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MATHEMATICS TRIPOS

Part IV

Geometric Group Theory

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0 Introduction

Contents:

1. free groups: “universal property”, study subgroups using topology,
2. group presentations and constructions, ways of making new groups from old,
3. Cayley graphs, viewing groups geometrically (e.g. \mathbb{Z}), connections to group actions,
4. geometric properties of groups, growth, other geometric invariants, “dictionary” between algebra and geometry,
5. amenable groups.

1 Free groups

Let S be a set, called an *alphabet*, and let S^{-1} be the set of formal inverses of elements in S , i.e. $S^{-1} = \{s^{-1} : s \in S\}$. A *word* in the alphabet S is a finite sequence of elements in $S \cup S^{-1}$ and the empty word. A word is *reduced* if it does not contain occurrences of $ss^{-1}, s^{-1}s$. Given a word, we can reduce it by removing any such subwords. For example if $S = \{a, b, c\}$, $aa^{-1}bcb^{-1}bc^{-1}$ is a word and we can reduce it to bcc^{-1} , and further to b . This induces an equivalence relation such that there is a unique reduced word in each class. We also write s^2 for ss .

Definition (free group). The *free group* on the set S , denoted $F(S)$, is the set of reduced words in S , with the operation of concatenation (followed by reduction if necessary).

Free groups satisfies the universal property

Theorem 1.1. *Given a free group $F(S)$ with an inclusion $\iota : S \rightarrow F(S)$, whenever G is a group with a function $\varphi : S \rightarrow G$, there is a unique group homomorphism $\bar{\varphi} : F(S) \rightarrow G$ such that the following diagram commutes*

$$\begin{array}{ccc} S & \xrightarrow{\iota} & F(S) \\ & \searrow \varphi & \downarrow \bar{\varphi} \\ & & G \end{array}$$

Proof. Given $\varphi : S \rightarrow G$, define $\bar{\varphi} : F(S) \rightarrow G$ by $\bar{\varphi}(s_{i_1}^{\alpha_1} \cdots s_{i_n}^{\alpha_n}) = \varphi(s_{i_1})^{\alpha_1} \cdots \varphi(s_{i_n})^{\alpha_n}$. Check this is a homomorphism. \square

Definition (rank). The cardinality of S is the *rank* of $F(S)$, denoted by $\text{rk}(F(S))$.

Corollary 1.2. *If $|S| = |T|$ the $F(S) \cong F(T)$.*

Proof. If $|S| = |T|$ then there exists a bijection $\phi : S \rightarrow T$. Consider

$$\begin{array}{ccc} S & \longrightarrow & F(S) \\ & \searrow \theta & \downarrow \bar{\theta} \\ & & F(S) \end{array}$$

where $\bar{\theta}$ is a homomorphism by the universal property. Similarly we have $\overline{\theta^{-1}} : F(T) \rightarrow F(S)$ and $\overline{\theta^{-1}} \circ \bar{\theta} : F(S) \rightarrow F(S)$ extends the identity map $S \rightarrow F(S)$ so must be the identity on $F(S)$. Same for the other way so $\bar{\theta}$ is an isomorphism. \square

Notation. Write F_n for the isomorphism class of $F(S)$ with $|S| = n$.

Exercise. If $F_n \cong F_m$ then $n = m$.

Corollary 1.3. *Every group is a quotient of a free group.*

Proof. Given G , consider $F(G)$. By the universal property exists a homomorphism $\pi : F(G) \rightarrow G$ extending the identity, so must be surjective. \square

Definition. Let G be a group, $A \subseteq G$ a subset. Define $\langle A \rangle$ to be the intersection of all subgroups containing A , i.e. the unique smallest subgroup containing A . We also call it the subgroup generated by A .

Definition. G is *generated* by $A \subseteq G$ if $\langle A \rangle = G$. Then A is a *generating set* of G . G is *finitely generated* if exists a finite generating set of G .

Notation. Write $\langle a_1, \dots, a_n \rangle$ to mean $\langle \{a_1, \dots, a_n\} \rangle$.

Example.

1. \mathbb{Z}_n, \mathbb{Z} can be generated by one element.
2. \mathbb{Z}^n can be generated by $\geq n$ elements.
3. $F_2 = \langle a, b \rangle = \langle a, ab \rangle$ so generating sets are not unique.

Definition. A group F is *freely generated* by $S \subseteq F$ if for any group G and any map $\varphi : S \rightarrow G$, exists a unique homomorphism $\tilde{\varphi} : F \rightarrow G$ extending φ .

Lemma 1.4. *If F is freely generated by S then F is generated by S .*

1.1 Subgroups of free groups

Let's see some examples of subgroups of free groups.

- Given any $e \neq w \in F_n$, $\langle w \rangle \cong \mathbb{Z}$.
- Given $T \subseteq S$, $\langle T \rangle$ is a free subgroup of $F(S)$ of rank $|T|$.
- If $S = \{a, b\}$, the set $\{a^{-n}ba^n : n \in \mathbb{N}\}$ freely generates a subgroup of F_2 , so isomorphic to F_∞ (exercise).

Remark. Subgroups of finitely generated groups are not necessarily finitely generated.

Revision of fundamental groups. See IID Algebraic Topology. Particularly relevant to this course is $\pi_1(\bigvee_{i=1}^n S^1) = F_n$, and a connected loop-free graph is contractible so has trivial π_1 .

It is the fact that if X is sufficiently nice and $Y \subseteq X$ is closed simply connected, then collapsing Y to a point does not alter $\pi_1(X)$. In particular, for graphs we can collapsing T , a maximal spanning tree, to get a bouquet of circles. Since maximal spanning tree always exists (use axiom of choice if the

graph is infinite), π_1 of a graph is a free group of rank equal to the number of edges not in the maximal spanning tree.

Recall the Galois correspondence between subgroups of $\pi_1(X)$ and covering spaces: we have a bijection between covering maps $p : (\tilde{X}, \tilde{x}_0) \rightarrow (X, x_0)$ and subgroups of $\pi_1(X, x_0)$.

Thus let $X = \bigvee_{i=1}^n S^1$. For any $H \leq \pi_1(X) \cong F_n$, there is a covering space \bar{X} with $\pi_1(\bar{X}) \cong H$. Since \bar{X} , being a cover of a graph, is a graph, we have H is free. This shows that every subgroup of a free group is free.

We work out the rank of H given its index in F_n . The index of H in F_n is exactly the degree of the covering map $\bar{X} \rightarrow X$, i.e. the number of vertices of \bar{X} . Each vertex of \bar{X} has degree $2n$ so the number of edges in \bar{X} is $[F_n : H] \cdot 2n \cdot \frac{1}{2} = [F_n : H] \cdot n$. To work out the number of edges not in a maximal spanning tree, use the graph theoretic fact that a tree on n vertices has exactly $n - 1$ edges (exercise), so the number of edges not in a maximal spanning tree is

$$[F_n : H] \cdot n - [F_n : H] - 1 = (n - 1)[F_n : H] + 1.$$

Theorem 1.5 (Nielsen-Schreier). *Every subgroup of a free group is free and if the subgroup has finite index then*

$$\text{rk}(H) = [F_n : H](\text{rk}(F_n) - 1) + 1.$$

Mnemonic:

$$\text{rk}(H) - 1 = (\text{rk}(F_n) - 1)[F_n : H].$$

Example. A degree 2 cover of $S^1 \vee S^1$ realises F_3 as a subgroup of index 2 in F_2 .

The group of *covering transformation*, or *deck transformation* of a cover is the group of isomorphisms $\bar{X} \rightarrow \bar{X}$.

A cover is *normal* if for any two lifts of the basepoint $x_0 \in X$, there is a covering transformation of \bar{X} sending one to the other.

Normal covering spaces correspond to normal subgroups of $\pi_1(X)$. If the cover is normal then the group of covering transformations is isomorphic to $\pi_1(X)$ quotiented by the corresponding subgroup.

Example. In the previous example we have $F_3 \trianglelefteq F_2$. We can have a nonnormal index 3, and a normal one.

2 Group presentations and constructions

Definition (normal closure). The *normal closure* of a subset $A \subseteq G$, denoted $\langle\langle A \rangle\rangle$, is the unique smallest normal subgroup of G containing A .

Given a free group $F(S)$ and $R \subseteq F(S)$, we write $\langle S|R \rangle$ for the group $F(S)/\langle\langle R \rangle\rangle$. R and S are called *generators* and *relators* respectively.

Definition (group presentation). A *presentation* of a group G is an isomorphism of G with a group of the form $\langle S|R \rangle$.

G is *finitely presented* if it admits a presentation $\langle S|R \rangle$ with S, R finite.

Example.

1. If $R = \emptyset$ then $\langle S|R \rangle \cong F(S)$.
2. $\langle a|a^n \rangle \cong \mathbb{Z}_n$.
3. $\langle a, b|aba^{-1}b^{-1} \rangle$ is a presentation of \mathbb{Z}^2 : let $\mathbb{Z}^2 = \{(c^n, d^m) : n, m \in \mathbb{Z}\}$. We have a homomorphism

$$\begin{aligned} \varphi : F(a, b) &\rightarrow \mathbb{Z}^2 \\ a &\mapsto c \\ b &\mapsto d \end{aligned}$$

Need to show $\ker \varphi = \langle\langle aba^{-1}b^{-1} \rangle\rangle$. \supseteq is clear since \mathbb{Z}^2 is abelian so have $F(a, b)/\langle\langle aba^{-1}b^{-1} \rangle\rangle \twoheadrightarrow F(a, b)/\ker \varphi$. The domain is a 2-generated abelian group. But the only 2-generated abelian group that surjects onto \mathbb{Z}^2 is itself.

