p \frac{dy_2}{dx} \Big]_a^b + \int_a^b y_2 \Big(-\frac{d}{dx}(p\frac{dy_1}{dx}) + q y_1 \Big) dx + \Big[y_2 p \frac{dy_1}{dx} \Big]_a^b \\ &= \int_a^b y_2 \mathcal{L} y_1 dx + \underbrace{\Big[p(y_2\frac{dy_1}{dx} - y_1\frac{dy_2}{dx}) \Big]_a^b}_{T_1}\end{aligned}$$

Note the Wronskian in T_1.

The operator

$$\mathcal{L} = -\frac{d}{dx}(p\frac{d}{dx}) + q$$

is thus self-adjoint if and only if $T_1 = 0$.

In the previous example, $y = 0$ at $x = a, b$. Other simple examples satisfying $T_1 = 0$ include:

- $y' = 0$ at $x = a, b$,
- $y + ky' = 0$ with k constant at $x = a, b$.
- Periodic boundary conditions satisfying

$$y(a) = y(b) \text{ or } y'(a) = y'(b).$$

- $p = 0$ at $x = a, b$.

4.2.2 Properties of Self-adjoint Operators

1. The eigenvalues are real: assume that λ_n is an eigenvalue and y_n is the associated eigenfunction. Then

$$\mathcal{L} y_n = \lambda_n y_n w,\ (\mathcal{L} y_n)^* = \lambda_n^* y_n^* w^*.$$

But here w and $\mathcal{L}$ are real by assumption (unlike in quantum mechanics, for example) so

$$\begin{aligned}\mathcal{L} y_n^* &= \lambda_n^* y_n^* w \\ \int_a^b y_n (\mathcal{L} y_n^*) dx &= \int_a^b y_n (\lambda_n^* w y_n^*) dx = \lambda_n^* \int_a^b w |y_n|^2 dx \\ \int_a^b y_n^* (\mathcal{L} y_n) dx &= \int_a^b y_n^* (\lambda_n y_n w) dx = \lambda_n \int_a^b w |y_n|^2 dx\end{aligned}$$

By self-adjointness,

$$\begin{aligned}
0 &= \int_a^b y_n (\mathcal{L} y_n)^* dx - \int_a^b y_n^* (\mathcal{L} y_n) dx \\
&= (\lambda_n^* - \lambda_n) \int w |y_n|^2 dx
\end{aligned}$$

so $\lambda_n^* = \lambda_n$.

2. The eigenfunctions of distinct eigenvalues are orthogonal: assume that y_n and y_m are eigenfunctions with distinct eigenvalues $\lambda_n \neq \lambda_m$:

$$\mathcal{L} y_n = \lambda_n w y_n, \ \mathcal{L} y_m = \lambda_m w y_m.$$

By self-adjointness

$$\begin{aligned}
0 &= \int_a^b y_m \mathcal{L} y_n dx - \int_a^b y_n \mathcal{L} y_m dx \\
&= \int_a^b y_m \lambda_n w y_n dx - \int_a^b y_n \lambda_m w y_m dx \\
&= (\lambda_n - \lambda_m) \int_a^b w y_n y_m dx \\
\Rightarrow \int_a^b w y_n y_m dx &= 0
\end{aligned}$$

Note the weight function in the integral.

3. There is an orthonormal set of eigenfunctions: let

$$Y_n(x) = \frac{y_n(x)}{\left(\int_a^b w y_n^2 dx \right)^{1/2}}$$

so

$$\delta_{m,n} = \int_a^b w(x) Y_m(x) Y_n(x) dx.$$

4. The set of eigenfunctions form a complete basis: suppose a function $f(x)$ satisfying the boundary condition as eigenfunction. *Completeness* means that any such $f(x)$ can be expressed as

$$f(x) = \sum_{n=1}^{\infty} a_n y_n(x) = \sum_{n=1}^{\infty} A_n Y_n(x).$$

where $Y_n(x)$ is an orthonormal basis.

5. These coefficients can always be determined using orthogonality:

$$\int_a^b f(x) w Y_m(x) dx = \sum_{n=1}^{\infty} A_n \int_a^b w Y_n(x) Y_m(x) dx = A_m$$

6. A corollary of the property of completeness is that there is always a *countably infinite* number of eigenvalues which satisfy the underlying self-adjoint problem.

7. Parseval's Theorem for such self-adjoint operators: consider

$$
\begin{aligned}
I &= \int_a^b w \underbrace{\left[f(x) - \sum_{n=1}^{\infty} A_n Y_n(x)\right]^2} dx \\
&= \int_a^b w f^2(x) dx - 2\sum_{n=1}^{\infty} A_n \int_a^b w f Y_n dx + \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} A_n A_m \int_a^b w Y_n Y_m dx \\
&= \int_a^b w f^2 dx - 2\sum_{n=1}^{\infty} A_n^2 + \sum_{n=1}^{\infty} A_n^2 \\
&= \int_a^b w f^2 dx - \sum_{n=1}^{\infty} A_n^2
\end{aligned}
$$

   - If the eigenfunctions are complete, $I = 0$ and so there is a Parseval's relation

$$\int_a^b w f^2 dx = \sum_{n=1}^{\infty} A_n^2.$$

   - If the eigenfunctions are not complete, e.g. if $\mathcal{L}$ is *not* self-adjoint, then square bracket term is positive, w is positive by construction. Therefore we get *Bessel's inequaility*:

$$\int_a^b w f^2 dx \geq \sum_{n=1}^{\infty} A_n^2.$$

8. Eigenfunction series representation are the "best" such representation in a particular sense: define the partial sum

$$s_N(x) = \sum_{n=1}^{N} A_n Y_n.$$

   Completeness implies that $f(x) = \lim_{N\to\infty} s_N(x)$ except at points of discontinuity of $f(x)$. For simplicity consider continuous $f(x)$. The mean square error ε_N in approximating $f(x)$ by $s_N(x)$ is

$$\varepsilon_N = \int_a^b w(f - s_N(x))^2 dx \to 0 \text{ as } N \to \infty.$$

   Differentiate,

$$
\begin{aligned}
\frac{\partial \varepsilon_n}{\partial A_m} &= -2\int_a^b w\Big(f - \sum_{n=1}^{N} A_n Y_n\Big) Y_m dx \\
&= -2\int_a^b w f Y_m dx + 2\sum_{n=1}^{N} A_n \int_a^b w Y_n Y_m dx \\
&= -2A_m + 2A_m \\
&= 0
\end{aligned}
$$

Therefore the coefficients extremise (in this case, minimise) the error in the mean square sense and so the best partial sum representation is the partial representation using the eigenfunctions of the underlying Sturm-Liouville operator.

4.2.3 Example with Non-trivial Weight

Problem: find $y(x)$ on $[0, \pi]$ such that

$$\begin{aligned} y'' + y' + (\frac{1}{4} + \lambda)y &= 0 \\ y(0) &= 0 \\ y(\pi) - 2y'|_\pi &= 0 \end{aligned}$$

Write in self-adjoint form:

$$-\frac{d}{dx}\left(e^x \frac{dy}{dx}\right) - \frac{e^x y}{4} = \lambda e^x y.$$

The auxillary equation $y \propto e^{\sigma x}$ is

$$\sigma^2 + \sigma + \frac{1}{4} + \lambda = 0$$

so

$$\sigma = -\frac{1}{2} \pm i\sqrt{\lambda}.$$

The homogeneous solution is

$$y(x) = Ae^{-x/2} \cos \mu x + Be^{-x/2} \sin \mu x$$

with $\mu^2 = \lambda$, satisfying

$$\begin{aligned} y(0) &= 0, \\ y(\pi) - 2y'(\pi) &= Be^{-\pi/2}(\sin \mu\pi + \sin \mu\pi - 2\mu \cos \mu\pi) = 0. \end{aligned}$$

Therefore

$$\begin{aligned} A &= 0, \\ \tan \mu\pi &= \mu. \end{aligned}$$

A simple plot shows that there are countably infinite eigenvalues that satisfy this equation. As $n \to \infty, \mu_n \approx \frac{2n+1}{2}$.

Notice that the eigenfucntions are proportional to $e^{-x/2} \sin \mu_n x$ so

$$\begin{aligned} I_{mn} \propto \alpha \int_0^\pi w y_n(x) y_m(x) dx \\ &= \int_0^\pi e^x e^{-x/2} \sin \mu_n x e^{-x/2} \sin \mu_m x dx \\ &= \int_0^\pi \sin \mu_n x \sin \mu_m x dx \\ &= 0 \text{ if } n \neq m \end{aligned}$$

4.2.4 Application in Inhomogeneous Boundary Value Problems

Consider this inhomogeneous problem:

$$(\mathcal{L} - \hat{\lambda} w)y = f(x) = wF(x) = w(x) \sum_{n=0}^{\infty} A_n Y_n(x).$$

By completeness, $y = \sum_{n=1}^{\infty} B_n Y_n(x)$ where B_n is the unknown.

$$\begin{aligned} w \sum_{n=1}^{\infty} A_n Y_n &= \sum_{n=1}^{\infty} B_n (\mathcal{L} Y_n - \hat{\lambda} w Y_n) \\ &= \sum_{n=1}^{\infty} B_n (\lambda_n - \hat{\lambda}) w Y_n \end{aligned}$$

Multiply across by Y_m and use property 4: $\delta_{mn} = \int_a^b w Y_n Y_m dx$:

$$A_m = B_m (\lambda_m - \hat{\lambda})$$

so

$$\begin{aligned} B_m &= \frac{A_m}{\lambda_m - \hat{\lambda}} \\ &= \frac{\int_a^b w(\xi) Y_m(\xi) F(\xi) d\xi}{\lambda_m - \hat{\lambda}} \\ &= \frac{\int_a^b Y_m(\xi) f(\xi) d\xi}{\lambda_m - \hat{\lambda}} \\ y(x) &= \int_a^b \sum_{n=1}^{\infty} \frac{Y_n(\xi) Y_n(x)}{\lambda_n - \hat{\lambda}} f(\xi) d\xi \\ &= \int_a^b G(x;\xi) f(\xi) d\xi \end{aligned}$$

which is the general form of the example in section 3.3.

5 Wave Equation

Wave equation is a canonical second order PDE: an example of *hyperbolic equation*. Whenever you have a system with low dissipation of energy and a restoring force you have a wave.

The linear wave equation takes the form

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \nabla^2 \phi$$

where c is the (phase) speed, i.e. speed of propagation of crests.

5.1 Physical Derivation

The simplest example is a heavy "massive" elastic string suspended between $x = 0$ and $x = L$. Assume those points are at $y = 0$ and assume all deflections are sufficiently small that we can consider them to be vertical. Therefore $y(x,t)$ describes the deflections.

Tension $T(x)$ in the string, restoring forces horizontally and vertically:

The horizontal forces satisfy

$$T(x)\cos\theta_1 = T(x+\delta x)\cos\theta_2,\ |\theta_1|, |\theta_2| \ll 1,$$

so

$$T(\theta) \approx T(x+\delta x)$$

so T is a constant.

We further assume that there is no horizontal motion and the mass of string per unit length is μ. The vertical force

$$\mu\delta x \frac{\partial^2 y}{\partial t^2} = T\sin\theta_2 - T\sin\theta_1 - \mu g\delta x$$

For small angles,

$$\begin{aligned}\sin\theta_2 &\approx \tan\theta_2 = \frac{\partial y}{\partial x}\Big|_{x+\delta x} \approx \frac{\partial y}{\partial x}\Big| x + \delta x \frac{\partial^2 y}{\partial x^2}\Big| x \\ \sin\theta_1 &\approx \tan\theta_1 = \frac{\partial y}{\partial x}\Big|_x\end{aligned}$$

so

$$\mu\delta x \frac{\partial^2 y}{\partial x^2} = T\delta x \frac{\partial^2 y}{\partial x^2} - \mu g\delta x + O(\delta x^2).$$

We now assume that the weight of the string plays no role:

$$g \to 0 \text{ or } \frac{T\partial^2 y}{\mu\partial x^2} \gg g$$

so

$$\frac{\partial^2 y}{\partial x^2} = \frac{T}{\mu}\frac{\partial^2 y}{\partial x^2} = c^2\frac{\partial^2 y}{\partial x^2}$$

5.2 Example: Wave on a Finite String

We want to find the displacement $y(x,t)$ such that

$$\frac{\partial^2 y}{\partial x^2} = c^2 \frac{\partial^2 y}{\partial x^2}$$
$$y(0,t) = y(L,t) = 0$$
$$y(x,0) = \phi(x)$$
$$\frac{\partial y(x,0)}{\partial x} = \psi(x)$$

We use *seperation of variables*. At its heart there is an existence and uniqueness proof. If we can find a solution by *any means* it is the unique solution.