4. More generally a finitely generated abelian group is always finitely presented.
5. The same is true for nilpotent groups. Recall that G is *nilpotent* if the *lower central series* of G terminates in a finite number of steps. The lower central series of G is

$$G_0 = G, G_{i+1} = [G_i, G].$$

6. $\langle a, b|aba^{-1}b^{-2}, a^{-2}b^{-1}ab \rangle = \{1\}$.

Remark.

1. It is difficult to tell which group is given by a particular presentation. Indeed there does not exist an algorithm that, upon input of a presentation, can determine whether the corresponding group is trivial. This is the *word problem*, introduced by Dehn in early 20th century. The classes of groups for which it does have a solution are often geometry.
2. There are uncountably many isomorphism classes of finitely generated groups (even 2-generated). For reference, see de la Harpe *Geometric Group Theory* IIB. But there are only countably many isomorphism classes of finitely presented groups.

The notion of finite presentation makes sense without fixing a specific surjection of a free group.

Theorem 2.1. *Given a not necessarily finite presentation $\langle (s_j)_{j \in J} | (r_i)_{i \in I} \rangle$ of a finitely presented group G , there exists a finite subset $J_0 \subseteq J$ and a finite set $(\tilde{r}_i)_{i \in I_0}$ of elements of the free group $F((s_j)_{j \in J_0})$ such that $\langle (s_j)_{j \in J_0} | (\tilde{r}_i)_{i \in I_0} \rangle$ is a finite presentation of G .*

“Proof”. de la Harpe has a proof but it seems to be wrong. \square

Our aim is to prove that finite index subgroups of finitely generated (resp finitely presented) groups are finitely generated (resp finitely presented).

Definition (Schreier transversal). Let $F(S)$ be a free group and $H \leq F(S)$ a subgroup. A (right) *Schreier transversal* for H in $F(S)$ is a set J of reduced words such that each right coset of H in G contains exactly one word of J , called a *representative* of this class, and all initial segments of these words are also in J .

For $g \in F(S)$, denote by \bar{g} the element of J such that $Hg = H\bar{g}$.

Theorem 2.2. *For any $H \leq F(S)$, there a Schreier transversal J . Moreover H is freely generated by the set*

$$\{ts(\overline{ts})^{-1} : t \in J, s \in S \text{ and } ts(\overline{ts})^{-1} \neq 1\}.$$

Proof. Take $X = \bigvee_S S^1$ so $\pi_1 X = F(S)$. Take \overline{X} to be the cover corresponding to $H \leq F(S)$. The vertices of \overline{X} correspond to cosets of H in $F(S)$ and choosing a path from a fixed basepoint to a vertex gives us a coset representative for that coset. Pick a maximal spanning tree $T \subseteq \overline{X}$. Choosing the unique path to each vertex in T gives us coset representatives with initial segments that are also such paths. Since $H \cong \pi_1 \overline{X}$ and it is freely generated by the set of loops with exactly one edge not in T , this generating set is of the required form. \square

Remark. The argument also shows that the set of Schreier transversals for H in $F(S)$ is in bijection with the set of maximal spanning trees in \overline{X} .

Write $\gamma(t, s) = ts(\overline{ts})^{-1}$. Explicitly, given $h \in H$ written as $s_1 s_2 \cdots s_n$ where $s_i \in S \cup S^{-1}$, we can write

$$h = \gamma(1, s_1) \gamma(\overline{s_1}, s_2) \cdots \gamma(\overline{s_1 \cdots s_{i-1}}, s_i) \cdots \gamma(\overline{s_1 \cdots s_{n-1}}, s_n)$$

(use $\gamma(t, s^{-1}) = \gamma(\overline{ts^{-1}}, s)^{-1}$). This is the *Reidemeister-Schreier rewriting process*.

Theorem 2.3. *Let G be a group with presentation $\langle S | R \rangle$ and let $\varphi : F(S) \rightarrow G$ correspond to this presentation. Let $G_1 \leq G$ and let H be the subgroup of $F(S)$ containing $\ker \varphi$ such that $\varphi(H) = G_1$. Then G_1 has presentation*

$$\langle \gamma(t, s) : t \in J, s \in S, \gamma(t, s) \neq 1 | trt^{-1} : t \in J, r \in R \rangle$$

where J is a Schreier transversal for H in $F(S)$.

Proof. Have $G_1 = H/\langle\langle R \rangle\rangle^{F(S)}$ and would like to find some possibly larger set of words R' in H such that $G_1 = H/\langle\langle R' \rangle\rangle^H$. Let H be generated freely by $\gamma(t, s)$'s. The subgroup $\langle\langle R \rangle\rangle^{F(S)}$ is generated by $\{grg^{-1} : g \in F(S), r \in R\}$, and writing each g as $g = h_g\bar{g}$ where $h_g \in H, \bar{g} \in J$, we have

$$grg^{-1} = (h_g\bar{g})r(h_g\bar{g})^{-1} = h_g(\bar{g}r\bar{g}^{-1})h_g^{-1}$$

and so can take $R' = \{trt^{-1} : t \in J, r \in R\}$. Thus G_1 has the required presentation. \square

Corollary 2.4. *Any subgroup of finite index in a finitely generated (resp. finitely presented) group is itself finitely generated (resp. finitely presented).*

Proof. If $[G : G_1] < \infty$ then $[F(S) : H] < \infty$ so J is finite. \square

2.1 Free product

One way to create new finitely generated/presented groups from old one is via free products. Given two groups A, B , a *normal form* is an expression of the form $g_1g_2 \cdots g_n$ where $n \geq 0$ such that if $n = 0$, take the identity element, $g_i \in (A \setminus \{1\}) \amalg (B \setminus \{1\})$ and consecutive elements g_i, g_{i+1} do not lie in the same group. n is the *length of normal form*. We define multiplication of normal forms inductively by

- $(g_1 \cdots g_n) \cdot 1 = 1 \cdot (g_1 \cdots g_n) = g_1 \cdots g_n$.
- For $n, m \geq 1$, set

$$(g_1 \cdots g_n)(h_1 \cdots h_m) = \begin{cases} g_1 \cdots g_n h_1 \cdots h_m & \text{if } g_n, h_1 \text{ in different groups} \\ g_1 \cdots g_{n-1} k h_2 \cdots h_m & \text{if } g_n, h_1 \text{ in same group, } g_n h_1 = k \neq 1 \\ (g_1 \cdots g_{n-1})(h_2 \cdots h_m) & \text{if } g_n, h_1 \text{ in same group, } g_n h_1 = 1 \end{cases}$$

Definition (free product). The set of normal forms with this multiplication forms a group $A * B$, called the *free product* of A and B .

Remark.

1. The groups A, B embed naturally into $A * B$.
2. If $A, B \leq G$ such that any $g \neq 1$ in G can be represented in a unique way as a product $g = g_1 \cdots g_n$ with $g_i \in A \cup B \setminus \{1\}$ and consecutive g_i, g_{i+1} not in the same group, then $G = A * B$.

Theorem 2.5. *If $A = \langle S_A | R_A \rangle, B = \langle S_B | R_B \rangle$ and $S_A \cap S_B = \emptyset$ then*

$$A * B = \langle S_A \cup S_B | R_A \cup R_B \rangle.$$

Proof. Let $\varphi : F(S_A) \rightarrow A, \psi : F(S_B) \rightarrow B$ be the homomorphisms with $\ker \varphi = \langle\langle R_A \rangle\rangle^{F(S_A)}, \ker \psi = \langle\langle R_B \rangle\rangle^{F(S_B)}$. Let $\theta : F(S_A \cup S_B) \rightarrow A * B$ be the homomorphism coinciding with φ on S_A and ψ on S_B . Need to show $\ker \theta = \langle\langle R_A \cup R_B \rangle\rangle^{F(S_A \cup S_B)}$. \supseteq is trivial. For \subseteq , consider $g = g_1 \cdots g_n \in \ker \theta$ in normal form (using $F(S_A \cup S_B) = F(S_A) * F(S_B)$). Then

$$\theta(g) = \theta(g_1) \cdots \theta(g_n) = 1$$

in $A * B$. Thus exists i such that $\theta(g_i) = 1$, so $g_i \in \ker \varphi$ or $g_i \in \ker \psi$. Proceed by induction. \square

Example. $D_\infty = \langle a, b | a^2 = 1, a^{-1}ba = b^{-1} \rangle$. It is the automorphism group of the graph C_∞ , where a is a reflection (say about the origin) and b is a translation. Then D_∞ is generated by a and $c = ba$, both have order 2. Can check (by acting on C_∞ 's vertices and edges) that $(ca)^n, (ca)^n c, a(ca)^n c, a(ca)^n$ give different elements of D_∞ . By remark above

$$D_\infty = \langle a, c | a^2, c^2 \rangle = \langle a | a^2 \rangle * \langle c | c^2 \rangle = \mathbb{Z}_2 * \mathbb{Z}_2$$

Remark. $\mathbb{Z}_2 * \mathbb{Z}_2$ is the only free product of non-trivial groups that does not contain a non-abelian free group. For example $\mathbb{Z}_2 * \mathbb{Z}_3 \supseteq [\mathbb{Z}_2, \mathbb{Z}_3] \cong F_2$.

2.2 Group actions

Theorem 2.6 (ping-pong lemma). *Let G act on X . Let $H_1, H_2 \leq G$ such that $|H_1| \geq 3, |H_2| \geq 2$ and let $H = \langle H_1, H_2 \rangle$. Suppose there exists nonempty $X_1, X_2 \subseteq X$ with $X_2 \not\subseteq X_1$ such that*

$$\begin{aligned} h(X_2) &\subseteq X_1 \text{ for all } h \in H_1 \setminus \{e\} \\ h(X_1) &\subseteq X_2 \text{ for all } h \in H_2 \setminus \{e\} \end{aligned}$$

*then $H \cong H_1 * H_2$.*

Proof. Let w be a nonempty reduced word in the alphabet $H_1 \setminus \{e\} \amalg H_2 \setminus \{e\}$. Need to show that the element defined by w in G is not e . Cases:

- if $w = a_1 b_1 a_2 b_2 \cdots a_k$ with $a_i \in H_1 \setminus \{e\}, b_i \in H_2 \setminus \{e\}$, then

$$w(X_2) = a_1 b_1 \cdots a_k(X_2) \subseteq a_1 b_1 \cdots b_{k-1}(X_1) \subseteq \cdots \subseteq a_1(X_2) \subseteq X_1.$$

As $X_2 \not\subseteq X_1$, $w \neq e$ in G .

- if $w = b_1 a_2 b_2 \cdots b_k$, then let $a \in H_1 \setminus \{e\}$ we have $a^{-1} w a \neq e$ by above.
- if $w = a_1 b_1 \cdots a_k b_k$ take $a \in H_1 \setminus \{e, a_1\}$ and $a^{-1} w a \neq e$.
- if $w = b_1 a_2 \cdots a_k$, take $a \in H_1 \setminus \{e, a_k\}$ and use awa^{-1} .

\square

Example. Let $\mathrm{SL}_2(\mathbb{Z})$ act on \mathbb{R}^2 in the usual way. Consider

$$H_1 = \left\{ \begin{pmatrix} 1 & 0 \\ 2n & 1 \end{pmatrix} : n \in \mathbb{Z} \right\} = \left\langle \begin{pmatrix} 1 & 0 \\ 2 & 1 \end{pmatrix} \right\rangle \cong \mathbb{Z}$$

$$H_2 = \left\{ \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix} : n \in \mathbb{Z} \right\} = \left\langle \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} \right\rangle \cong \mathbb{Z}$$

and

$$X_1 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : |x| < |y| \right\}$$

$$X_2 = \left\{ \begin{pmatrix} x \\ y \end{pmatrix} : |x| > |y| \right\}$$

If $|x| > |y|$ then

$$|2nx + y| \geq |2n| \cdot |x| - |y| \geq 2|x| - |y| > |x|$$

so the two subgroups map the subsets into each other. Thus by ping-pong lemma

$$\langle H_1, H_2 \rangle \cong F_2 \leq \mathrm{SL}_2(\mathbb{Z}).$$

Check this subgroup has finite index.

Amalgamated free product Suppose $A \leq G, B \leq H$ and there is an isomorphism $\varphi : A \rightarrow B$. Then the *free product of G and H with amalgamation A and B (via φ)* is

$$G *_A H = G * H / \langle\langle \varphi(a)a^{-1} : a \in A \rangle\rangle.$$

The intuition is to glue G and H along A . G, H embed as subgroups and elements in $G *_A H$ admits normal forms. It is related to Seifert-van Kampen theorem.

Example. $\mathrm{SL}_2(\mathbb{Z}) = \mathbb{Z}_4 *_2 \mathbb{Z}_6$, see de la Harpe.

2.3 HNN extension

Suppose $A, B \leq G$, $\varphi : A \rightarrow B$ is an isomorphism. We want A and B to be isomorphic via conjugation, but it might not be the case in G so we extend G to a larger group. The *HNN extension* of G (with A, B, φ) is

$$G *_\varphi = G * \langle t \rangle / \langle\langle t^{-1}at\varphi(a)^{-1} : a \in A \rangle\rangle.$$

G embed in $G *_\varphi$ and its elements admit normal forms. It has application to calculation of fundamental groups of surface bundles.

2.4 Semidirect product

G is the *semidirect product of N by H* if $N \trianglelefteq G, H \leq G, N \cap H = \{e\}$ and $G = NH$. Write $N \rtimes H = G$. Equivalently, if $H \leq G$ and exists a

homomorphism $\varphi : G \rightarrow H$ (such that the inclusion is a section) such that $\ker \varphi = N$, then $G = N \rtimes H$, i.e. the short exact sequence of groups

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 1$$

splits.

Alternatively, given two groups H and N and a homomorphism $\alpha : H \rightarrow \text{Aut}(N)$, we can construct $G = N \rtimes_{\alpha} H$ as follow: as a set $G = N \times H$. The multiplication is defined by

$$(n_1, h_1)(n_2, h_2) = (n_1\alpha(h_1)(n_2), h_1h_2).$$

The subgroups $N \times \{e\}, \{e\} \times H$ satisfy the conditions above. Conversely, given subgroups N and H , we can recover $\alpha : H \rightarrow \text{Aut}(N), \alpha(h)(n) = hnh^{-1}$.

Example.

1. Direct product $H \times H$.
2. $D_{2n} \cong \mathbb{Z}_n \rtimes \mathbb{Z}_2$.
3. $\pi_1(\text{Klein bottle}) \cong \mathbb{Z} \rtimes \mathbb{Z}$ where the action is the only nontrivial automorphism.

More generally, a *group extension* is a group G given by

$$1 \longrightarrow N \longrightarrow G \longrightarrow H \longrightarrow 0$$

We say G is the *extension of N by H* .

Note.

1. If H is free then G splits.
2. Not all extensions split: $2\mathbb{Z} \rightarrow \mathbb{Z} \rightarrow \mathbb{Z}_2$.

2.5 Wreath product

Definition (wreath product). The *wreath product* of G and H , $G \wr H$, is $\bigoplus_H G \rtimes H$ where, thinking of $\bigoplus_H G$ as the set of finitely supported function $H \rightarrow G$, the action of H is given by

$$h(f)(h_1) = f(h^{-1}h_1).$$

Example. Lamplight group $\mathbb{Z}_2 \wr \mathbb{Z}$, which is not finitely presented. An element $f \in \bigoplus_{\mathbb{Z}} \mathbb{Z}_2$ is a function $\mathbb{Z} \rightarrow 0, 1$ with compact support and the action of \mathbb{Z} is shifting.

Theorem 2.7 (Kaloujnine-Krasner). *If D is a group and Q is a finite group then $D \wr Q$ contains an isomorphic copy of every extension of D by Q .*

Proof. If G is an extension of D by Q then let $\pi : G \rightarrow Q$ where $\ker \pi = D$. We are going to define $\varphi : G \rightarrow D \wr Q$ and show its an injective homomorphism. Choose transversal for D in G , writing it as a map $T : Q \rightarrow G$. For $a \in G$, define $f_a : Q \rightarrow D$ by

$$f_a(x) = T(x)^{-1}aT(\pi(a^{-1})x)$$

and define $\varphi(a) = (f_a, \pi(a))$. To show this is a homomorphism, if $a, b \in G$ then

$$\begin{aligned} (f_a \cdot \pi(a)f_b)(x) &= f_a(x)f_b(\pi(a)^{-1}x) \\ &= T(x)^{-1}aT(\pi(a^{-1})x) \cdot \underbrace{T(\pi(a)^{-1}x)^{-1}bT(\pi(b^{-1})\pi(a)^{-1}x)}_{=e} \\ &= T(x)^{-1}abT(\pi((ab)^{-1})x) \\ &= f_{ab}(x) \end{aligned}$$

For injectivity, suppose $a \in \ker \varphi$ then $\pi(a) = e$ so $a \in D$ and $e = f_a(x) = T(x)^{-1}aT(x)$ so $a = e$. \square

Remark. The proof works verbatim for not necessarily finite groups G and H by using $\prod_H G \rtimes H$.

2.6 Sketch of Bass-Serre theory

Reference: J-P. Serre, Trees.

Theorem 2.8. *Let $G = G_1 *_A G_2$. Then G acts without inversion of edges on a tree X such that the quotient graph $G \backslash X$ is a segment. Moreover this segment can be lifted to one in X such that the stabilisers of its vertices are G_1, G_2 and stabiliser of the edge is A .*

Sketch proof. Let $X^0 = G/G_1 \amalg G/G_2$ and the positively oriented edges $X^1_+ = G/A$. It is well-defined since $A = G_1 \cap G_2$. Check G acts on X via left multiplication.

X is connected: suffices to show gG_1 is connected to G_1 . Express g as $g_1g_2 \cdots g_n$ with $g_i \in G_1 \amalg G_2$ and no consecutive elements in the same G_i . Suppose $g_n \in G_2$, then $gG_2 = g_1 \cdots g_{n-1}G_2$. Proceed by induction.

Acyclicity follows from the uniqueness of normal form for amalgamated free product. \square

Serre gave a converse to the statement: if exists such an action then the group is an amalgamated product.

group G	G acting on a graph without inversion of edges
$G_1 *_A G_2$	$G \backslash X$ segment
$G = H *_\varphi$	$G \backslash X$ loop
fundamental group of a graph of groups $\pi_1(G, Y)$	$G \backslash X = Y$ a graph

Table 1: Correspondence

Consequence: Kurosh subgroup theorem: if $G = A * B$ then $H \leq G$ has the form $H = F(S) * (*\text{conjugate of subgroups of } A * ())$

3 Cayley graphs

We'll focus on finitely generated groups.