Suppose

$$y(x,t) = X(x)T(t),$$

substitute

$$X\ddot{T} = c^2 X'' T$$
$$\frac{1}{c^2}\frac{\ddot{T}}{T} = \frac{X''}{X} = -\lambda$$

so we have two equations:

$$X'' = -\lambda X$$
$$\ddot{T} = -\lambda c^2 Y$$
$$X(0) = 0$$
$$X(L) = L$$

$\lambda \geq 0$ as otherwise X has at most 1 zero. Solve X to get

$$X = \alpha \cos \sqrt{\lambda} x + \beta \sin \sqrt{\lambda} x$$

so

$$\lambda_n = \frac{n^2 \pi^2}{L^2},\ n \in \mathbb{N}.$$

Note that these are real eigenvalues of the Sturm-Liouville system

$$-\frac{d}{dx}(\frac{dX}{dx}) = \lambda X.$$

The associated eigenfunctions are the *normal modes*:

$$X_n(x) = \beta_n \sin \frac{n\pi x}{L}.$$

The lowest mode, i.e. $n = 1$, is the *fundamental mode* while $n = 2, 3, ...$ are the *overtones*.

Note how the eigenvalues *quantise* the admissible solutions. For each of the X_n there is an associated $T_n(t)$:

$$\ddot{T}_n = -\frac{n^2 \pi^2 c^2}{L^2} T_n$$

which has solution

$$T_n = \gamma_n \cos\frac{n\pi ct}{L} + \delta_n \sin\frac{n\pi ct}{L}.$$

Combining the two parts,

$$y_n(x,t) = X_n(x)T_n(t) = \sin\frac{n\pi x}{L}\left(A_n\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi ct}{L}\right).$$

The wave equation is linear so we can add all these solutions together

$$\begin{aligned} y(x,t) &= \sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}\left(A_n\cos\frac{n\pi ct}{L} + B_n\sin\frac{n\pi ct}{L}\right) \\ &= \sum_{n=1}^{\infty} b_n(t)\sin\frac{n\pi x}{L} \end{aligned}$$

which is naturally a Fourier sine series. The A_n and B_n are determined from initial conditions:

$$\begin{aligned} y(x,0) = \phi(x) &= \sum_{n=1}^{\infty} A_n\sin\frac{n\pi x}{L} \\ \Rightarrow A_n &= \frac{2}{L}\int_0^L \phi(x)\sin\frac{n\pi x}{L}dx \\ \frac{\partial y(x,0)}{\partial t} = \psi(x) &= \sum_{n=1}^{\infty}\frac{n\pi c B_n}{L}\sin\frac{n\pi x}{L} \\ \Rightarrow B_n &= \frac{2}{n\pi c}\int_0^L \psi(x)\sin\frac{n\pi x}{L}dx \end{aligned}$$

5.3 Algorithm for Seperation of Variables

The algorithm for seperation of variables is clear:

1. seperate the variables: write $y(x,t) = X(x)T(t)$,
2. determine the admissable form for eigenvalues and associated eigenfunctions from boundary conditions (i.e. conditions on x),
3. determine the form of the other seperated function using the eigenvalues and find a particular set of solution y_n,
4. sum over all y_n to form a series representation,
5. determine the coefficients from the initial condititions (i.e. conditions on t).

Exercise. Determine the full solution for a plucked string

$$\psi(x) = 0, \phi(z) = \begin{cases} \phi_0\frac{x}{d} & 0 \le x \le d \\ \phi_0\frac{L-x}{L-d} & d \le x \le L \end{cases}$$

6 Bessel's Equation

6.1 Derivation of Bessel's Equation

Consider the wave equation on the unit disc

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} + c^2 \frac{\partial^2 u}{\partial y^2} = c^2 \nabla^2 u,\ x^2 + y^2 \leq 1$$

As usual, use seperation of varaibles:

1. $u(x, y, t) = V(x, y)T(t)$, $\ddot{T} = -\lambda c^2 T, \nabla^2 V = -\lambda V$ with $\lambda \geq 0$, since we are in the unit disc,

   $$V(x, y) = R(r)\Theta(\theta)$$

   so

   $$\frac{\partial^2 V}{\partial r^2} + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} + \frac{1}{r}\frac{\partial V}{\partial r} + \lambda V = 0$$
   $$R''\Theta + \frac{1}{r^2} R\Theta'' + \frac{1}{r} R'\Theta + \lambda R\theta = 0$$
   $$r^2 R''\Theta + rR'\Theta + \lambda r^2 R\Theta = -R\Theta''$$

   Divide by $R\Theta$, we get

   $$f_2(r) = f_1(\theta) = \mu$$

   Due to circular gemoetry Θ must be periodic with period 2π. Thus $\mu = m^2$, where m is an integer. This implies that

   $$r^2 R'' + rR' + (\lambda r^2 - m^2)R = 0.$$

   Now dividing across by $-r$ to put this in Sturm-Liouville form:

   $$-\frac{d}{dr}(r\frac{dR}{dr}) + \frac{m^2}{r} R = \lambda r R,\ r \leq 1$$

   which has non-trivial weigtht. Furthermore let $z = \sqrt{\lambda} r$ this can be reposed as

   $$z^2 \frac{d^2 R}{dz^2} + z\frac{dR}{dz} + (z^2 - m^2)R = 0.$$

   This is *Bessel's equation.*

Solution to this equation are called *Bessel functions of the first kind and second kind.*

6.2 Properties of Bessel Functions

1. $J_m(z)$, the Bessel function of the frist kind of order m which is regular at the origin (and is 0 there for $m > 0$) $m = 0$ for axis symmetric solution (i.e. solutions independent of Θ).

2. $Y_m(z)$ the Bessel function of the second kind which are singular at the origin.[1] Sometimes called Neumann function N_m or Weber function.

[1] Note $Y_m(z)$ here is *not* orthonormal.

Some properities of the Bessel functions include:

- $J_\nu(z)$ has a series expansion (ν not an integer in general):

$$J_\nu(z) = \left(\frac{z}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-z^2/4)^k}{k!\Gamma(\nu+k+1)}$$

  where Γ is the Gamma function.

- J_ν and $J_{-\nu}$ are linearly independent for non-integer ν.
- $Y_\nu(z)$ is defined by

$$\frac{J_\nu(z)\cos\nu\pi - J_{-\nu}(z)}{\sin\nu\pi}.$$

- For integer $\nu = m$:

$$\begin{aligned} J_{-m}(z) &= (-1)^m J_m(z) \\ Y_{-m}(z) &= (-1)^m Y_m(z) \\ Y_m(z) &= \lim_{\nu\to m} \frac{J_\nu(z)\cos\nu\pi - J_{-\nu}}{\sin\nu\pi}. \end{aligned}$$

- $Y_0(z) \approx \frac{2}{\pi}\log z$, $J_0 \approx 1$ as $z \to 0$.
- $Y_m(z) \approx -\frac{1}{\pi}\left(\frac{2}{\pi}\right)^{-m}(m-1)!$, $J_m \approx \frac{(z/2)^m}{m!} \to 0$ for $m > 0$.
- As $z \to \infty$,

$$\begin{aligned} J_\nu(z) &= \left(\frac{2}{\pi z}\right)^{1/2} \cos\left(z - \frac{i\pi}{2} - \frac{\pi}{4}\right) + O(z^{-3/2}) \\ Y_\nu(z) &= \left(\frac{2}{\pi z}\right)^{1/2} \sin\left(z - \frac{i\pi}{2} - \frac{\pi}{4}\right) + O(z^{-3/2}) \end{aligned}$$

6.3 Application to a Simple Drum Problem

From the problem in Section 6.1 it is clear $J_m(\sqrt{\lambda_{mn}}r)$ and $Y_m(\sqrt{\lambda_{mn}}r)$ are appropriate eigenfunctions for $R(r)$ in Bessel's equation.

To solve a problem we need boundary conditions. Let use consider

$$\frac{\partial^2 u}{\partial t^2} = c^2\nabla^2 u \text{ on } r \leq 1$$

such that $u = 0$ when $r = 1$ and u is finite when $r = 0$. Thus the solution cannot be Y_m.

The eigenvalues λ_{mn} are solutions of $J_m(\sqrt{\lambda_{mn}}) = 0$ there is clearly a countably infinite number such that

$$0 < \lambda_{m1} < \lambda_{m2} < \cdots < \lambda_{mn} = j_{mn}^2$$

so the general form for the spherically varying part:

$$V(r_1\theta) = V_{mn}(r,\theta) = J_n(j_{mn}r)(A_n\cos n\theta + B_n\sin n\theta)$$

so we can see that the general solution is

$$
\begin{aligned}
u(r,\theta,t) = &\sum_{n=1}^{\infty} J_0(j_{0n}r)(A_{0n}\cos j_{0n}ct + B_{0n}\sin j_{0n}ct) \\
&+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(j_{mn}r)(A_{mn}\cos m\theta + B_{mn}\sin m\theta)\cos j_{mn}ct \\
&+ \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} J_m(j_{mn}r)(A_{mn}\cos m\theta + B_{mn}\sin m\theta)\sin j_{mn}ct
\end{aligned}
$$

The orthogonality condition for Bessel functions is

$$
\int_0^1 J_k(j_{kn}r)J_k(j_{km}r)rdr = \frac{1}{2}\big(J_k'(j_{kn})\big)^2\delta_{mn} = \frac{1}{2}\big(J_{k+1}(j_{kn})\big)^2\delta_{mn}
$$

and so all the coefficients can be determined using orthogonality of sinces, cosines and Bessel functions.

7 Wave Property

Let us return to the Cartesian problem of wave equation

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},\ y(0,t) = y(L,t) = 0,\ y(x,0) = \phi(x),\ \frac{\partial y(x,0)}{\partial t} = \psi(x)$$

which has general solution

$$y(x,t) = \sum_{n=1}^{\infty} \Big(a_n \cos \frac{n\pi ct}{L} + b_n \sin \frac{n\pi ct}{L} \Big) \sin \frac{n\pi x}{L}$$

where

$$a_n = \frac{2}{L} \int_0^L \phi(x) \sin \frac{n\pi x}{L} dx$$
$$b_n = \frac{2}{n\pi c} \int_0^L \psi(x) \sin \frac{n\pi x}{L} dx$$

7.1 Wave Energy

Suppose the mass per unit length is μ. The total kinetic energy is thus

$$K = \int_0^L \frac{1}{2} \mu \Big(\frac{\partial y}{\partial t} \Big)^2 dx$$

The potential energy is $T \times$ extenstion $= T(\delta s - \delta x)$ which equals to

$$T \left(\sqrt{1 + \Big(\frac{\partial y}{\partial x} \Big)^2} - 1 \right) \delta x.$$

Thus the total potential energy is

$$V = T \int_0^L \sqrt{1 + \Big(\frac{\partial y}{\partial x}^2 \Big)} - 1 dx$$
$$\approx \frac{T}{2} \int_0^L \Big(\frac{\partial y}{\partial x} \Big)^2 dx$$

using a binomial expansion and noting that since $\partial y / \partial x$ is small

$$\Big(\frac{\partial y}{\partial x} \Big)^4 \ll \Big(\frac{\partial y}{\partial x} \Big)^2$$

Thus the total energy is

$$E = K + V = \frac{\mu}{2} \int_0^L \Big(\frac{\partial y}{\partial t} \Big)^2 + c^2 \Big(\frac{\partial y}{\partial x} \Big)^2 dx$$

The solution is

$$
\begin{aligned}
\frac{\partial y}{\partial t} &= \sum_{n=1}^{\infty} \frac{n\pi c}{L}\Big(b_n \cos\frac{n\pi ct}{L} - a_n \sin\frac{n\pi ct}{L}\Big)\sin\frac{n\pi x}{L} \\
\frac{\partial y}{\partial x} &= \sum_{n=1}^{\infty} \frac{n\pi}{L}\Big(a_n \sin\frac{n\pi ct}{L} + b_n \cos\frac{n\pi ct}{L}\Big)\cos\frac{n\pi x}{L} \\
K &= \frac{\mu}{2}\int_0^L \sum_{n=1}^{\infty}\sum_{m=1}^{\infty} \sin\frac{n\pi x}{L}\sin\frac{m\pi x}{L} \\
&\times \Big[\frac{n\pi c}{L}\Big(b_n \cos\frac{n\pi ct}{L} - a_n \sin\frac{n\pi ct}{L}\Big)\Big] \\
&\times \Big[\frac{m\pi c}{L}\Big(b_m \cos\frac{m\pi ct}{L} - a_m \sin\frac{m\pi ct}{L}\Big)\Big]dx \\
&= \frac{\mu L}{4}\sum_{n=1}^{\infty}\frac{n^2\pi^2c^2}{L^2}\Big(a_n^2\sin^2\frac{n\pi ct}{L} + b_n^2\cos^2\frac{n\pi ct}{L} \\
&- 2a_nb_n\cos\frac{n\pi ct}{L}\sin\frac{n\pi ct}{L}\Big) \\
V &= \frac{TL}{4}\sum_{n=1}^{\infty}\frac{n^2\pi^2}{L^2}\Big(a_n^2\cos^2\frac{n\pi ct}{L} + b_n^2\sin^2\frac{n\pi ct}{L} \\
&+ 2a_nb_n\sin\frac{n\pi ct}{L}\cos\frac{n\pi ct}{L}\Big)
\end{aligned}
$$

Since $T^2 = \mu c^2$, these expression yield

$$E = \frac{\mu c^2\pi^2}{4L}\sum_{n=1}^{\infty} n^2(a_n^2 + b_n^2)$$

which is, surpringly (assuming you don't have trust in physicists), independent of time.