Definition (Cayley graph). Let $G = \langle S \rangle$, $S \subseteq G$ finite. The *Cayley graph* of G with respect to S ($\text{Cay}(G, S)$) is given by

$$\begin{aligned} V(\text{Cay}(G, S)) &= G \\ E(\text{Cay}(G, S)) &= \{(g, gs) : g \in G, s \in S\}. \end{aligned}$$

Example.

- $\mathbb{Z}^2, S = \{(1, 0), (0, 1)\}$. Grid
- In fact we don't have to require S generating G . $\mathbb{Z}^2, S = \{(1, 0)\}$. Parallel lines.

The Cayley graph has the following properties:

1. $\text{Cay}(G, S)$ is a $2|S|$ -regular graph.
2. $\text{Cay}(G, S)$ is connected if and only if $\langle S \rangle = G$.
3. Relators in elements of S give rise to cycles.
4. When $\langle S \rangle = G$, paths from e to g give words in S representing g .
5. $\text{Cay}(G, S)$ allows us to view G as a metric space, with *word metric*

$$d_S(g, h) = \min\{\text{length of path from } g \text{ to } h \text{ in } \text{Cay}(G, S)\}.$$

Define *word length* to be $|g| = d_S(e, g)$. Note $d_S(g, h) = |g^{-1}h|$.

6. It follows that G acts on $\text{Cay}(G, S)$ via left-multiplication by isometry.

Theorem 3.1. $\text{Cay}(F(S), S)$ is a tree.

Let \bar{X} be a covering space of $X = \bigvee_S S^1$, corresponding to $N \trianglelefteq F(S)$. Then \bar{X} is exactly $\text{Cay}(F(S)/N, \pi(S))$ where $\pi : F(S) \rightarrow F(S)/N$ is the quotient map.

Definition (quasi-isometric embedding, quasi-isometry). Let $(X, d_X), (Y, d_Y)$ be metric spaces. A map $f : X \rightarrow Y$ is a *quasi-isometric embedding* if exists $\lambda \geq 1, C \geq 0$ such that for all $a, b \in X$,

$$\frac{1}{\lambda}d_X(a, b) - C \leq d_Y(f(a), f(b)) \leq \lambda d_X(a, b) + C.$$

f is a *quasi-isometry* if in addition exists $D \geq 0$ such that for all $y \in Y$ exists $x \in X$ such that $d_Y(f(x), y) \leq D$. Write $X \simeq_{\text{QI}} Y$ and we usually say (λ, C, D) is a quasi-isometry.

Quasi-isometry preserves large-scale structure of a space.

Proposition 3.2. *Quasi-isometry is an equivalence relation on metric spaces.*

Example.

1. A non-empty bounded metric space is quasi-isometric to a point. In particular all finite groups have Cayley graphs quasi-isometric to a point.
2. $\mathbb{R} \times [0, 1] \simeq_{\text{QI}} \mathbb{R}$.
3. $\text{Cay}(\mathbb{Z}, S) \simeq_{\text{QI}} \mathbb{R}$.
4. $\text{Cay}(\mathbb{Z}^n, S) \simeq_{\text{QI}} \mathbb{R}^n$.

Example. Cayley graph cannot determine the group. For example $C_4 \not\cong C_2 \times C_2$ but taking S to be the set of all elements, both Cayley graphs are the complete graph on four vertices.

In fact Cayley graph doesn't even determine the group with respect to a minimal generating set. For example $C_2 \times C_3 = \langle (1, 0), (0, 1) \rangle$, $S_3 = \langle (12), (123) \rangle$. The resulting Cayley graphs are isomorphic as undirected graphs but nonisomorphic as directed graphs.

On the other hand we can obtain nonisomorphic Cayley graphs of G by choosing different generating sets. For example $C_2 \times C_3 = \langle (1, 0), (0, 1) \rangle = \langle (1, 1), (1, 0) \rangle$.

Example. 3 regular trees $T_3 \cong T_4$ by contracting edges (graph)

Example. On the other hand, by using quasi-isomorphism invariants we can show some Cayley graphs are not isomorphic.

1. boundedness is an invariant so for example $\mathbb{R} \not\cong_{\text{QI}} *$.
2. $\mathbb{R} \not\cong_{\text{QI}} [0, \infty)$: suppose $\varphi : \mathbb{R} \rightarrow [0, \infty)$ is a quasi-isomorphism (λ, C, D) . Then $\varphi(t), \varphi(-t) \rightarrow \infty$ as $t \rightarrow \infty$. For any $x \in [0, \infty)$, let

$$\begin{aligned} M_x &= \max\{n \in \mathbb{Z} : \varphi(n) < x\} \\ N_x &= \min\{n \in \mathbb{Z} : \varphi(n) < x\} \end{aligned}$$

They exist because for all $x \in [0, \infty)$, there are only finitely many $n \in \mathbb{Z}$ with $\varphi(n) < x$. We thus have

$$\begin{aligned} \varphi(M_x) < x \leq \varphi(M_x + 1) \\ \varphi(N_x) < x \leq \varphi(N_x - 1) \end{aligned}$$

Since $d_{\mathbb{R}}(M_x, M_x + 1) = d_{\mathbb{R}}(N_x, N_x + 1)$,

$$\begin{aligned} d_{[0, \infty)}(\varphi(M_x), \varphi(N_x)) &\leq d_{[0, \infty)}(\varphi(M_x), x) + d_{[0, \infty)}(x, \varphi(N_x)) \\ &\leq d_{[0, \infty)}(\varphi(M_x), \varphi(M_x + 1)) + d_{[0, \infty)}(\varphi(N_x - 1), \varphi(N_x)) \\ &\leq (\lambda \cdot 1 + C) + (\lambda \cdot 1 + C) \\ &\leq 2\lambda + 2C \end{aligned}$$

which is bounded independent of x . But $d_{\mathbb{R}}(M_x, N_x) \rightarrow \infty$ as $x \rightarrow \infty$ as more and more elements will land in $[0, x)$ as $x \rightarrow \infty$. Absurd.

3. $\mathbb{R}^m \not\cong_{\text{QI}} \mathbb{R}^n$ for $m \neq n$.
4. $T_3 \not\cong_{\text{QI}} \mathbb{R}$.

Proposition 3.3. *Let G be a finitely generated group, S, S' two finite generating sets of G . Then $\text{Cay}(G, S) \cong_{\text{QI}} \text{Cay}(G, S')$.*

Proof. Consider the identity map $\varphi = \text{id} : |\text{Cay}(G, S)| \rightarrow |\text{Cay}(G, S')|$. Let

$$\begin{aligned}\lambda &= \max\{|a|_{S'} : a \in S\} \\ \lambda' &= \max\{|a|_S : a \in S'\end{aligned}$$

then

$$\begin{aligned}d_{S'}(\varphi(g), \varphi(h)) &\leq \lambda d_S(g, h) \\ d_S(g, h) &\leq \lambda' d_{S'}(\varphi(g), \varphi(h))\end{aligned}$$

Set $\lambda = \max \lambda, \lambda'$. □

Recall that

1. a metric space is *proper* if all closed balls are compact.
2. a metric space is *geodesic* if for any x, y there is a path between them with length $d(x, y)$.
3. an action G on X is *proper* if for all $K \subseteq X$ compact, $|\{g \in G : gK \cap K \neq \emptyset\}| < \infty$. This implies that X/G is Hausdorff and locally compact.

Theorem 3.4 (Švarc–Milnor lemma). *Let X be a proper geodesic metric space and G acts on X properly by isometry. Assume also the quotient X/G is compact. Then G is finitely generated and picking $x_0 \in X$ defines a quasi-isomorphism $\varphi_{x_0} : G \rightarrow X, g \mapsto gx_0$.*

Proof. Since the quotient space is compact, there is a closed ball $\overline{B} = \overline{B}(x_0, D)$ such that $G\overline{B} = X$. Since X is proper, \overline{B} is compact. Define

$$S = \{g \in G : g \neq e, g\overline{B} \cap \overline{B} \neq \emptyset\}$$

which is finite by properness. For $A, B \subseteq X$, define

$$d(A, B) = \inf\{d_X(a, b) : a \in A, b \in B\}.$$

Pick some $g \in G \setminus (S \cup \{e\})$ such that $d(\overline{B}, g\overline{B}) = R > 0$. Consider

$$H = \{g \in G \setminus (S \cup \{e\}) : d(\overline{B}, g\overline{B}) \leq R\}.$$

Note as H is a subset of

$$\{g \in G : g\overline{B}(x_0, D + R) \cap \overline{B}(x_0, D + R) \neq \emptyset\}$$

it is finite. Thus

$$\inf\{d(\overline{B}, g\overline{B}) : g \in G \setminus (S \cup \{e\})\} = \min\{d(\overline{B}, h\overline{B}) : h \in H\}$$

so the infimum, say $2d$, is achieved. Thus if $d(\overline{B}, g\overline{B}) < 2d$ then $g \in S \cup \{e\}$.

To prove that $G = \langle S \rangle$ we translate paths in X to words in G . Take $g \in G$. Let $k = \lfloor \frac{d_X(x_0, gx_0)}{d} \rfloor$. Take a sequence of points $y_0 = x_0, y_1, \dots, y_{k+1} = gx_0$ on the geodesic from x_0 to gx_0 such that $d_X(y_i, y_{i+1}) \leq d$ for all i . Take a corresponding sequence $h_i \in G$ such that for all i , $y_i \in h_i \bar{B}$. Taking $h_0 = e, h_{k+1} = g$. We have

$$d(h_i \bar{B}, h_{i+1} \bar{B}) \leq d_X(y_i, y_{i+1}) \leq d$$

so $d(\bar{B}, h_i^{-1} h_{i+1} \bar{B}) \leq d$ so $h_i^{-1} h_{i+1} \in S \cup \{e\}$, i.e. $h_{i+1} = h_i s$. Inductively $g = h_{k+1} = s_0 \cdots s_k$.

All word metrics on G are quasi-isometric so take S as above. Clearly we have that $2D$ -neighbourhood of the image of the map $\varphi_{x_0} : g \mapsto gx_0$ is X so just need to show φ_{x_0} is a quasi-isometric embedding. By construction

$$|g|_S \leq k + 1 \leq \frac{d_X(x_0, gx_0)}{d} + 1.$$

On the other hand if $|g|_S = m$ and $t_1 \cdots t_m = g$ in G , $t_i \in S$ then

$$\begin{aligned} d_X(x_0, gx_0) &\leq d_X(t_1^{-1} x_0, x_0) + d_X(x_0, t_2 \cdots t_m x_0) \\ &\leq d_X(x_0, t_1 x_0) + d_X(x_0, t_2 \cdots t_m x_0) \\ &\leq \sum_{i=1}^m d_X(x_0, t_i x_0) \\ &\leq 2Dm \\ &= 2D|g|_S \end{aligned}$$

Finally apply left-invariance of d_S and d_X . □

Corollary 3.5. *Let M be a compact connected Riemannian manifold and let \tilde{M} be its universal. $\pi_1 M$ acts on \tilde{M} isometrically so $\pi_1 M$ is finitely generated and is quasi-isometric to \tilde{M} .*

Corollary 3.6. *Let G be a connected real Lie group and let Γ be a cocompact lattice in G , i.e. a discrete subgroup such that G/Γ is compact. Then Γ is finitely generated and $\Gamma \cong_{\text{QI}} G$.*

Corollary 3.7. *Let G be a finitely generated group.*

1. *If H is a finite index subgroup of G then $G \cong_{\text{QI}} H$.*
2. *If $H \trianglelefteq G$ finite then $G \cong_{\text{QI}} G/N$.*
3. *If G, H are commensurable finitely generated groups then $G \cong_{\text{QI}} H$.*

Definition (commensurable). G and H are commensurable if exists $K_1 \leq G, K_2 \leq H$ finite index such that $K_1 \cong K_2$.

Proof.

1. H acts on $\text{Cay}(G)$.
2. G acts on $\text{Cay}(G/N)$.
3. Immediate.

□

| **Corollary 3.8.** *All finitely generated free groups are quasi-isometric.*

Proof. $F_n \leq F_2$ of finite index for all $n \geq 2$.

□

One may wonder if all quasi-isometric groups are commensurable. In other words, when does geometric similarity forces algebraic similarity?

Example. There exist groups which are quasi-isometric but not commensurable. Take

$$G = \mathbb{Z}_4 \wr \mathbb{Z}, H = (\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbb{Z}.$$

Not commensurable: the only elements of finite order in H , and hence in a finite index subgroup thereof, are of order 2. On the other hand, given a finite index subgroup $K \leq G$, K necessarily contains elements of order 4 as

$$[\bigoplus_{\mathbb{Z}} \mathbb{Z}_4 : K \cap \bigoplus_{\mathbb{Z}} \mathbb{Z}_4] = [K \cdot \bigoplus_{\mathbb{Z}} \mathbb{Z}_4 : K] < \infty$$

and the subgroup generated by all elements of order 2 in $\bigoplus_{\mathbb{Z}} \mathbb{Z}_4$ has infinite index.

Quasi-isometric: take the generating set

$$S = \{(0, 1)\} \cup \{(f_1, 0), (f_2, 0), (f_3, 0)\}$$

of $\mathbb{Z}_4 \wr \mathbb{Z}$ where $f_i(0) = i, f_i(n) = 0$ otherwise. Similarly take

$$S' = \{(0, 1)\} \cup \{(f_{(0,1)}, 0), (f_{(1,0)}, 0), (f_{(1,1)}, 0)\}$$

of $(\mathbb{Z}_2 \times \mathbb{Z}_2) \wr \mathbb{Z}$. Then the Cayley graphs are actually isomorphic.

Question: in which case does quasi-isometry imply commensurability? Typical rigidity question.

Since quasi-isometry only sees finite index subgroups, it is convenient to define

| **Definition** (virtual property). A group is called *virtually* P for some property P if it has a finite index subgroup that is P .

Example.

- $\mathbb{Z}_2 \times \mathbb{Z}$ is virtually \mathbb{Z} .
- $\text{SL}_2(\mathbb{Z})$ is virtually free.

Theorem 3.9. *Let G be a finitely generated group such that $\text{Cay}(G) \cong_{\text{QI}} \mathbb{Z}$. Then G is virtually \mathbb{Z} .*

Sketch proof. First show there is an element of infinite order in G . We will find $g \in G$ and $A \subseteq G$ such that $gA \subsetneq A$ (then $g^n \neq e$ for all n). Let $\varphi : \text{Cay}(G) \rightarrow \mathbb{R}$ be a quasi-isometry. As G acts on $\text{Cay}(G)$ by isometry, any $g \in G$ determines a quasi-isometry $\psi_g : \mathbb{R} \rightarrow \mathbb{R}$. Take $[0, \infty) \subseteq \mathbb{R}$, then $\psi_g([0, \infty))$ is either a bounded distance¹ from $[\psi_g(0), \infty)$ or $(-\infty, \psi_g(0)]$. If $\psi_g([0, \infty))$ is bounded distance from $[\psi_g(0), \infty)$ then setting

$$A = V(\text{Cay}(G)) \cap \varphi^{-1}([0, \infty)).$$

If $\psi_g(0) \gg 0$ then $gA \subsetneq A$: this is possible since $\psi_g(0)$ is bounded below in terms of the quasi-isometry constants, while $\inf \psi_g([0, \infty))$ (the “left-most point” the image of $[0, \infty)$ reaches before going off to infinity) is bounded above.

So need g so that $\psi_g([0, \infty))$ is bounded distance from $[\psi_g(0), \infty)$ and $\psi_g(0) \gg 0$. To find such g , take $h, k \in G$ such that e, g, k far apart in $\text{Cay}(G)$ (if and only if $\varphi(e), \varphi(h), \varphi(k)$ far apart in \mathbb{R}). Consider images of $[0, \infty)$ under ψ_e, ψ_h, ψ_k — at least two of these images will be of bounded distance from each other, so at least two of A, hA, kA are nested. So take g to be one of the elements of $h, k, k^{-1}h, h^{-1}k$.

Let $H = \langle g \rangle$. Want to show H has finite index in G . We have $d(e, g^n) \rightarrow \infty$ as $n \rightarrow \pm\infty$ and $d(g^n, g^m) = d(e, g^{n-m})$. Define

$$\begin{aligned} f : \mathbb{Z} &\rightarrow \mathbb{R} \\ n &\mapsto \varphi(g^n) \end{aligned}$$

Then $|f(n) - f(n-1)|$ is bounded independent of n and for all $r \geq 0$ exists $K \in \mathbb{N}$ such that $|f(n) - f(m)| \leq r$ implies $|m - n| \leq K$. It is an exercise to check that exists $C > 0$ such that for all $x \in \mathbb{R}$, exists $n \in \mathbb{Z}$ such that $|x - f(n)| \leq C$. Then exists C' such that for all $g' \in G$, exists $g^n \in H$ such that $d(g', g^n) \leq C'$, i.e. $\text{Cay}(G)/H$ is finite, so H is a finite index subgroup in G . \square

Other examples:

- If both groups are virtually abelian then quasi-isometry implies commensurability. This is an exercise.
- It is also true if just one of the groups is assumed to be abelian. This is much deeper. See next chapter.
- True if both groups are virtually free.
- True if only one of the groups is assumed to be virtually free.

¹ X is bounded distance from Y if exists $M \geq 0$ such that for all $x \in X$ exists $y \in Y$ such that $d(x, y) \leq M$ and vice versa.

4 Geometric property of groups

4.1 Growth

Notation. Let $f, g : X \rightarrow \mathbb{R}$ where $X \subseteq \mathbb{R}$, we write

- $f \preceq g$ if exists $a, b > 0$ and x_0 such that $f(x) \leq ag(bx)$ for $x \geq x_0$.
- $f \asymp g$ if $f \preceq g, g \preceq f$.

Definition (growth function). Let X be a discrete metric space and $x_0 \in X$ a basepoint. The *growth function* is the function

$$\beta_{X, x_0}(r) = |\overline{B}_X(x_0, r)|.$$

Lemma 4.1. *The equivalence class of growth function under \asymp is a quasi-isometry invariant for groups. In particular $\beta_{G, g} = \beta_{G, h}$ for all $g, h \in G$ and write β_G for this equivalence class.*

We write $\beta_{G, S}$ for the growth function relative to the generating set S .

Proposition 4.2.

1. If G is infinite then $\beta_{G, S}|_{\mathbb{N}}$ is strictly increasing.
2. $\beta_{G, S}(r+t) \leq \beta_{G, S}(r) \cdot \beta_{G, S}(t)$.
3. $\beta_{G, S}(r) \leq |S|^r$.

Example.

1. $\beta_{\mathbb{Z}^k}(r) \asymp r^k$.
2. $\beta_{F_k}(r) \asymp (2k)^r$.