The period of oscillation is $\frac{2\pi}{\omega} = \frac{2\pi L}{\pi c} = \frac{2L}{c}$ so the average kinetic energy over a period is

$$\overline{K} = \frac{c}{2L}\int_0^{2L/c} K dt = \frac{\mu L}{4}\sum_{n=1}^{\infty}\frac{n^2\pi^2c^2}{L^2}\Big(\frac{a_n^2}{2} + \frac{b_n^2}{2}\Big) = \frac{E}{2}$$

similarly

$$\overline{V} = \frac{c}{2L}\int_0^{2L/c} V dt = \frac{E}{2}$$

i.e. over a period there is an equipartition between kinetic and potential energy.

7.2 Reflection and Transmission

If the medium through which the waves are travelling has different properties then the properties of the waves will change and there may be *reflections* and *transmissions.*

Consider a string with mass per unit length

$$\mu = \begin{cases} \mu_- & x < 0 \\ \mu_+ & x > 0 \end{cases}$$

The horizontal forces between is still horizontal. Wave speed on either side of $x = 0$ is

$$c_\pm = \sqrt{\frac{\tau}{\mu_\pm}}.$$

The incident wave

$$\begin{aligned} w_I &= \operatorname{Re}\left(I \exp\left(i\omega\left(t - \frac{x}{c_-}\right)\right)\right) \\ &= I_r \cos\omega\left(t - \frac{x}{c_-}\right) - I_i \sin\omega\left(t - \frac{x}{c_-}\right) \\ &= A_I \cos\left(\omega\left(t - \frac{x}{c_-}\right) + \phi_I\right) \end{aligned}$$

where $I = I_r + iI_i$ and we define amplitude

$$A_I = \sqrt{I_r^2 + I_i^2} = |I|$$

and phase

$$\phi_I = \arccos\frac{I_r}{|I|} = \arcsin\frac{I_i}{|I|}$$

Similarly the transmitted wave is

$$w_T = \operatorname{Re}\left(T \exp\left(i\omega\left(t - \frac{x}{c_+}\right)\right)\right)$$

and the reflected wave is

$$w_R = \operatorname{Re}\left(R \exp\left(i\omega\left(t + \frac{x}{c_-}\right)\right)\right)$$

notice the plus sign.

We determine the coefficients by matching conditions at the origin. The displacement at $x = 0$ must be continuous for all time:

$$w_I\big|_{x=0^-} + w_R\big|_{x=0^-} = w_T\big|_{x=0^+}$$

In the derivation above we asserted that the angular frequency ω is the same on both sides. If you are not convinced, expand the above relation

$$\operatorname{Re}\left(I \exp(i\omega_- t)\right) + R \exp(i\omega_- t)\Big) = \operatorname{Re}\left(T \exp(i\omega_+ t)\right)$$

and it must be that $\omega_- = \omega_+$. Also

$$\begin{aligned} I + R &= T \\ I_r + R_r &= T_r \\ I_i + R_i &= T_i \end{aligned}$$

i.e. coefficients of cosines and sines are the same.

In the absence of inertia at $x = 0$, the vertical forces are continuous

$$\tau \frac{\partial y}{\partial x}\Big|_{x=0^-} = \tau \frac{\partial y}{\partial x}\Big|_{x=0^+}$$

so

$$\begin{aligned} \frac{R}{c_-} - \frac{I}{c_-} &= -\frac{T}{c_+} \\ \frac{R_r}{c_-} - \frac{I_r}{c_-} &= -\frac{T_r}{c_+} \\ \frac{R_i}{c_-} - \frac{I_i}{c_-} &= -\frac{T_i}{c_+} \end{aligned}$$

Solve to get

$$\begin{cases} R = \left(\frac{c_+ - c_-}{c_+ + c_-}\right) I \\ T = \left(\frac{2c_+}{c_+ + c_-}\right) I \end{cases}$$

The waves have the following properties:

1. $$\frac{R_i}{R_r} = \frac{T_i}{T_r} = \frac{I_i}{I_r}$$

   so there is a simpe relationship between the phases of the incoming, reflected and transmitted waves.

   $$\phi_I = \arccos \frac{I_r}{\sqrt{I_r^2 + R_i^2}} = \arccos \frac{I}{\sqrt{1 + I_i^2/I_r^2}}$$

2. $$\frac{I^2}{c_-} - \frac{R^2}{c_-} = \frac{T^2}{c_+}$$

   This is the statement that the flux of kinetic energy through the system matches at $x = 0$. This is an important concept in physics called *impedance*.

3. Different limiting cases of μ:

   - If $\mu_+ = \mu_-$, then $c_+ = c_-, R = 0, T = I$, i.e. perfect transmission.
   - On the other hand if $\mu_+ \gg \mu_-$ then $c_+ \ll c_-, T \approx 0, R \approx -I$. This is the reflection of fixed end. The reflected wave has a phase shift of π.
   - if $\mu_+ \ll \mu_-$ then $c_+ \gg c_-, T \approx 2I, R \approx I$. There is no phase shift and there is large amplitude disturbance to the right.

Exercise. Show in the last two cases above most of the energy is reflected.

8 Diffusion Equation

The diffusion equation is an exmaple of *parabolic* PDE, which is of enormous importance in physics and chemistry.

8.1 Fick's Law

In steady state, *Fick's First Law* states that the transport per unit area, *flux*, may be related to the negative of the spatial gradient of the quantity by a *diffusion coefficient* (which may or may not be constant):

$$\mathbf{J}_A = -D_{AB}\boldsymbol{\nabla} c_A$$

where c_A is the concentration of species A in species B.

In general D_{AB} may be a function but in this course we only consider the constant case.

Fick's Second Law is about heat. Heat is defined to be

$$Q = \int_V c_p \rho \theta dV$$

where θ is the temperatue in Kelvin, c_P is specific heat capacity at constant pressure and ρ is the mass density. Differentiate with respect to time

$$\frac{dQ}{dt} = \int_V c_P \rho \frac{\partial \theta}{\partial t} dV$$

By Fourier's Law (an empirical result), which is a special case of Fick's First Law, the heat flux is given by

$$\mathbf{q} = -k\boldsymbol{\nabla}\theta$$

where k is *thermal conductivity.* Integrate over S, the surface of V, with outward normal $\hat{\mathbf{n}}$, the total transport of heat out is

$$\begin{aligned} -\frac{dQ}{dt} &= \int_S -k\boldsymbol{\nabla}\phi \cdot \hat{\mathbf{n}} ds \\ &= \int_V \boldsymbol{\nabla} \cdot (-k\nabla) dV \\ &= -\int_V c_P \rho \frac{\partial \theta}{\partial t} dV \end{aligned}$$

As V is arbitrary,

$$\frac{\partial \theta}{\partial t} = \frac{1}{c_P \rho} \boldsymbol{\nabla} \cdot (k\boldsymbol{\nabla}\theta)$$

and $c_P\rho$ is the *volumetric heat capacity.*

Under the simplifying assumption that k is constant,

$$\frac{\partial \theta}{\partial t} = \frac{k}{c_P \rho}\boldsymbol{\nabla}^2\theta = D\boldsymbol{\nabla}^2\theta = \kappa\boldsymbol{\nabla}^2\theta$$

where D or κ is *thermal diffusivity.*

8.2 Random Walk Interpertation

Heat diffusion can also be understood, initially conjured by Eisntein, in terms of random walk.

Consider a lattice at

$$\ldots, -\delta x, 0, \delta x, \ldots, x-\delta x, x, x+\delta x, \ldots$$

Let $c(x,t)$ be the concentration of x at t. Every particle takes a random walk. In δt the probability a particle takes a step to right is p, same for left and the probability of staying in the same place is $1-2p$. Thus

$$c(x,t+\delta t) - c(x,t) = p\Big[c(x+\delta x,t) - 2c(x,t) + c(x-\delta x,t)\Big]$$

Talylor expand,

$$\delta t \frac{\partial c}{\partial t}\Big|_{x,t} + O(\delta t^2) = p(\delta x^2)\frac{\partial^2 c}{\partial x^2}\Big|_{x,t} + O(\delta x^3)$$

Thus

$$\frac{\partial c}{\partial t} + O(\delta t) = \frac{p\delta x^2}{\delta t}\frac{\partial^2 c}{\partial x^2} + O(\delta x^3/\delta t)$$

Now take limit

$$\begin{aligned}\delta t &\to 0\\ \delta x &\to 0\\ \frac{p\delta x^2}{\delta t} &\to D\end{aligned}$$

The equation becomes

$$\frac{\partial \theta}{\partial t} = D\frac{\partial^2 c}{\partial x^2}.$$

Note that $\frac{\partial \theta}{\partial t} = D\nabla^2\theta$ is first order in time so to solve the problem we need to know boundary conditions and $\theta(x,0)$, i.e. initial condition.

8.3 Similarity Solutions

We can find a solution *without* applying boundary conditions by considering a *similarity variable*:

$$\begin{aligned}\eta &= \frac{x}{2\sqrt{Dt}}\\ \frac{\partial}{\partial t} &= \frac{\partial \eta}{\partial t}\frac{\partial}{\partial \eta} = -\frac{\eta}{2t}\frac{\partial}{\partial \eta}\\ \frac{\partial}{\partial x} &= \frac{\partial \eta}{\partial x}\frac{\partial}{\partial \eta} = \frac{1}{2\sqrt{Dt}}\frac{\partial}{\partial t}\\ \frac{\partial^2}{\partial x^2} &= \frac{1}{4Dt}\frac{\partial^2}{\partial \eta^2}\end{aligned}$$

so

$$-\frac{\eta}{2t}\frac{\partial \theta}{\partial \eta} = D\frac{1}{4Dt}\frac{\partial}{\partial \eta}\frac{\partial \theta}{\partial \eta}$$

Introduce

$$X = \frac{\partial \theta}{\partial \eta}$$

and it follows that

$$-2\eta X = \frac{\partial X}{\partial \eta}$$

Solve the simple equation to get

$$\theta(x,t) = \frac{2C_3}{\sqrt{\pi}} \int_0^{\frac{x}{2\sqrt{Dt}}} e^{-u^2} du = C_3 \operatorname{erf}\left(\frac{x}{2\sqrt{Dt}}\right)$$

where

$$\operatorname{erf}(y) = \frac{2}{\sqrt{\pi}} \int_0^y e^{-u^2} du$$

is the *error function*. Note that

- $\operatorname{erf}(0) = 0$ and $\operatorname{erf}(y) \to \pm 1$ as $y \to \pm\infty$.
- The solution does not change along the line of constant η, i.e. constant $\frac{x}{2\sqrt{Dt}}$.

8.4 Seperation of variables

8.4.1 Cartesian Geometry

We wish to solve a problem with boundary conditions and initial conditions so consider a bar of length $2L$,

$$\begin{aligned} \theta(x,0) &= \begin{cases} \theta_0 & 0 < x \le L \\ 0 & -L \le x < 0 \end{cases} \\ \theta(L,t) &= \theta_0 \\ \theta(-L,t) &= 0 \\ \frac{\partial \theta}{\partial t} &= D\frac{\partial^2 \theta}{\partial x^2} \end{aligned}$$

This problem has *inhomogeneous boundary conditions*, although it is still linear so we can define

$$\begin{aligned} \theta &= \theta_s(x) + \hat{\theta}(x,t) \\ 0 &= \frac{d^x \theta_s}{dx^2} \\ \frac{\partial \theta}{\partial t} &= D\frac{\partial^2 \theta}{\partial x^2} \end{aligned}$$

where $\theta_s(L) = \theta_0$ and $\theta_s(-L) = 0$ so $\hat{\theta}(L,t) = 0 = \hat{\theta}(-L,t)$.

So we are solving

$$\begin{aligned}\theta_s(x) &= \theta_0\left(\frac{x+L}{2L}\right)\\ \hat{\theta}(x,t) &= \theta-\theta_s\\ \hat{\theta}(x,0) &= \theta_0\left(H(x)-\frac{x+L}{2L}\right)\\ \hat{\theta}(-L,t) &= \hat{\theta}(L,t)=0\end{aligned}$$

satisfying the heat equation.