Remark. The proposition implies that

$$\lim_{n \rightarrow \infty} \beta_{G, S}(n)^{1/n} \geq 1.$$

The limit exists by Fekete's lemma: if (a_n) is a subadditive sequence then $\lim_{n \rightarrow \infty} \frac{a_n}{n}$ exists.

Definition ((sub)exponential/polynomial growth). We say G has *exponential growth* if $\lim_{n \rightarrow \infty} \beta_{G, S}(n)^{1/n} > 1$. Otherwise we say G has *subexponential growth*.

G has *polynomial growth* if exists D such that $\beta_G(r) \leq r^D$.

Proposition 4.3.

1. If H is a finitely generated subgroup of G then $\beta_H \preceq \beta_G$.
2. If H is a finite index subgroup of G then $\beta_H \asymp \beta_G$.
3. If $N \trianglelefteq G$ then $\beta_{G/N} \asymp \beta_G$.

4. If $N \trianglelefteq G$ is finite then $\beta_{G/N} \asymp \beta_G$.

Proof. 2 and 4 are easy consequences Švarc-Milnor. For 1, take T to be a finite generating set of H and take $S \supseteq T$ to be a finite generating set of G . Then $\text{Cay}(H, T)$ is a subgraph of G so $d_S(e, h) \leq d_T(e, h)$ for all $h \in H$. Thus the closed ball of radius r about e in $\text{Cay}(G, S)$ contains the corresponding ball of $\text{Cay}(H, T)$.

For 3, take S to be a finite generating set of G . Let $T = SN/N$ be a finite generating set of N . $\pi : G \rightarrow G/N$ maps closed r -ball about e onto closed r -ball around e in G/N . \square

We can ask many questions about growth, for example

1. what types of growth can groups display?
2. which group have which types of growth?

We have seen that virtually abelian groups have polynomial growth. This generalises to

Proposition 4.4. *Let G be a 2-step nilpotent finitely generated group, i.e. $[[G, G], G] = \{e\}$. Then G has polynomial growth.*

Proof. Suppose G is generated by g_1, \dots, g_m . As G is 2-step nilpotent, $[G, G] \subseteq Z(G)$. We bound the size of all products of n generators using a normal form. Note we can exchange two elements at the cost of a commutator:

$$gh = hg \cdot g^{-1}h^{-1}gh = hg[g, h].$$

Commutators are central so we can move them to the right. Thus in $\leq n^2$ moves we can express the element as $g_1^{\alpha_1} \dots g_m^{\alpha_m} \cdot C$, where C is a product of $\leq n^2$ commutators. It is an easy exercise to check that $[G, G]$ is finitely generated, in this case by $[g_i^{\pm 1}, g_j^{\pm 1}]$. Thus the commutators are words of length 1 in generators of $[G, G]$, which has polynomial growth (say degree D). Thus G has polynomial growth of degree $\leq m + 2D$. \square

Theorem 4.5. *All finitely generated virtually nilpotent groups have polynomial growth.*

Proof. Exercise. \square

Can we push this result further? The natural class of groups to consider after nilpotent groups is solvable groups. Unfortunately, there do exist solvable groups of exponential growth, for example the lamplight group $\mathbb{Z}_2 \wr \mathbb{Z}$. In fact,

Theorem 4.6 (Gromov). *A finitely generated group has polynomial growth if and only if it is virtually nilpotent.*

Remark. The proof uses what is now called *asymptotic cones*, the limit of the objects $(X, \frac{d}{n})$ as $n \rightarrow \infty$.

Tits alternative: in the language of growth, a group either has exponential growth or is virtually solvable. Then we can use algebraic techniques to show it's virtually nilpotent.

c.f. paper by Wilkie van den Dries (rewrite the old paper), Kleiner (different proof, elementary but hard), Ozawa (functional analysis, representation theory)
 For ultralimits and asymptotic growth, see Druţu and Kapovich Chapter 7.
 There is also a proof based on approximate groups.

| **Corollary 4.7.** *Being virtually nilpotent is quasi-isometry invariant.*

Does there exist groups whose growth is between polynomial and exponential? The answer is yes, and such groups are said to have intermediate growth

| **Theorem 4.8** (Grigorchuk, 1983). *There exists a finitely generated group G such that*

$$2^{r^{\alpha_1}} \preceq \beta_G(r) \preceq 2^{r^{\alpha_2}}$$

| for $0 < \alpha_1 < \alpha_2 < 1$.

For more on intermediate growth see de la Harpe.

4.2 Ends

How many ways are there to go to infinity in a Cayley graph? Informally, for F_2 there are “infinitely many” while for \mathbb{Z}^2 there is “only one” way. For $\mathbb{Z} \times \mathbb{Z}/2$ there are “two” ways. Of course for finite groups there is no way to move to infinity. We are going to formalise this notion using *ends* and prove that, perhaps surprisingly, the above are all the possibilities that can arise from the Cayley graph of a finitely generated group.

| **Definition** (proper map). A map $f : X \rightarrow Y$ between topological spaces is *proper* if $f^{-1}(C)$ is compact whenever C is compact.

| **Definition** (ray). Let X be a topological space. A *ray* in X is a proper continuous map $r : [0, \infty) \rightarrow X$.

| **Definition** (convergence to the same end). Let $r_1, r_2 : [0, \infty) \rightarrow X$ be rays. r_1, r_2 *converge to the same end* if for all compact $C \subseteq X$, exists $N \in \mathbb{N}$ such that $r_1([N, \infty))$ and $r_2([N, \infty))$ are contained in the same path component of $X \setminus C$.

(pic of Calyay graph of F_2)

This defines an equivalence relation on rays, the equivalence classes of which are called the set of *ends* of X and is denoted $\text{Ends}(X)$. If $|\text{Ends}(X)| = m$ we say X has m ends.

We can topologise $\text{Ends}(X)$ by declaring a set $B \subseteq \text{Ends}(X)$ to be closed if $\text{end}(r_n) \in B$ for all n and $\text{end}(r_n) \rightarrow \text{end}(r)$ imply that $\text{end}(r) \in B$, where $\text{end}(r_n) \rightarrow \text{end}(r)$ if for all $C \subseteq X$ compact, exists a sequence of natural numbers (N_n) such that $r_n([N_n, \infty))$ and $r([N_n, \infty))$ lie in the same path component of $X \setminus C$ for n sufficiently large.

Definition. A k -path from x to y in a metric space X is a sequence of points $x_1 = x, x_2, \dots, x_n = y$ such that $d(x_i, x_{i+1}) \leq k$ for all i .

The following lemma justifies the heuristics at the beginning of the section of thinking ends as ways of escaping to infinity from a fixed point.

Lemma 4.9. *Let X be a proper geodesic metric space, $k > 0$ and r_1, r_2 rays in X . Let $G_{x_0}(X)$ be the set of (proper) geodesic rays starting at $x_0 \in X$. Then*

1. $\text{end}(r_1) = \text{end}(r_2)$ if and only if for all $R > 0$, exists $T > 0$ such that for all $t > T$, $r_1(t)$ can be connected to $r_2(t)$ by a k -path in $X \setminus B(x_0, R)$.
2. the natural map $G_{x_0}(X) \rightarrow \text{Ends}(X)$ is surjective.

Proof.

1. Every compact subset of X is contained in an open ball about x_0 and vice versa. Given a k -path from x_1 to x_n in $X \setminus B(x_0, R+k)$, concatenate any geodesics from x_i to x_{i+1} to get a continuous path in $B(x_0, R)$.
2. Let $r : [0, \infty) \rightarrow X$ be a ray. Let $c_n : [0, d_n] \rightarrow X$ be a geodesic from x_0 to $r(n)$ where $d_n = d(x_0, r(n))$. Extend c_n to $[d_n, \infty)$ by setting $c_n(t) = r(n)$ for $t \in [d_n, \infty)$. By Arzela-Ascoli, there exists a convergent subsequence of c_n converging to $c : [0, \infty) \rightarrow X$ a geodesic ray with $\text{end}(c) = \text{end}(r)$. □

Let X be a metric space. Given $f, g : X \rightarrow X$, say $f \sim g$ if $\sup_{x \in X} d_X(f(x), g(x))$ is finite. The set of equivalent classes of quasi-isometries of X forms a group, which we denote by $\text{QI}(X)$. A quasi-isometry $\varphi : X \rightarrow Y$ induces an isomorphism $\varphi_* : \text{QI}(X) \rightarrow \text{QI}(Y)$.

Proposition 4.10. *Let X and Y be proper geodesic metric spaces. A quasi-isometry $f : X \rightarrow Y$ induces a homeomorphism $\bar{f} : \text{Ends}(X) \rightarrow \text{Ends}(Y)$ which will be defined below. Then*

$$\begin{aligned} \text{QI}(X) &\rightarrow \text{Homeo}(\text{Ends}(X)) \\ f &\mapsto \bar{f} \end{aligned}$$

is a homomorphism.

Proof. Let r be a geodesic ray in X from x_0 , f_*r be the ray in Y obtained by concatenating some choice of geodesic segments $[f(r(n)), f(r(n+1))]$. f is a quasi-isometry implies that f_*r is a (proper) ray. $\text{end}(f_*r)$ is independent of the choice of geodesic segments. Define

$$\begin{aligned} \bar{f} : \text{Ends}(X) &\rightarrow \text{Ends}(Y) \\ \text{end}(r) &\mapsto \text{end}(f_*r) \end{aligned}$$

The image of a k -path under f is a $(\lambda k + c)$ -path so \bar{f} is well-defined and continuous by the previous lemma part 1.

Lemma part 2 ensures that \bar{f} is defined on all of $\text{Ends}(X)$. The rest are exercise. □

Definition (ends of a group). Let G be a finitely generated group. Then $\text{Ends}(G) = \text{Ends}(\text{Cay}(G))$.

Theorem 4.11. *Let G be a finitely generated group.*

1. G has 0, 1, 2 or finitely many ends.
2. G has 0 end if and only if G is finite.
3. G has 2 ends if and only if G is virtually \mathbb{Z} .
4. G has infinitely many ends if and only if G can be expressed as $A *_C B$ or $A *_C$ with C finite, $|A/C| \geq 3$, $|B/C| \geq 2$.

Proof. We prove 1. Fix a generating set S of G and work with $\text{Cay}(G, S)$. G acts on $\text{Cay}(G, S)$ by isometry, giving a homomorphism $G \rightarrow \text{Homeo}(\text{Ends}(\text{Cay}(G, S)))$. Let H be its kernel. Suppose $|\text{Ends}(G)| < \infty$, so H has finite index in G . Assume further $e_0, e_1, e_2 \in \text{Ends}(G)$ are distinct for contradiction. Fix geodesic rays $r_1, r_2 : [0, \infty) \rightarrow \text{Cay}(G)$ with $r_1(0) = r_2(0) = e_G$ such that $\text{end}(r_i) = e_i$. The ray corresponding to e_0 is defined slightly differently. Since H has finite index in G , exists $\mu > 0$ such that for all $g \in G$, exists $h \in H$ with $d(g, h) \leq \mu$. Thus exists a ray $r_0 : [0, \infty) \rightarrow \text{Cay}(G)$ with

- $\text{end}(r_0) = e_0$,
- $d(r_0(n), e_G) \geq n$,
- $r_0(n) \in H$ for all n .

Set $h_n = r_n(n)$. Fix $N > 0$ such that $r_i[N, \infty)$ lie in different path components of $\text{Cay}(G) \setminus B(e_G, N)$. If $t, t' > 2N$ then $d(r_1(t), r_2(t')) > 2N$ since any path joining $r_1(t)$ and $r_2(t')$ must pass through $B(e_G, N)$.

H acts trivially on $\text{Ends}(G)$ so $\text{end}(h_n r_i) = \text{end}(r_i)$ for all i . Let $n > 3N$. Then $h_n r_i(0) = h_n$ lie in a different path component of $\text{Cay}(G) \setminus B(e_G, N)$ from $r_i[N, \infty)$ for $i = 1, 2$, so $h_n r_i$ must through $B(e_G, N)$. Thus exists t_i such that $h_n r_i(t_i) \in B(e_n, N)$ for $i = 1, 2$. Since h_n is an isometry, $d(r_1(t_1), r_2(t_2)) < 2N$, contradicting $d(r_1(t_1), r_2(t_2)) > 2N$. \square

Remark. We have seen that being virtually nilpotent is a geometric property. In fact,

1. being virtually free is geometric, as it is equivalent to being quasi-isometric to a tree. c.f. Antolin.
2. being finitely presentable is geometric. c.f. Bridson-Haefliger Prop 8.24.

5 Amenability

5.1 Paradoxical decomposition

The motivating example for this chapter is *paradoxical decomposition*, which is the key argument in Banach-Tarski.

Definition (equidecomposable). Let G act on a set X and $A, B \subseteq X$. Say that A and B are (*finitely*) G -*equidecomposable* if exist partitions

$$\begin{aligned} A &= A_1 \cup A_2 \cup \cdots \cup A_n \\ B &= B_1 \cup B_2 \cup \cdots \cup B_n \end{aligned}$$

and $g_1, \dots, g_n \in G$ such that $g_i A_i = B_i$ for all i . We write $A \sim B$. If $A \sim C$ for some $C \subseteq B$ then write $A \lesssim B$.

A *realisation* h of $A \sim B$ is a bijection $h : A \rightarrow B$ such that there exists a decomposition as above with $h(a_i) = g_i(a_i)$ for all i and for all $a_i \in A_i$.

Note that if $h : A \rightarrow B$ is a realisation of $A \sim B$ and $S \subseteq A$ then $S \sim h(S)$. For fixed X , G -equidecomposability is an equivalence relation.

Theorem 5.1. *Suppose G acts on X and $A, B \subseteq X$. Then $A \lesssim B$ and $B \lesssim A$ implies $A \sim B$.*

Proof. Schöder-Bernstein. □

Corollary 5.2. *Let G act on X . Then TFAE:*

1. *there exist proper disjoint subsets $A, B \subseteq X$ such that $A \sim X \sim B$.*
2. *there exist proper disjoint subsets $A, B \subseteq X$ such that $A \cup B = X$ and $A \sim X \sim B$.*

Proof. For 1 \implies 2, since $X \sim B \subseteq X \setminus A$, have $X \lesssim X \setminus A$. Trivially $X \setminus A \lesssim X$. Thus $A \sim X \sim X \setminus A$. □

Definition (G -paradoxical). Let G act on X . If the condition in the previous corollary holds, we say X is (*finitely*) G -*paradoxical*.

Proposition 5.3.

1. F_2 is F_2 -paradoxical (left multiplication).
2. If F_2 acts on X freely then X is F_2 -paradoxical.

Proof.

1. Let $F_2 = \langle a, b \rangle$. Let $W(y)$ be the set of reduced words starting in y where $y \in \{a^{\pm 1}, b^{\pm 1}\}$. Then

$$F_2 = \{e\} \cup W(a) \cup (a^{-1}) \cup W(b) \cup W(b^{-1})$$

as a disjoint union. We can also write

$$F_2 = W(a) \cup aW(a^{-1}) = W(b) \cup bW(b^{-1})$$

so define

$$\begin{aligned} A &= W(a) \cup W(a^{-1}) \\ B &= W(b) \cup W(b^{-1}) \end{aligned}$$

and A, B satisfies the first condition in the corollary.

2. Take M to be a set of representatives of F_2 -orbits of X . Set

$$X_y = \{zm : z \in W(y), m \in M\}.$$

Then $X_a, X_{a^{-1}}, X_b, X_{b^{-1}}$ are disjoint and

$$X = X_a \cup aX_{a^{-1}} = X_b \cup bX_{b^{-1}},$$

giving the desired decomposition. □

Note that in the second part we need the axiom of choice.

Proposition 5.4. $F_2 \leq \text{SO}(3, \mathbb{R})$, with generators

$$\left(\begin{array}{ccc} 1 & 0 & 0 \\ 0 & \frac{1}{3} & \frac{-2\sqrt{2}}{3} \\ 0 & \frac{2\sqrt{2}}{3} & \frac{1}{3} \end{array} \right), \left(\begin{array}{ccc} \frac{1}{3} & \frac{-2\sqrt{2}}{3} & 0 \\ \frac{2\sqrt{2}}{3} & \frac{1}{3} & 0 \\ 0 & 0 & 1 \end{array} \right)$$

Proof. Exercise. Uses a ping-pong lemma argument. □

Theorem 5.5 (Hausdorff paradox). *There exists a countable set $D \subseteq S^2$ such that $S^2 \setminus D$ is $\text{SO}(3, \mathbb{R})$ -paradoxical.*

Proof. Every non-trivial element in $\text{SO}(3, \mathbb{R})$ fixes exactly two points of S^2 . Let D be the union of fixed points of $F_2 \subseteq \text{SO}(3, \mathbb{R})$. F_2 then acts freely on $S^2 \setminus D$. □

Proposition 5.6. *For any countable $D \subseteq S^2$, S^2 and $S^2 \setminus D$ are $\text{SO}(3)$ -equidecomposable.*

Proof. Let ℓ be a line through the origin that misses D . As D is countable, exists θ such that for all $n > 0$, the image $\rho^n(D)$ of D under rotation ρ^n by $n\theta$ about ℓ does not intersect D . Set $\bar{D} = \bigcup_{n=0}^{\infty} \rho^n(D)$. Then

$$\begin{aligned} S^2 &= \bar{D} \cup (S^2 \setminus \bar{D}) \\ &\sim \rho(\bar{D}) \cup \rho(S^2 \setminus \bar{D}) \\ &\sim \rho(\bar{D}) \cup (S^2 \setminus \bar{D}) \\ &= S^2 \setminus D \end{aligned}$$

□

Together this shows

Theorem 5.7 (Banach-Tarski). S^2 is $\text{SO}(3, \mathbb{R})$ -paradoxical.

Theorem 5.8. Let $E(3)$ be the group of isometries of \mathbb{R}^3 . Then any solid ball in \mathbb{R}^3 is $E(3)$ -paradoxical, as is \mathbb{R}^3 .

Moral: cannot put a finitely-additive probability measure that is invariant under rotations on subsets of S^2 .

Theorem 5.9 (Tarski). Let G act on X and $E \subseteq X$. Then there is a finitely-additive measure $\mu : \mathcal{P}(X) \rightarrow [0, \infty]$ with $\mu(E) = 1$ that is G -invariant if and only if E is not G -paradoxical.

5.2 Amenable group

Definition (amenable). Let G be a discrete (resp. locally compact) group. A measure on G is a finitely-additive left-invariant measure μ on $\mathcal{P}(G)$ (resp. Borel sets of G) with $\mu(G) = 1$. G is *amenable* if it has such a measure.