Now apply seperation of variables:

1. Let $\hat{\theta}(x,t)=X(x)T(t)$,
2. $X\dot{T}=DTX''$ so $\frac{\dot{T}}{T}=D\frac{X''}{X}$. $X''=-\lambda X, \dot{T}=-D\lambda T$ where $\lambda>0$. Note that in comparison with the wave equation, it has first time derivative. Thus any state is transient.
3. Apply the boundary conditions:

$$\begin{aligned}X(x) &= A\cos\sqrt{\lambda}x+B\sin\sqrt{\lambda}x\\ 0 &= A\cos\sqrt{\lambda}L-B\sqrt{\lambda}L\\ 0 &= A\cos\sqrt{\lambda}L+B\sqrt{\lambda}L\end{aligned}$$

   so $A=\sin\sqrt{\lambda}L=0, \lambda=\frac{n^2\pi^2}{L^2}$ we require the solution to be odd.
4. Find the eigenvalues,

$$T_n=c_n\exp\left(-\frac{Dn^2\pi^2}{L^2}t\right)$$

5. the full solution is

$$\hat{\theta}(x,t)=\sum_{n=1}^{\infty}b_n\sin\frac{n\pi x}{L}\exp\left(-\frac{Dn^2\pi^2}{L^2}t\right)$$

6. Apply the initial condition:

$$\hat{\theta}(x,0)=\sum_{n=1}^{\infty}\sin\frac{n\pi x}{L}=\theta_0\left(H(x)-\frac{x+L}{2L}\right)$$

   so

$$\begin{aligned}Lb_m &= \theta_0\int_0^L\sin\frac{m\pi x}{L}dx\\ &-\frac{\theta_0}{2}\int_{-L}^{L}\sin\frac{m\pi x}{L}dx\\ &-\frac{\theta_0}{2L}\int_{-L}^{L}x\sin\frac{m\pi x}{L}dx\\ &=\frac{L\theta_0}{m\pi}\end{aligned}$$

So

$$\hat{\theta} = \theta_0 \sum_{n=1}^{\infty} \frac{1}{n\pi} \sin \frac{n\pi x}{L} \exp\left(-\frac{Dn^2\pi^2}{L^2} t\right)$$

and the full solution is

$$\theta(x,t) = \theta_0 \left(\frac{x+L}{2L}\right) + \hat{\theta}(x,t).$$

Now consider an infinite bar with

$$\theta(x,0) = \begin{cases} \theta_0 & x > 0 \\ 0 & x < 0 \end{cases}$$

with similarity solution

$$\theta_e = \frac{\theta_0}{2}\left(1 + \operatorname{erf}\left(\frac{x}{2\sqrt{Dt}}\right)\right)$$

is a solution that satsifies this initial condition.

We will show that at early time θ_e is very similary to θ with boundary conditions, i.e. $t \ll L^2/D$. The exponential terms have almost disappeared

8.5 Annular Geometry

Consider an annulus with out radius R_o and inner radius R_i. At $t = 0^+$ the fluid in the pipe flows. The outer temperature is θ_f and the inner temperature is θ_g.

This problem is clearly axis-symmetric so consider the scaled temperature

$$\psi(r,t) = \frac{\theta(r,t) - \theta_g}{\theta_f - \theta_g}$$

therefore the problem reduces to

$$\begin{aligned} \frac{\partial \psi}{\partial t} &= \frac{D}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \psi}{\partial r}\right) \\ \psi(r,0) &= 0,\ R_i < r < R_0 \\ \psi(R_i,t) &= 1 \\ \phi(R_o,t) &= 0 \end{aligned}$$

Write ψ as a sum of steady state solution and time-dependent solution:

$$\psi(r,t) = \psi_s(r) = \hat{\psi}(r,t)$$

solve to get

$$\psi_s = \frac{\log r/R}{\log R_i/R_o}$$

Now apply seperation of variables,

$$\begin{aligned} \hat{\psi}(r,t) &= R(r)T(t) \\ \dot{T} &= -\lambda D T \\ r^2R'' + rR' + \lambda r^2 R &= 0 \end{aligned}$$

which is Bessel's equation of order zero. So we set

$$R_m(r) = A_m J_0(s_m r) + B_m Y_0(s_m r)$$

s_m are determined from the boundary contitions. Note since the domain does not include the origin the solution involves Y_m.

$$R_m(r) = \left[\frac{J_0(s_m r)}{J_0(s_m R_i)} - \frac{Y_0(s_m r)}{Y_0(s_m R_i)}\right]$$

Condition at $r = R_o$ yields eigenvalues s_m since $\hat{\psi}(R_o, t) = 0, R_m(R_o) = 0$ so

$$Y_0(s_m R_i)J_0(s_m R_o) - Y_o(s_m R_o)J_o(s_m R_i) = 0$$

must have a countably infinite number of solutions since the operator is self-adjoint.

Also, the eigenfunctions are orthogonal so

$$\frac{1}{a_m^2} = \int_{R_i}^{R_o} r \left[\frac{J_0(s_m r)}{J_0(s_m R_i)} - \frac{Y_0(s_m r)}{Y_0(s_m R_i)}\right]^2 dr$$

The other seperated solution is

$$T_m(t) = \exp\left(-D s_m^2 t\right)$$

and thus the general solution is

$$\psi(r,t) = \psi_s + \sum_{m=1}^{\infty} c_m \exp\left(-D s_m^2 t\right) R_m(r)$$

c_m is determined from initial conditions:

$$c_n = -\int_{R_i}^{R_o} r\psi(sR_n(r)dr$$

9 Laplace's Equation

There are many steady state problems, and they often involve finding the solution to *Lapace's eqaution*:

$$\nabla^2\psi = 0$$

in some domain D. It is a canonical example of the 3rd qualitatively different linear PDE: *elliptic* PDEs.

9.1 Motivation & Notation

Subject to boundary condition on ∂D of D we can define the problem:

1. Dirichlet conditions: if ψ is given on the boundary, the solution to ψ is unique.
2. Neumann conditions: if $\mathbf{n}\cdot\nabla\psi$ is given on boundary where $\mathbf{n}$ is the outward normal, ψ is unique up to an additive constant.

Examples include:

1. In fluid mechaincs, in the absence of sources, sinks and vortices, the velocity $\mathbf{u}$ can be written as a velocity potential $\mathbf{u} = \nabla\phi$ and if the fluid is incompressible, i.e. $\nabla \cdot \mathbf{u} = 0$, then $\nabla^2\phi = 0$.
2. The study of Laplace's equation is often called *potential theory* because there are many solutions where a vector quantity (a conservative force) can be expressed as a gradient of potential $\mathbf{F} = -\nabla\psi$. Work done around any closed path is zero:
$$\oint_C \mathbf{F}\cdot d\mathbf{r} = 0.$$
Thus by vector calculus ψ satisfies the Laplace's equaiotn. By convention $\mathbf{F}$ is called a *force field.*
3. Harmonic functions (another name for solutions of Laplace's equations) are extremely important in mathematics. Consider a complex function $f(z)$ defined in some region $R \subseteq \mathbb{C}$. Write $z = x + iy$.
$$f(z) = u(x, y) + iv(x, y)$$
The requirement that $f(z)$ is analytic/holomorphic in R (single valued and differentiable, loosely a generalisation of the concept of continuity), can be shown to require that *both* u and v satisfy the 2D version of Laplace's equation:
$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$
$$\frac{\partial^2 v}{\partial x^2} + \frac{\partial^2 v}{\partial y^2} = 0$$

9.2 Laplace's Equation in 3D Cartesian Coordinates

In 3D $\psi(x,y,z) = X(x)Y(y)Z(x)$ such that

$$\psi_{xx} + \psi_{yy} + \psi_{zz} = 0$$

satsify

$$\frac{X''}{X} + \frac{Y''}{Y} + \frac{Z''}{Z} = 0.$$

Thus write

$$X'' = -\lambda_\ell X$$
$$Y'' = -\mu_m Y$$
$$Z'' = (\lambda_\ell + \mu_m)Z$$

where λ_ℓ and μ_m are positive. Now procees to solve in using SOV:

1. Seperate variables.
2. Find the eigenvalues λ_ℓ and μ_m and the associated eigenfunctions $X_\ell(x), Y_m(y)$ by applying boundary conditions on x and y.
3. Solve for $Z_{\ell,m}$ and thus construct a particular solution $P_{\ell,m} = X_\ell(x)Y_m(y)Z_{\ell,m}(z)$
4. Sum
$$\phi(x,y,z) = \sum_{\ell,m} a_{\ell,m} X_\ell(x)Y_m(y)Z_{\ell,m}(z)$$
5. Determine the $a_{\ell,m}$ using the boundary conditions on z.

9.3 Example: Steady Heat Conduction

Consider a semi-infinite rod of rectangular cross-section heated at one end with fixed temepratura and fixed (lower) temperature on the other side:

$$\frac{\partial \psi}{\partial t} = k\boldsymbol{\nabla}^2\psi = 0$$
$$\psi(x,y,0) = \Theta(x,y),\ \psi \to 0 \text{ as } y \to \infty$$
$$\psi(0,y,z) = \psi(a,y,z) = \psi(x,0,z) = \psi(x,b,z) = 0$$

Same steps as above:

1. Let $\psi = X_\ell(X)Y_m(y)Z_{\ell,m}(z)$. So $X'' = -\lambda_\ell X, X(0) = X(a) = 0$. The boundary conditions quantise $\lambda_\ell = \frac{\ell^2\pi^2}{a^2}$ so
$$X_\ell = \sqrt{\frac{2}{a}}\sin\frac{\ell\pi x}{a},\ \ell = 1,2,3,...$$
Similarly
$$Y'' = -\mu_m Y,\ Y(0) = Y(b) = 0$$
implies
$$\mu_m = \frac{m^2\pi^2}{b^2}$$
$$Y_m = \sqrt{\frac{2}{b}}\sin\frac{m\pi y}{b}$$

2.

$$Z''_{\ell,m} = \left(\frac{\ell^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}\right) Z_{\ell,m}$$

so

$$Z_{\ell,m} = \alpha \exp\left[\left(\frac{\ell^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}\right)^{1/2} \pi z\right] + \beta \exp\left[-\left(\frac{\ell^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}\right)^{1/2} \pi z\right]$$

The generation solution is

$$\psi(x,y,z) = \frac{2}{\sqrt{ab}} \sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} a_{\ell,m} \sin\frac{\ell\pi x}{a} \sin\frac{m\pi y}{b} \exp\left[-\left(\frac{\ell^2\pi^2}{a^2} + \frac{m^2\pi^2}{b^2}\right)^{1/2} \pi z\right]$$

3. The boundary conditions at $z = 0$ determines the coefficients $a_{\ell,m}$ due to orthogonality of sines:

One observation of that Fourier series is clearly useful. Sine functions because of Dirichlet conditions, Neuman conditions cosines (no need to normalise).

For a finte bar we would have had $Z_{\ell,m}$ in terms of sinhs and coshs. Solution for $\Theta = 1$:

$$\begin{aligned} a_{pq} &= \frac{2}{\sqrt{ab}} \int_0^b \int_0^a \sin\frac{p\pi x}{a} \sin\frac{q\pi y}{b} dxdy \\ &= \begin{cases} \frac{8\sqrt{ab}}{\pi^2 pq} & \text{if } p \text{ and } q \text{ are both odd} \\ 0 & \text{otherwise} \end{cases} \end{aligned}$$

so

$$\phi(x,y,z) = 16 \sum_{\ell=1}^{\infty} \sum_{m=1}^{\infty} \ldots$$

Some obervations:

- The bar is hotter in the middle.
- As $z \to \infty$, the solution is dominated by the lower harmonics, i.e. when ℓ and m are small so the far field solution is determined by $\ell = m = 1$.
- if $a = b$ then $k_{m,\ell} = k_{\ell,m}$ and the eigenvalues are degenerate. The eigenfunctions are still orthogonal and the problem is still well-posed.

9.4 Laplace's Equation in Polar Coordinates

9.4.1 Plane Polar Coordinates

In polar coordinates $\psi(r, \theta)$ and Laplaces's equation becomes

$$\boldsymbol{\nabla}^2\psi = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2}$$

Follow the seperation of variables algotithm:

1. $\psi = R(r)\Theta(\theta)$ such that

$$\Theta'' = -\lambda\Theta$$
$$\frac{r}{R}(rR')' = \lambda$$

2. $\Theta(\theta + 2\pi) = \Theta(\theta)$ so for n a positive integer $\lambda = n^2$ is an eigenvalue with associated eigenfunction

$$\Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$$

For $n^2 = 0 = \lambda$,

$$\Theta_0(\theta) = \frac{a_0}{2} + b_0\theta = \frac{a_0}{2} = c$$

where $b_0 = 0$ for periodicity and the scaling is to mimic Fourier series representation. The constant is also known as c sometimes.

3. For $n \neq 0$,

$$r(rR')' - n^2 R_n = 0$$
$$r^2 R_n'' + rR_n' - n^2 R_n = 0$$

This is an equidimensional equation so use ansatz $R_n \propto r^\beta$:

$$\beta^2 - n^2 = 0$$
$$\beta = \pm n$$

so

$$R_n(r) = c_n r^n + d_n r^{-n},\ n = 1, 2, ...$$

If $n = 0$, $(rR')' = 0$ so

$$c_0 + d_0 \log r = R_0 = \psi_0(r, \theta)$$

since Θ_0 is simply a constant.

4. Combine the results above:

$$\psi(r, \theta) = c_0 + d_0 \log r + \sum_{n=1}^{\infty} (a_n \cos n\theta + b_n \sin n\theta)(c_n r^n + d_n r^{-n})$$

Note: only 3 of a_n, b_n, c_n and d_n are needed for a complete description as the solution can always be scaled.

9.4.2 Example: Laplace's Equation in a Unit Disc

We want to solve the Laplace's equation for $0 \leq r \leq 1$ such that

$$\psi(1, \theta) = f(\theta)$$

and we require regularity at the origin.

Regularity at $r = 0$ implies that $d_0 = d_n = 0$ so

$$\psi(r,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta) r^n$$

at $r = 1$,

$$f(\theta) = \psi(1,\theta) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\theta + b_n \sin n\theta)$$

Use Fourier series

$$a_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\cos n\theta d\theta$$
$$b_n = \frac{1}{\pi}\int_0^{2\pi} f(\theta)\sin n\theta d\theta$$

Note. Higher harmonics, i.e. larger n have influence which is localised near $r = 1$ due to r^n factor.

9.4.3 Cylindrical Polar Coordinates

$$\boldsymbol{\nabla}^2\psi = 0 = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2\psi}{\partial\theta^2} + \frac{\partial^2}{\partial z^2}\psi$$

Seperation of variables algorithm:

1. $\psi = R(r)\Theta(\theta)Z(z)$ such that

   $$\Theta'' = -n^2\Theta$$
   $$Z'' = k^2 Z$$

   so

   $$r\frac{d}{dr}\left(r\frac{dR}{dr}\right) + (k^2r^2 - n^2)R = 0$$

   Note substituting $kr = x$ obtain Bessel's equation of order n.

2. Solve to get

   $$\Theta_n(\theta) = a_n \cos n\theta + b_n \sin n\theta$$
   $$Z_k(z) = C_k e^{-kz} + d_z e^{kz}$$

3. Substitute $kr = x$:

   $$R_{n,k}(r) = \alpha_{n,k} J_n(kr) + \beta_{n,k} Y_n(kr)$$

4. The full solution is

   $$\psi(r,\theta,z) = \sum_{n=0}^{\infty}\sum_{j=1}^{\infty}\left(\alpha_{j,n}J_n(k_jr) + \beta_{jn}Y_n(h_jr)\right) \times (a_n \cos n\theta + b_n \sin n\theta)(c_j e^{-k_jz} + d_j e^{k_jz})$$

9.4.4 Example: Heat Conduction in a Wire

Imagine a semi-infinite rod of circular cross section of radius a heated at one end (for simplicity at some spatially constant high temperature) with fixed lower temperature on the outer cylindrical surface.

The Laplace's equation is

$$\begin{aligned} \nabla^2\psi &= 0 \\ \psi(a,\theta,z) &= 0 \\ \psi(r,\theta,0) &= \Theta_0 \\ \psi &\to 0 \text{ as } z \to \infty \end{aligned}$$

The symmetry implies that there is no θ dependence so $b_n = 0 = a_n$ for $n > 0$. Similarly all $d_j = 0$. In addition, regularity at origin implies $\beta_{jn} = 0$ so

$$\psi(r,\theta,z) = \sum_{j=1}^{\infty} A_j J_0\left(\frac{k_j r}{a}\right) e^{-k_j z/n}$$

It is an exercise to show that

$$\psi(r,\theta,z) = \sum_{j=1}^{\infty} \frac{2\Theta_0}{k_j J_1(k_j)} J_0\left(k_j \frac{r}{a}\right) e^{-k_j z/a}$$

10 Legendre's Equation

Laplace's equation is often solved on the sphere, which is an example of *orthogonal polynomials.*

10.1 Laplace's Equation in Spherical Polar

Recall that spherical polar is parameterised by r, θ, ϕ where r is the distance from the origin, θ is the angle $\mathbf{r}$ makes with the positive z-axis (so $\theta = \frac{\pi}{2}-$ latitude in radians) and ϕ is the angle the projection of $\mathbf{r}$ onto the xy-plane makes with positive x-axis (ϕ is the longitude).

In spherical polar, Laplace's equation becomes

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\psi}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\psi}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\psi}{\partial\phi^2} = 0$$

In this course, we restrict out attention to axisymmetric solutions so ψ is independent of ϕ.

Seperation of variables algorithm:

1. $\psi(r,\theta) = R(r)\Theta(\theta)$ substitute into the equation and multiply across by $\frac{r^2}{R\Theta}$,

$$\begin{aligned} \frac{1}{R}\frac{d}{dr}\left(r^2\frac{dR}{dr}\right) + \frac{1}{\Theta\sin\theta}\frac{d}{d\theta}\left(\sin\theta\frac{d\Theta}{d\theta}\right) &= 0 \\ (r^2R')' - \lambda R &= 0 \\ (\sin\theta\Theta')' + \lambda\sin\theta\Theta &= 0 \end{aligned}$$

10.2 Legendre's Equation

Make the substitution $x = \cos\theta$ in the equations above, $0 \leq \theta \leq \pi$ and $-1 \leq x \leq 1$ and $\frac{d}{d\theta} = -\sin\theta\frac{d}{dx}$ so

$$-\sin\theta\frac{d}{dx}(\sin\theta(-\sin\theta\frac{d\Theta}{dx})) + \lambda\sin\theta\Theta = 0$$

which can be written in self-adjoint form

$$-\frac{d}{dx}\left((1-x^2)\frac{d\Theta}{dx}\right) = \lambda\Theta$$

This is known as *Legendre's equation.*

10.3 Legendre Polynomial

We require a bounded solution of Legendre's equation on $[-1.1]$. Sustitute a series solution

$$\Theta = \sum_{n=0}^{\infty} a_n x^n$$

$$(1-x^2)\frac{d^2\Theta}{dx^2} - 2x\frac{d\Theta}{dx} + \theta\Theta = 0$$
$$(1-x^2)\sum_{n=2}^{\infty} a_n(n)(n-1)x^{n-2} - 2\sum_{n=1}^{\infty} na_nx^n + \lambda\sum_{n=0}^{\infty} a_nx^n = 0$$

Consider the coefficients of x^n:

$$a_{n+2}(n+2)(n+1) - n(n-1)a_n - 2na_n + \lambda a_n = 0$$
$$a_{n+2} = \frac{n(n+1)-\lambda}{(n+2)(n+1)}a_n$$

There are two linearly independent solution L_e and L_0, L_e has $a_0 \neq 0, a_1 = 0$ and L_0 has $a_0 = 0, a_1 \neq 0$.

$$L_e = a_0\left(1 + \frac{-\lambda x^2}{2!} + \frac{(-\lambda)(6-\lambda)x^4}{4!} + \frac{(-\lambda)(6-\lambda)(20-\lambda)x^6}{6!} + ...\right)$$
$$L_0 = a_1\left(x + \frac{(2-\lambda)x^3}{3!} + ...\right)$$

Note $\frac{a_{n+2}}{a_n} \to 1$ as $n \to \infty$. THe series converges for $|x| < 1$ and diverges for $x = \pm 1$. Therefore for the solution to be bounded the series must terminate so

$$\lambda = m(m+1)$$

for some integer m. It follows that there are countably many eigenvalues. This defines the *Legendre polynomial* of degree m.

10.4 Properties of Legendre Polynomials

By convention they are scaled so $P_m(1) = 1$. Each $P_n(x)$ has n zeros in $[-1.1]$. When n is odd, $P_n(x)$ is odd about $x = 0$ and when n is even, $P_n(x)$ is even about $x = 0$. The first four of them is

m	λ	$P_n(x)$
0	0	1
1	2	x
2	6	$\frac{3x^2-1}{2}$
3	12	$\frac{5x^3-3x}{2}$
4	20	$\frac{35x^4-20x^2+3}{8}$

In addition the polynomials are orthogonal:

$$\int_{-1}^{1} P_n(x)P_m(x)dx = \frac{2}{2n+1}\delta_{mn}$$

So bounded functions on $[-1, 1]$ can be represented using the Legendre polynomials

$$f(x) = \sum_{n=0}^{\infty} a_nP_n(x)$$
$$a_n = \frac{2n+1}{2}\int_{-1}^{1} f(x)P_n(x)dx$$

10.5 General axisymmetric solution

Let's return to the original question

$$\psi(r,\theta) = R(r)\Theta(\theta)$$
$$\Theta_n(\theta) = P_n(x) = P_n(\cos\theta), \lambda = n(n+1)$$
$$(r^2R_n') - n(n+1)R_n = 0$$

The last equation is equidimensional so use ansatz $R_n \propto r^\beta$ so

$$\beta(\beta+1) = n(n+1)$$

which has two solutions $\beta = n$ and $\beta = -n-1$. Therefore particular solution is

$$\psi_n(r,\theta) = (a_n r^n + b_n r^{-(n+1)})P_n(\cos\theta)$$

and summing up

$$\psi(r,\theta) = \sum_{n=0}^{\infty}(a_n r^n + b_n r^{-(n+1)})P_n(\cos\theta)$$

a_n and b_n are determined from boundary conditions, remembering $\cos\theta = x$ and normalisation constant $\frac{2}{2n+1}$.

Example. Axisymmetric Laplace's equation inside the unit sphere:

$$\nabla^2\psi = 0, r < 1, \psi(1,\theta) = f(\theta)$$

The solution has to be regular at the origin so $b_n = 0$ for all n. Then at $r = 1$,

$$f(\theta) = \sum_{n=0}^{\infty} a_n P_n(\cos\theta), 0 \le \theta \le \pi$$

$f(\theta)$ can in general be reposed as $F(x)$ where

$$F(x) = \sum_{n=0}^{\infty} a_n P_n(x), x = \cos\theta, -1 \le x \le 1$$

so

$$a_n = \frac{2n+1}{2}\int_{-1}^{1} F(x)P_n(x)dx$$

10.6 Generating Function

Consider a point unit charge located at $z = 1$ in a Cartesian coordinate.

The potential is

$$\psi(r,\theta) = \frac{1}{\rho} = \frac{1}{\sqrt{1-2r\cos\theta + r^2}}$$

ψ satisfies Laplace's equation so it must be regular near the origin:

$$\psi(r,\theta) = \sum_{n=0}^{\infty} a_n P_n(\cos\theta) r^n.$$

By uniqueness of solution, these two must agree. Scale the polynomials so that $P_n(1) = 1$ so

$$\frac{1}{\sqrt{1-2r+r^2}} = \frac{1}{1-r} = \sum_{n=0}^{\infty} a_n r^n$$

so $a_n = 1$ and the solution is

$$\sum_{n=0}^{\infty} P_n(x) r^n = \frac{1}{\sqrt{1-2r+r^2}}$$

differentiate n times so

$$\frac{1}{n!}\frac{d^n}{dr^n}\left(\frac{1}{\sqrt{1-2rx+r^2}}\right)_{r=0} = P_n(x)$$

Exercise. Derive the orthogonality condition.

$$\begin{aligned}\int_{-1}^{1} \frac{1}{1-2rx+r^2} dx &= \sum_{m=0}^{\infty} r^m \sum_{n=0}^{\infty} r^n \int_{-1}^{1} P_n(x) P_m(x) dx \\ &= \sum_{n=0}^{\infty} \frac{2}{2n+1} r^{2n}\end{aligned}$$

10.7 Example: Unbounded Force Field

Consider a neutral conducting sphere in previously uniform electric field. Find the new potential V

$$\mathbf{E} = -\boldsymbol{\nabla} V$$

$$\boldsymbol{\nabla}^2 V = -\frac{\rho}{\rho_0} = 0$$

$$V(r,\theta) = \sum_{n=0}^{\infty} (a_n r^n + \frac{b_n}{r^{n+1}}) P_n(\cos\theta)$$

For field as $r \to \infty$, $V \to -E_0 z = -E_0 r \cos\theta$ so

$$V \to -E_0 r P_1(\cos\theta)$$

since $P_1(x) = x$. Thus we have $a_1 = -E_0$ and $a_n = 0$ for $n = 0, n > 1$. Thus

$$V(r_0) = 0 = \frac{b_0}{r_0} + \left(\frac{b_1}{r_0^2 - E_0 r_0}\right) P_1 \cos\theta + \sum_{n=2}^{\infty} \frac{b_n P}{denominator}$$

Since this is true on the whole sphere, $b_n = 0$ for $n \geq 2$, $b_0 = 0$, $b_1 = \phi E_0 r_0^2$. So

$$V = -E_0 r \cos\theta \left(1 - \frac{r_0^3}{r^3}\right)$$

This show that the effect of the sphere is local.

10.8 Connection with Electrostatic Multipoles

Consider chareges placed non-uniformly but axisymmetrically inside a sphere. Far away we must have

$$V(r,\theta) = \sum_{n=0}^{\infty} b_n r^{-(n+1)} P_n(\cos\theta)$$

For $n = 0$, $V \propto b_0/r$ isotropic monopole froma point charge
$n = 1$, $V \propto b_1 \cos\phi/r^2$ dipole field
$n = 2$ $V \propto \frac{b_2(3\cos^2\theta - 1)}{2r^3}$ quadrupole.
06/11/17

11 Green's Function

11.1 Construction of the Green's Function

From the properties of the δ function, $\mathcal{L}G = 0$ for $x \neq \xi$. Algorithm for Green's function:

1. construct a solution for region I $x < \xi$ from two linearly independent solutions y_1 and y_2 of the homogeneous problem $\mathcal{L}y = 0$ i.e.

$$G(x;\xi) = A(\xi)y_1(x) + B(\xi)y_2(x), a \le x < \xi$$

2. in region II $x > \xi$,

$$G(x;\xi) = C(\xi)Y_1(x) + D(\xi)Y_2(x), \xi < x \le b$$

   where Y_1 and Y_2 are also linearly indepdent solution of $\mathcal{L}y = 0$.

   There are four constant, so need four condition.

3. Apply the homogeneous boundary conditions at $x = a$ to eliminate either A or B:

$$G(a;\xi) = 0 \Rightarrow Ay_1(a) + By_2(a) = 0$$

4. Similarly

$$G(b;\xi) = 0 \Rightarrow CY_1(a) + DY_2(a) = 0$$

5. The third condition is that $G(x;\xi)$ must be continuous there:

$$Ay_1(\xi) + By_2(\xi) = CY_1(\xi) + DY_2(\xi)$$

6. The fourth condition is the *jump condition*:

$$\frac{dG}{dx}\Big|_{x=\xi-}^{x=\xi+} = \frac{1}{\alpha(\xi)} = \lim_{x\to\xi+}\frac{dg}{dx} - \cdots = \frac{1}{\alpha(\xi)}$$

   so

$$CY_1'(\xi) + DY_2'(\xi) - Ay_1'(\xi) - By_2'(\xi) = \frac{1}{\alpha(\xi)}$$

7.

$$\begin{aligned} y(x) = Y_1(x)\int_a^x C(\xi)f(\xi)d\xi + Y_2(x)\int_a^x D(\xi)f(\xi)d\xi \\ + y_1(x)\int_x^b A(\xi)f(\xi)d\xi + y_2(x)\int_x^b B(\xi)f(\xi)d\xi \end{aligned}$$

Condition ot $x = \xi$, proof by contradiction: Assume G is discontinuous at $x = \xi$. possible discontinuity is a finite jump $F\,H(\xi)$.

$$\frac{dG}{dx} \propto \delta(x-\xi),\ \frac{d^2T}{dx^2} \propto \delta'(x-\xi)$$

However

$$\alpha G'' + \beta G' + \gamma G = \delta(x - \xi)$$

there is no discontinuity of the strngth so G is continuous at $x = \xi$.

Jump condition: integrate over an arbitrary small interval around $x = \xi$:

$$\int_{\xi-\epsilon}^{\xi+\epsilon} \delta(x-\xi)dx = 1 + \int_{\xi-\epsilon}^{\xi+\epsilon} \alpha(x)\frac{d^2G}{dx^2}dx + ...$$

T_3: G is continuous, γ is bounded so $T_3 \to 0$ as $\epsilon \to 0$.

T_2: dG/dx is bounded, β is bounded so $T_2 \to 0$ as $\epsilon \to 0$.

T_1: α is condinuous, $\alpha(x) \to \alpha(\xi)$ as $\epsilon \to 0$ so

$$T_1 \to \alpha(\xi) \lim_{\epsilon \to 0} \int_{\xi-\epsilon}^{\xi+\xi} \frac{d^2G}{dx^2}dx = \alpha(\xi)\frac{dG}{dx}\Big|_{x=\xi-}^{x=\xi+}$$

Jump condition is established.

11.2 Example of Construction

$$-y'' - y = f(x), y(0) = y(1) = 0$$

Algorithm:

1. For $0 \le x < \xi$, $G'' + G = 0$ so

$$G = A\cos x + B\sin x$$

2. For $\xi < x \le 1$, $G'' + G = 0$ so

$$G = C\cos(1-x) + D\sin(1-x)$$

This solution is equivalent to the "obvious" solution but it is easier to apply boundary conditions.

3. $G(0;\xi) = 0$ implies that $A = 0$ and $G(1,\xi) = 0$ implies that $C = 0$.

4. Continuity at $x = \xi$ implies that

$$B = D\frac{\sin(1-\xi)}{\sin\xi}$$

5. Thus

$$G = \begin{cases} \frac{D\sin(1-\xi)}{\sin\xi}\sin x & 0 < x < \xi \\ D\sin(1-x) & \xi < x < 1 \end{cases}$$

Here $\alpha(x) = 1$ so

$$D[-\cos(1-x)]_{\xi+} - D\,... = -1$$

$$D = \frac{\sin\xi}{\sigma 1}$$

so

$$G(x;\xi) = \begin{cases} \frac{\sin(1-\xi)\sin x}{\sin 1} & 0 \le x < \xi \\ \frac{\sin(1-x)\sin\xi}{\sin 1} & \xi < x \le 1 \end{cases}$$

and

$$y(x) = \frac{\sin(1-x)}{\sin 1} \int_a^x dx$$

Note the symmetry of G with respect to x and ξ.

Note for the two linearly independent solutions

$$\begin{aligned} \hat{y}_1 &= \sin x \\ \hat{y}_2 &= \sin(1-x) \end{aligned}$$

of the $\mathcal{L}y = 0$ which satisfy boundary conditions at $x = a$ and $x = b$ respectively,

$$G(x;\xi) = \frac{}{denominator}$$

where

$$J = \alpha(x) W(\hat{y}_1, \hat{y}_2) = \alpha(x)[\hat{y}_1(x)\hat{y}_2'(x) - \hat{y}_2(x)\hat{y}_1'(x)]$$

Note. Be careful with $\xi < x$ and $\xi > x$ region of substitution.

11.3 Inhomogeneous Boundary Conditions

Homogeneous boundary conditions are essential to construction of Green's function. This method is a way to construct a *particular integral* with homogeneous boundary conditions. It is easy to construct a *complementary function* satisfying inhomogeneous boundary conditions and then add this to the particular integral in Green's function form.

Example. $-y'' - y = f(x) = \mathcal{L}y, y(0) = 1, y(1) = 1$

Inhomogeneous boundary conditions: $y_c(0) = 0$ implies that $C_1 = 0$, $y_c(1) = 1$ implies that $c_2 = \frac{1}{\sin 1}$.

...

12 Green's Function Properties

12.1 Equivalent of Eigenfunction Expansion

Remember section 4.5:

$$(\mathcal{L} - \hat{\lambda} w)y = f(x)$$

where $\hat{\sigma}$ is not an eigenvalue of $\mathcal{L}y$. The solution is

$$y(x) = \int_a^b \left(\sum n = 1^\infty \frac{Y_n(\xi)Y_n(x)}{\lambda_n - \hat{\lambda}} \right) f(\xi) d\xi$$

where Y_n are normalised eigenvectors with eigenvalues σ_n.

In chapter 14

$$\mathcal{L} = -\frac{d^2}{dx^2}, \hat{\lambda} = 1, w = 1$$

so

$$G_e(x; \xi) = ...$$

For $G_c = G_e$, we need to find the Fourier sine series representation of $G_c(x; \xi)$:

$$G_c(x; \xi) = \sum_{n=1}^{\infty} b_n(\xi) \sin n\xi x$$

i.e.

$$b_n = \frac{2 \sin n\pi\xi}{n^2\pi^2 - 1}$$

$$\begin{aligned}
b_n(\xi) &= 2 \int_0^1 G_c(x; \xi) \sin n\pi x dx \\
&= \int_0^\xi \frac{\sin x \sin(1-\xi)}{\sin 1} \sin n\pi x dx + \int_\xi^1 \frac{\sin(1-x)\sin\xi}{\sin 1} \sin n\pi x dx \\
&= \sin(n\pi\xi)(\frac{1}{n\pi - 1} - \frac{1}{n\pi + 1} \\
&= \frac{2 \sin n\pi\xi}{n^2\pi^2 - 1}
\end{aligned}$$

so $G_c($ and G_e are equivalent.

12.2 Physical Interpretation

Steady forced wave equation:

$$T\frac{\partial^2 y}{\partial x^2} - \mu y = \mu \frac{\partial^2 y}{\partial t^2} = 0$$

The solution is

$$\frac{\partial^2 y}{\partial x^2} = \frac{\mu(x)}{T} g = f(x)$$

Assume μ is constant,

$$y = \frac{\mu g}{2T} x(x - L)$$

Imagine a point mass m concentrated at $x = \xi$. Resolving forces horizontally show that T is constant along the string and resolving forces vertically show that

$$\begin{aligned} mg &= T(\sin\theta_1 + \sin\theta_2) \\ &= -T(\frac{y(\xi)}{\xi} + \frac{y(\xi)}{L-\xi} \text{ using approximation } \sin\theta = \tan\theta \end{aligned}$$

so

$$y(\xi) = \frac{mg}{T}\xi(\xi - L) = f(\xi)\xi(\xi - L)$$
$$y = \begin{cases} f(\xi)\frac{x(\xi-L)}{L} & 0 \le x \le \xi \\ f(\xi)\frac{\xi(\xi-L)}{L} & \xi \le x \le L \end{cases}$$

The Green's function of this problem is

$$\mathcal{L}G = \frac{d^2}{dx^2}G = \delta(x - \xi)$$

1. $\frac{d^2G}{dx^2} = 0$ in region I so $Ax + B$,
2. $\frac{d^2G}{dx^2} = 0$ in region II so $c(x - L) + D$,
3. Boundary condition at 0, $B = 0$
4. Boundary condition at L, $D = 0$
5. Continuity at $x = \xi$, $c = \frac{A\xi}{\xi-L}$
6. Apply jump condition, $A = \frac{\xi-L}{L}$
7. The result is
$$G(x;\xi) = \begin{cases} \frac{x(\xi-L)}{L} & 0 \le x \le \xi \\ \frac{\xi(x-L)}{L} & \xi \le x \le L \end{cases}$$

12.3 Continuum Generalisation

The problem is linear so it is possible to have N different point masses m_i locatee at ξ_i in which case

$$y(x) = \sum_{i=1}^{N} \frac{m_i g}{T} G(x;\xi_i) = \sum_{i=1}^{N} f(\xi_i) G(x;\xi_i)$$

It is at least plausible that we can take a continumm limit:

$$y(x) = \int_0^L f(\xi)G(x;\xi)d\xi$$

Exercise. Show you can recover the parabolic constant μ case from the Green's function integral.

12.4 Higher Order Differential Operators

If $\mathcal{L}y = f(x)$ is an nth order ODE (where we assumem the coefficient of $\frac{d^n}{dx^n}$ is 1 for simplicity and $n > 2$) with homogeneous boundary conditions on $[a, b]$ then

$$y(x) = \int_a^b f(\xi)G(x;\xi)d\xi$$

where G satisfies the homogeneous boundary conditions, $\mathcal{L}G = \delta(x - \xi)$, G and its first $n-2$ erivatives are continuous at $x = \xi$ and

$$\frac{d^{(n-1)}}{dx^{(n-1)}}G(\xi^+) - \frac{d^{(n-1)}}{dx^{(n-1)}}G(\xi^-) = 1$$

12.5 Application of Green's Function to Initial Value Problem

Green's functions can also be used to solve initial value problems. Consider

$$\mathcal{L}y = f(t), t \geq a, y(a) = y'(a) = 0$$

As before we wantto find

$$\mathcal{L}G = \delta(t - \tau)$$

1. Construct G for $a \leq t < z$:

$$G = Ay_1(t) + By_2(t)$$

with $\mathcal{L}y_1 = \mathcal{L}(y_2) = 0$ and they are linearly independent.

2. Apply *both* boundary conditions to this solution:

$$Ay_1(a) + By_2(a) = 0$$
$$Ay_1'(a) + By_2'(a) = 0$$

y_1, y_2 are linearly independent, so Wronskian is non-zero. Thus $A = B = 0$. Thus $G(t;\tau) = 0$ for $a \leq t \leq \tau$.

3. Construct G for $t \geq \tau$:

$$G = Cy_1(t) + Dy_2(t)$$

4. Apply conditions at τ:

$$Cy_1(\tau) + Dy_2(\tau) = 0$$
$$Cy_1'(\tau) + Dy_2'(\tau) = \frac{1}{\alpha(\xi)}$$

and so we have the Green's function

$$y(t) = \int_a^t f(\tau)G(t;\tau)d\tau$$

since $G \neq 0$ only when $t > \tau$.

Note the causality: the solution at t depdends only on the forcing on $a \leq \tau \leq t$.

Example. Consider the problem

$$\frac{d^2y}{dt^2} + y = f(t), y(0) = \dot{y}(0) = 0$$

The Green's function has the form

$$G = \begin{cases} 0 & \leq t \leq \tau \\ C\cos(t-\tau) + D\sin(t-\tau) & t > \tau \end{cases}$$

The continuity implies that $C = 0$. The jump condition says $D = 1$. Thus

$$G = \begin{cases} 0 & 0 \leq t \leq \tau \\ \sin(t-\tau) & t \geq \tau \end{cases}$$

so

$$y(t) = \int_0^t f(\tau)\sin(t-\tau)d\tau$$

Exercise. Show that if $f(t) = \sin t$,

$$y = \frac{1}{2}\sin t - \frac{1}{2}t\cos t.$$

13 Fourier Transforms

13.1 Connection of Fourier Transforms and Fourier Series

Consider the complex form of Fourier series of a $2L$-periodic function

$$f(x) = \sum_{n=-\infty}^{\infty} c_n^{(L)} \exp(\frac{in\pi x}{L})$$

where

$$c_n^{(L)} = \frac{1}{2L} \int_{-L}^{L} f(x) \exp(-\frac{in\pi x}{L}) dx$$

Question. What happens as $L \to \infty$?

Need the integrals to converge so assume that $f(x)$ is both absolutely integrable and square integrable:

$$\int_{-\infty}^{\infty} |f(x)| dx < \infty$$
$$\int_{-\infty}^{\infty} |f(x)|^2 dx < \infty$$

Now consider a fixed n but $L \to \infty$,

$$\lim_{L\to\infty} 2Lc_n^{(L)} = \lim_{L\to\infty} \int_{-\infty}^{\infty} f(x) \exp(-\frac{in\pi x}{L}) dx = \int_{-\infty}^{\infty} f(x) dx \leq \int_{-\infty}^{\infty} |f(x)| dx < \infty$$

This means that for fixed n,

$$\lim_{L\to\infty} c_n^{(L)} = 0.$$

Now let n vary too and think about the real line. Keep n/L fixed as L increases. The result is denser distribution of points in the real line with wavelengths $2L/n$.

Imagine repacing the discrete variable $n\pi/L$ by a continuous variable k. Then we define the *Fourier transformation*

$$\tilde{f}(k) := \lim_{L\to\infty} 2Lc_{kL/\pi}^{(L)} = \int_{-\infty}^{\infty} f(x) \exp(-ikx) dx$$

Fourier transformation is a linear operator which maps between physical spaces to *spectral spaces* (the k-wavenumber space). e^{-ikx} is the *kernel* (there exists other kernels of transformation).

13.2 Fourier Transform and its inverse

There is ambiguity in the definition of Fourier transform.

Convention. We will use the convetion

$$\tilde{f}(k) = \int_{-\infty}^{\infty} f(x) \exp(-ikx) dx$$
$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{f}(k) \exp(ikx) dk$$

Assume that

$$\tilde{f}(k) = A \int_{-\infty}^{\infty} f(x) \exp(\mp ikx) dx$$

($A = 1$, minus sign in this course). Now remember the definition of Fourier series:

$$\frac{f(x_+ + x_-)}{2} = \sum_{n=\infty}^{\infty} \frac{1}{2L} \int_{-L}^{L} f(y) \exp\left(-\frac{in\pi y}{L}\right) dy \exp\left(\frac{in\pi y}{L}\right)$$

Now let $h = \pi/c$ and let $n = \pm m$ (where $\mp$ is chosen the opposite of that in the Fourier transormation,

$$\frac{f(x_+ + x_-)}{2} = \frac{1}{2\pi} \sum_{m=\infty}^{\infty} h \exp(\pm imh) \int_{-L}^{L} f(y) \exp(\mp imhy) dy$$

Let $L \to \infty, h \to 0, mh \to k$ to obtain

$$\frac{f(x_+ + x_-)}{2} = \frac{1}{2\pi A} \int_{-\infty}^{\infty} \tilde{f}(k) \exp(\pm ikx) dx$$

defining the inverse transform.

13.3 Properties of Fourier Transform

1. Linearity:

$$\lambda f \tilde{+} \mu g(k) = \lambda \tilde{f}(k) + \mu \tilde{g}(k)$$

2. Translation:

$$g(x) = f(x - \lambda)$$
$$\tilde{g}(k) = \exp(-ik\lambda) \tilde{f}(k)$$

3. Dual to translation is frequency shift:

$$g(x) = \exp(i\lambda x) f(x)$$
$$\tilde{g}(k) = \tilde{(}k - \lambda)$$

4. Scaling:

$$g(x) = f(\lambda x)$$
$$\tilde{g}(k) = \frac{1}{|\lambda|} \tilde{f}\left(\frac{k}{\lambda}\right)$$

5.

$$g(x) = x f(x)$$
$$\tilde{g}(k) = \frac{i d\tilde{f}(k)}{dt}$$

13.3.1 Fourier Series of differentiation

Property 5 above is dual to a very important proporty of Fourier Transforms.

Let $f(x)$ be a continuous, piecewise continuous differentiable with a well-defined Fourier transform and $\lim_{x\to\infty} f(x) = 0$. Consider the Fourier transform of $g = df/dx$. Consider

$$\int_{-L}^{L} \frac{df}{dx} \exp(-kx)dx = f(x)\exp(-ikx)\Big|_{-L}^{L} + ik\int_{-L}^{L} f(x)\exp(-ikx)dx$$

Let $L \to \infty$, we get

$$\int_{-\infty}^{\infty} \frac{df}{dx} \exp(-ikx)dx = ik\tilde{f}(k) = \tilde{g}(k)$$

so differentiation in physical space is multiplication in spectral space. We can thus transform differentiation in physical space, a hard problem, into multiplication in spectral space, an easier problem, and finally inverse transform the solution to physical space.

13.3.2 Dirichlet's Discontinuous Formula

Consider the box-car function

$$f(tx) = \begin{cases} 1 & |x| \le a \\ 0 & \text{otherwise} \end{cases}$$

Calculate the transform:

$$\tilde{f}(t) = \int_{-a}^{a} \exp(-ikx)dx = \int_{-a}^{a} \cos kx dx = \frac{2\sin ka}{k}$$

Inverse transform:

$$\begin{aligned} f(x) &= \frac{1}{\pi}\int_{-\infty}^{\infty} \frac{\sin ka}{k} \exp(ikx)dk \\ &= \frac{2}{\pi}\int_{0}^{\infty} \frac{\sin ka \cos k\pi}{k} dk \\ &= \frac{1}{\pi}\int_{0}^{\infty} \frac{k(a+x)}{k} dk + \frac{1}{\pi}\int_{0}^{\infty} \frac{k(a-x)}{k} dk \end{aligned}$$

Recall from IA Differential Equations

$$\int_{0}^{\infty} \frac{\sin \lambda x}{x} dx = \frac{\pi}{2}\operatorname{sgn}(\lambda)$$

14 Properties of Fourier Transform

14.1 Convolution & Parseval's Theorem

Very often we encounter problems involving products of Fourier transforms:

$$\tilde{h}(k) = \tilde{f}(k)\tilde{g}(k)$$

with $f(x$ and $f(x)$ known, $h(x)$ unknown.

Asume $f(x)$ and $g(x)$ are piecewise continuously differentiable and one of the Fourier transforms is absolutely integrable. Apply the definitions:

$$h(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(k)\tilde{g}(k)\exp(ikx)dk = \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{g}(k)\exp(ikx)\int_{-\infty}^{\infty} f(u)\exp(-iku)dudt$$

since f and $\tilde{g}$ are absolutely integrable, we can invert the order of integration:

$$\begin{aligned} h(x) &= \int_{-\infty}^{\infty} f(u)\frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{g}(t)\exp(ik(x-u))dkdu \\ &= \int_{-\infty}^{\infty} f(x)g(x-u)du \end{aligned}$$

which by definition is the *convolution* of f and g.

Exercise. It has a dual:

$$\begin{aligned} h(x) &= f(x)g(x) \\ \Rightarrow \tilde{h}(k) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \tilde{f}(u)\tilde{g}(k-u)du \end{aligned}$$

14.2 Corollary Yielding Parseval's Theorem

Let $g(x) = f^*(-x)$, then

$$\begin{aligned} \tilde{g}(k) &= \int_{-\infty}^{\infty} f^*(-x)\exp(-ikx)dx \\ \Rightarrow \hat{g}^*(k) &= \int_{-\infty}^{\infty} f(-x)\exp(ikx)dx \end{aligned}$$

Let $y = -x$, we get

$$\tilde{g}^*(k) = \int_{-\infty}^{\infty} f(y)\exp(-iky)dy = \tilde{f}(k)$$

Substitute into convolution formula and inversion formula,

$$\int_{-\infty}^{\infty} f(u)f^*(u-x)du = h(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{f}(k)|^2\exp(ikx)dx$$

It is true for all x and in particular for $x = 0$,

$$\int_{-\infty}^{\infty} |f(u)|^2 du = \frac{1}{2\pi}\int_{-\infty}^{\infty} |\tilde{f}(k)|^2 dk$$

14.3 Fourier Tranform & delta-function

Assume $f(x)$ is continuous, square integrable and absolutely integrable. Transform and invert:

$$\begin{aligned} f(x) &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(ikx)\int_{-\infty}^{\infty} f(u)\exp(-iku)dudk \\ &= \int_{-\infty}^{\infty} f(u)\frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(ik(x-u))dkdu \end{aligned}$$

Compare this to the sampling property

$$f(x) = \int_{-\infty}^{\infty} f(u)\delta(x-u)du$$

which gives an altervative definition of δ-function:

$$\delta(x-u) = \delta(u-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\exp(ik(x-u))dk$$

The Fourier transform of a constant is easy to define in terms of δ-function as a dual property.

The Fourier transform of a δ-function $f(t) = \delta(x)$ is

$$\tilde{f}(k) = \int_{-\infty}^{\infty}\delta(x)\exp(-ikx)dx = 1$$

The Fourier transform of a constant funciton $f(x) = 2$ is

$$\tilde{f}(k) = \int_{-\infty}^{\infty}\exp(-ikx)dx = 2\pi\delta(k)$$

Note. This is the δ-function with argument k, not the transform of it.

By translation property,

$$\begin{aligned} f(x) &= \delta(x-a) \\ \tilde{f}(k) &= \int_{-\infty}^{\infty}\delta(x-a)\exp(-ikx)dx = \exp(-ika) \end{aligned}$$

Note. Localised in physical space, spread out in spectral space and vice versa.

14.4 Fourier Transform of Trigonometric Functions

Fourier transform of $H(x)$: assume $H(0) = \frac{1}{2}$. Let $g(x) = H(x)$, then $g(x) + g(-x) = 1$ for all x and

$$\tilde{g}(k) + \tilde{g}(-k) = 2\pi\delta(k)$$

but $H'(x) = \delta(x)$ so $ik\tilde{g}(k) = 1$. Now remember $k\delta(k) = 0 = \delta(k) = 0$ so

$$\tilde{g}(k) = \pi\delta(k) + \frac{1}{ik}$$

Remember Dirichlet derivative formula:

$$\frac{1}{2}\operatorname{sgn}(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{\exp(ikx)}{ik} dx$$

so

$$f(x) = \frac{1}{2}\operatorname{sgn}(x) \leftrightarrow \tilde{f}(k) = \frac{1}{ik}$$
$$f(x) = \frac{1}{2}\operatorname{sgn}(x-a) \leftrightarrow \tilde{f}(k) = \frac{\exp(-ika)}{ik}$$

Note that these are generalised functions.

15 Applications of Fourier Transforms

15.1 ODEs

Consider
$$\frac{d^2}{dx^2}y - A^2y = -f(x), -\infty < x < \infty$$
$y \to 0, y' \to 0$ as $|x| \to \infty$ and $A > 0$ is constant.
$$-(k^2 + A^2)\tilde{y}(k) = -\tilde{f}(k) \Rightarrow \tilde{y}(k) = \tilde{f}(t)\left(\frac{1}{A^2 + k^2}\right)$$

Consider $h(x) = \frac{\exp(-\mu|x|)}{2\mu}$, $\mu > 0$,
$$\begin{aligned}\tilde{h}(k) &= \operatorname{Re}\frac{1}{\mu}\int_0^\infty \exp(-x(\mu + ik))dx \\ &= \frac{1}{\mu}\operatorname{Re}\frac{-\exp(-x(\mu + ik))}{\mu + ik}\Big|_0^\infty \\ &= \frac{1}{\mu}\operatorname{Re}\frac{1}{\mu + ik} \\ &= \frac{1}{\mu^2 + k^2}\end{aligned}$$

Therefore
$$y(x) = \frac{1}{2A}\int_{-\infty}^{\infty} f(u)\exp(-A|x-u|)du$$

Exercise. Show that she solution can be found using Green's function. What are the boundary conditions on Green's function as $|x| \to \infty$?

15.2 Application of Fourier Transforms to Linear Systems.

Suppose there is a linear operator $\mathcal{L}$ forced by input $I(t)$ to give an output O:
$$\mathcal{L}[O] = I$$

Consider a general signal with frequency ω. It can be written as
$$\begin{aligned}G(t) &= \operatorname{Re}|A|\exp(i(\omega t + \phi)) \\ &= (|A|\cos\phi)\cos\omega t + (-|A|\sin\phi)\sin\omega t\end{aligned}$$

where $|A|$ is the amplitude and ϕ is the phase. Consider the signal
$$I(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\tilde{I}(\omega)\exp(i\omega t)d\omega,$$

the synthesis of a pulse. The Fourier transorm in the *resolution* of the pulse
$$\tilde{I}(\omega) = \int_{-\infty}^{\infty} I(t)\exp(-i\omega t)dt$$

Now imageine an amplifier ($\mathcal{L}$ operator) takes as input forcing the signal I, does something to it and makes an output:

$$O(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{R}(\omega)\tilde{I}(\omega) \exp(i\omega t) d\omega$$

where $\tilde{R}(\omega)$ is the *transfer function* of the operator.

The transfer function is the Fourier transform of another function the *response function* $R(t)$

$$R(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \tilde{R}(\omega) \exp(i\omega t) d\omega$$
$$\tilde{R}(\omega) = \int_{-\infty}^{\infty} R(t) \exp(-i\omega t) dt$$

Since by convolution Fourier transform of $O(t)$ is the product of Fourier transforms, convolution theorem says

$$O(t) = \int_{-\infty}^{\infty} I(u) R(t-u) du$$

Causality: wlog $I(t) = 0$ for $t < 0$ so the operator doesn't do anything without input (no hum). $R(t) = 0$ for $t < 0$ so

$$O(t) = \int_{0}^{t} I(u) R(t-u) du$$

which is the Green's function for initial value problem.

15.3 General form of Transfer Function

For simplicity assume that the operator does not depende on derivatives of the input:

$$\mathcal{L}_n[O(t)] = I(t) = \left(\sum_{i=0}^{n} u_{n-i} \frac{d^i}{dt^i} \right) O(t)$$

apply Fourier transforms:

$$\tilde{I} = \left(a_n + a_{n-1} i\omega + \cdots + a_1 (i\omega)^{n-1} + a_0 (i\omega)^n\right) \tilde{O}(\omega)$$
$$\tilde{R}(\omega) = \frac{1}{a_n + a_{n-1} i\omega + \cdots + a_1 (i\omega)^{n-1} + a_0 (i\omega)^n}$$

By Fundamental Theorem of Algebra, the denominator can be factored into a product of roots so

$$\tilde{R} = \frac{1}{a_0 (i\omega - c_1)^{k_1} \cdots (i\omega - c_j)^{k_j}}$$

where $\sum_{j=1}^{j} k_i = n$. For simplicity (require that the system to be stable), $\mathrm{Re}(c_j) < 0$. Applying partial fractions, $\tilde{R}$ can bge written as a simple sum of the form

$$\frac{\Gamma_{m_j}}{(i\omega - c_j)^m}, 1 \leq m \leq k_j$$

where Γ_{m_j} is a constant. Thus the general form of $\tilde{R}(\omega)$ is

$$\tilde{R}(\omega) = \sum_j \sum_m \frac{\Gamma_{m_j}}{(i\omega - c_j)^m}$$

Need to know the function $h_m(t)$ such that

$$\hat{h}_m(\omega) = \frac{1}{(i\omega - \alpha)^{m+1}}, m \geq 0, \operatorname{Re}\alpha < 0$$

But these are quiet easy: let $h_0(t) = \exp(\alpha t)$ for $t > 0$ and zero otherwise,

$$\tilde{h}_0(\omega) = \int_0^\infty \exp(\alpha - i\omega)t dt = \frac{1}{i\omega - \alpha}$$

Similarly $h_1(t) = t\exp(\alpha t)$ for $t > 0$ and zero otherwise and then

$$\tilde{h}_1(\omega) = \cdots$$

since we assume $\operatorname{Re}\alpha < 0$. By induction,

$$h_m(t) = \begin{cases} \frac{t^m e^{\alpha t}}{m!} & t > 0 \\ 0 & t \leq 0 \end{cases}$$

where $\operatorname{Re}\alpha < 0$.

As an application, consider a damped oscillator

$$\left(\frac{d^2}{dx^2} + 2p\frac{d}{dt} + p^2 + q^2\right) y = f(t), p > 0$$

Taking Fourier transforms:

$$(i\omega)^2\tilde{y} = 2ip\omega\tilde{y} + (p^2 + q^2)\tilde{y} = \tilde{f}$$

16 Discrete Fourier Transform

16.1 The Nyquist Frequency

Consider a signal which is sampled at evenly spaced time intervl Δ. There exists an underlying continuous function $h(t)$. Define $h_n = h(n\Delta)$ where $n \in \mathbb{Z}$. $f_s = \frac{1}{\Delta}$ is the sampling rate or sampling frequency. Note this frequency is different from the one we discussed before: if $h(t)$ has period then it has frequency $f = \frac{1}{T}$, as opposed to angular frequency $\omega = 2\pi f$. The sampling frequency chosen leads to the definition of the *Nyquist critical frequency* $f_c = \frac{1}{2\Delta}$.

Consider a signal in hte form of cosine with this frequency:

$$h(t) = A\cos 2\pi f_c t = A\cos\frac{\pi t}{\Delta}$$

h_n corresponds to the peaks and troughs of the wave. Therefore to catch the key properties of the wave we must sample at at least twice this frequency. For example, CDs are sampled at 44.1kHz so $f_c = 22.05$kHz while the audible range is $2 - 20000$Hz.

The margin of over-sampling is important because odd things happen near f_c. Consider

$$\begin{aligned} h(t,\phi) &= A\cos(2\pi f_c t + \phi) \\ &= A\cos\frac{\pi t}{\Delta}\cos\phi - A\sin\frac{\pi t}{\Delta}\sin\phi \end{aligned}$$

Note tha if sampling at $n\Delta$ the second term is always zero so the signal is indistinguishable from

$$\begin{aligned} g(t,\phi) &= A\cos(w\pi f_c t)\cos\phi \\ &= \frac{A}{2}(\cos) \end{aligned}$$

For this reason $g(t,\phi)$ and $h(t,\phi)$ are called *aliases* of each other.

16.2 Sampling Theorem

Rigorous results can be derived for band-width limited signals: $g(t)$ does not contain any frequency of magnitude greater than $\omega_{\max} = 2\pi f_{\max}$. Thus by the definition of Fourier transform

$$\tilde{g}(\omega) = \int_{-\infty}^{\infty} g(t)\exp(i\omega t)dt$$

and $\tilde{g}(\omega) = 0$ for $|\omega| > |\omega_{max}|$.

therefore

$$g(t) = ...$$

Now set the sampling interval $\Delta = \frac{1}{2f_{max}} = \frac{\pi}{\omega_{max}}$ and $t_n = n\Delta$,

$$g(t_n) = g_n = \frac{1}{2\pi}\int_{-\omega_{max}}^{\omega_{max}} \tilde{g}(\omega)\exp(\frac{w\pi\omega n}{\omega_{max}}d\omega$$

$$g_{-n} = \frac{\omega_{max}}{\pi} c_n$$

where c_n are the complex Fourier coefficients for series representation of a $2\omega_{max}$-periodic extesnsion $\tilde{g}_p(\omega)$ oft the Fourier transform $\tilde{g}_\omega$.

Therefore since the actual Fourier transform $\tilde{g}(\omega)$ is band-width limited, the actual Fourier transform of the underlying function is the product of the periodically repeating $\tilde{g}_p(\omega)$ and a box-car function

$$\begin{aligned}
\tilde{h}(\omega) &= \begin{cases} 1 \\ 0 \end{cases} \\
\tilde{g}(\omega) &= \tilde{g}_p(\omega)\tilde{h}(\omega) \\
&= \frac{\pi}{\omega_{max}} \sum_{n=\infty}^{\infty} g_n \exp - \frac{in\pi\omega}{\omega_{max}} \tilde{h}(\omega)
\end{aligned}$$

Invert, assuming we can swap integration and summation,

$$\begin{aligned}
g(t) &= \frac{1}{2\pi} \int_{-\infty}^{a} \tilde{g}(\omega) \exp(i\omega t) d\omega \\
&= \frac{1}{2\omega_{max}} \sum_{n=\infty}^{\infty} g_n \int_{-\omega_m}^{\omega_m} \exp i\omega(t - \frac{n\pi}{\omega_m}) d\omega \\
&= \frac{1}{2} \sum_{n=\infty}^{\infty} g_n \frac{\sin(\omega_m t - \pi n)}{\omega_m t - \pi n} \\
&= \Delta \sum_{n=\infty}^{\infty} g(n\Delta) \frac{\sin 2\pi f_{max}(t - n\Delta)}{\pi(t - n\Delta)}
\end{aligned}$$

The above equation is the *Shannon-Whittaker formula* This means that band-width limited function for continuous time can be represented exactly by this representation based on sampling at discrete time. This is the *Shannon sampling theorem.*

Note that the formula dependes on Δ and f_{max} alone.

Central role played by *Whittaker Cardinal sinc function*

$$\operatorname{sinc} t = \frac{\sin t}{t}.$$

What happenns for non-bandwidth limited functions? The answer is aliasing: power is "folded back" into the frequency below the Nyquist frequency.

16.3 Discrete Fourier Transform

Sampling theorem needs an infinite number of measurements but in the real world we only have finitely many, say N, measurements of a function $h(t)$. Let N be even. If the sampling interval is Δ then

$$h_m = h(t_m)$$

where $t_m = m\Delta$, $m = 0, 1, \dots, N - 1$. The best we can hope for is N estimates of Fourier transform at some frequency, clearly in the range $[-f_c, f_c]$ where $f_c = \frac{1}{2\Delta}$ is the Nyquist frequency discussed before. Therefore the most sensible choice is

$$f_n = \frac{n}{N\Delta}$$

where $n = -N/2, -N/2+1, \ldots, N/2$ so $\Delta_f = \frac{1}{N\Delta}$. Note that although there seem to be $2N+1$ numbers, $f_{\pm N/2} = \pm f_c$

Consider Fourier transform at fixed frequency f_n:

$$\begin{aligned}
\tilde{h}(f_n) &= \int_{-\infty}^{\infty} h(t)\exp(-2\pi i f_n t)dt \\
&\approx \Delta \sum_{m=0}^{N-1} h_m \exp(-\pi i f_n f_m) \\
&= \Delta \sum_{m=0}^{N-1} h_m \exp\left(-\frac{2\pi i}{N} mn\right)
\end{aligned}$$

This is the definition of *discrete Fourier transform.*

Properties of DFT: it maps N complex numbers h_m into N complex numbers $\tilde{h}(f_m)$ with no dependence on dimensional parameter.

$\tilde{h}_d(f_m)$ is periodic in n with period N. In derivation we chose to let n run from $-N/2$ to $N/2$. It is clear the the two end measurements are the same.

We can choose to let n run from 0 to $N-1$. Since

$$\tilde{h}_d(f_{-n}) = \tilde{h}_d(F_{N-n}),$$

there is a correspondence.

Note the actual Fourier transform at frequency f_n

$$\tilde{h}(f_n) \approx \Delta \tilde{h}_d$$

$$\begin{aligned}
h_m &= h(t_m) \\
&= \frac{1}{2\pi}\int_{-\infty}^{0} \tilde{h}_m \exp i\omega t_m d\omega
\end{aligned}$$