Remark. Clearly if G acts on itself by multiplication and G is paradoxical then G is not amenable. In particular F_2 is not amenable, nor is any group containing F_2 . One might naturally wonder if it is the only obstruction to amenability. This is the von Neumann conjecture: any non-amenable group contains F_2 . It is disproved by Ol'shanskii, who constructed a counterexample called Tarski monster. It has the strange property that for p a fixed prime, every non-trivial proper subgroup has order p .

Definition. Let G be a finitely generated group, and let $\ell^\infty(G)$ be the space of bounded functions on G . A linear functional $M : \ell^\infty(G) \rightarrow \mathbb{R}$ is a *left-invariant mean* on G if

- $M(f) \geq 0$ if $f(g) \geq 0$ for all $g \in G$.
- $M(\chi_G) = 1$ for the characteristic function on G .
- $M(g(f)) = M(f)$ for all $g \in G$, where $g(f)(h) = f(g^{-1}h)$.

Proposition 5.10. G is amenable if and only if G admits a left-invariant mean.

Proof. If G is amenable then define

$$M(f) = \int f d\mu.$$

Conversely if M is a left-invariant mean on G define

$$\mu(A) = M(\chi_A).$$

□

Proposition 5.11. *Let G amenable act on X . Then there exists a finitely-additive probability measure on $\mathcal{P}(X)$ that is G -invariant. In particular X is not G -paradoxical.*

Proof. Let μ be the measure realising amenability of G . Fix $x_0 \in X$ and define

$$\begin{aligned} \nu : \mathcal{P}(X) &\rightarrow [0, 1] \\ A &\mapsto \mu\{g \in G : g(x_0) \in A\} \end{aligned}$$

□

To summarise

Theorem 5.12. *TFAE:*

1. G amenable.
2. G admits a left-invariant measure.
3. G is not paradoxical.

Example. All finite groups are amenable via normalised counting measure.

Proposition 5.13.

1. If G is amenable and $H \leq G$ then H is amenable.
2. If G is amenable and $N \trianglelefteq G$ then G/N is amenable.
3. If $N \trianglelefteq G$ and G/N are amenable then G is amenable.
4. If $\{G_i\}$ is a direct system of amenable groups then so is $\varinjlim G_i$.

Proof.

1. Let $\mu : \mathcal{P}(G) \rightarrow [0, 1]$ realise the amenability of G . Let M be a right transversal of $H \leq G$ and define

$$\begin{aligned} \nu : \mathcal{P}(H) &\rightarrow [0, 1] \\ A &\mapsto \mu(AM) \end{aligned}$$

2. Define

$$\begin{aligned} \lambda : \mathcal{P}(G/N) &\rightarrow [0, 1] \\ A &\mapsto \mu(AN) \end{aligned}$$

3. Let ν_1, ν_2 realise amenability of $N, G/N$ respectively. For $A \leq G$ define

$$\begin{aligned} f_A : G &\rightarrow \mathbb{R} \\ g &\mapsto \nu_1(N \cap g^{-1}A) \end{aligned}$$

Note for $n \in N$, by translation invariance

$$f_A(gn) = \nu_1(N \cap n^{-1}g^{-1}A) = \nu_1(N \cap g^{-1}A) = f_A(g)$$

so f_A descends to a function on G/N . Now define the measure on G to be

$$\begin{aligned} \mu : \mathcal{P}(G) &\rightarrow [0, 1] \\ A &\rightarrow \int f_A d\nu_2 \end{aligned}$$

To show left-invariance, note

$$f_{hA}(g) = \nu_1(N \cap g^{-1}hA) = f_A(h^{-1}g) = hf_A(g)$$

The action of G on functions $G/N \rightarrow \mathbb{R}$ factors through G/N :

$$hn.f_A(g) = f_A(n^{-1}h^{-1}g) = \nu_1(N \cap g^{-1}hnA) = \nu_1(N \cap g^{-1}hA) = hf_A(g)$$

so

$$\mu(hA) = \int h^{-1}f_A d\nu_2 = \int f_A d\nu_2 = \mu(A)$$

by G/N -invariance of ν_2 .

4. Omitted. □

5.3 Amenability from a geometric viewpoint

We first derive a combinatorial characterisation of amenability.

Definition (Følner condition). A finitely generated group G is said to satisfy the *Følner condition* if for all finite subsets $A \subseteq G$, for all $\varepsilon > 0$, exists finite nonempty subset $F \subseteq G$ such that

$$\frac{|aF \Delta F|}{|F|} \leq \varepsilon$$

for all $a \in A$ where Δ denotes symmetric difference.

Theorem 5.14. *Suppose G is a finitely generated group. Then TFAE:*

1. G is amenable.
2. G satisfies the Følner condition.

Proof. For 1 \implies 2, see Theorem 16.62 in Druţu and Kapovich, or Theorem 4.2.3 in Juschenko's "Amenability".

For 2 \implies 1, $[0, 1]^{\mathcal{P}(G)}$ is compact in product topology. For $A \subseteq G$ finite, $\varepsilon > 0$, define $M_{A, \varepsilon}$ to be the set of finitely additive probability measure μ on G such that $|\mu(B) - \mu(aB)| \leq \varepsilon$ for all $B \subseteq G$, for all $a \in A$. $M_{A, \varepsilon}$ is closed in $[0, 1]^{\mathcal{P}(G)}$. To show it is nonempty, define $\mu(B) = \frac{|B \cap F|}{|F|}$ where F is a Følner set

for A, ε . Check

$$\begin{aligned}
 |\mu(B) - \mu(aB)| &= \left| \frac{|B \cap F|}{|F|} - \frac{|aB \cap F|}{|F|} \right| \\
 &= \left| \frac{|B \cap F|}{|F|} - \frac{|B \cap a^{-1}F|}{|F|} \right| \\
 &\leq \frac{|F \Delta a^{-1}F|}{|F|} \\
 &= \frac{|aF \Delta F|}{|F|} \\
 &\leq \varepsilon
 \end{aligned}$$

As

$$\bigcap_{i=1}^n M_{A_i, \varepsilon_i} \supseteq M_{\bigcup_{i=1}^n A_i, \min \varepsilon_i} \neq \emptyset,$$

$\{M_{A, \varepsilon}\}$ has finite intersection property so $\bigcap_{A, \varepsilon} M_{A, \varepsilon}$ is nonempty. \square

Definition (Cheeger constant). Let X be a graph. The *Cheeger constant* $h(X)$ is defined by

$$h(X) = \inf \frac{|\partial A|}{|A|}$$

over all nonempty finite $A \subseteq V(X)$, where ∂A is the set of vertices in $V(X) \setminus A$ that are connected by an edge to some element in A .

Note that under this definition $h(\text{finite group}) = 0$. This fits our purpose of studying amenability. Usually, a more useful definition is to take A such that $|A| \leq \frac{1}{2}|V(X)|$ for finite graphs.

Proposition 5.15. *Let G be a finitely generated group. Then TFAE:*

1. G satisfies the Følner condition.
2. $h(\text{Cay}(G, S)) = 0$ for all generating sets S .
3. $h(\text{Cay}(G, S)) = 0$ for some generating set S .

Informally this is saying non-amenability is equivalent to a “connectivity” property.

Proof. 2 \implies 3 is clear. 3 \implies 1 is an exercise. For 1 \implies 2, note that the Folner condition can be equivalently phrased in terms of right translates: take the Folner set F corresponding to A^{-1} and then

$$\frac{|F^{-1}a \Delta F^{-1}|}{|F^{-1}|} = \frac{|a^{-1}F \Delta F|}{|F|} < \varepsilon.$$

Now take $A = S^{\pm 1}$ for some generating set S of G and $\varepsilon > 0$. Let F be the Folner set such that

$$\frac{|Fs \Delta F|}{|F|} \leq \varepsilon$$

for all $s \in S^{\pm 1}$. Then

$$\begin{aligned} \frac{|\partial F|}{|F|} &= \frac{|\{gs : g \in F, s \in S^{\pm 1}, gs \notin F\}|}{|F|} \\ &\leq \frac{|\bigcup_{s \in S^{\pm 1}} (Fs \Delta F)|}{|F|} \\ &\leq |S^{\pm 1}| \cdot \varepsilon \\ &\leq 2|S| \cdot \varepsilon \end{aligned}$$

so indeed its infimum is 0. □

Example. $h = 0$ is quasi-isometry-invariant.

| **Corollary 5.16.** *Amenability is a quasi-isometry invariant.*

| **Corollary 5.17.** *All finitely generated groups of subexponential growth are amenable.*

Proof. Recall that if G has subexponential growth then at most

$$\beta_{G,S}(n)^{1/n} = |B(n)|^{1/n} \rightarrow 1.$$

If exists ε such that for all k , $\frac{|B(k+1)|}{|B(k)|} > 1 + \varepsilon$ then

$$|B(k+1)| > (1 + \varepsilon)^k \cdot |B(1)|$$

and letting $k \rightarrow \infty$, $\beta(k)^{1/k} \rightarrow 1$. Thus for all $N > 0$ exists k_N such that

$$\frac{|B(k_N+1)|}{|B(k_N)|} < 1 + \frac{1}{N}$$

and so

$$\frac{|\partial B(k_N)|}{|B(k_N)|} = \frac{|B(k_N+1) \setminus B(k_N)|}{|B(k_N)|} < \frac{1}{N}.$$

□

The converse is not true, by the following corollary and the existence of solvable groups of exponential growth:

| **Corollary 5.18.** *All solvable groups are amenable.*

Proof. All abelian groups have polynomial growth and are thus amenable. Any solvable group can be written as the direct limit of abelian groups. Use Proposition 5.13. □

Take the closure of finite groups and abelian groups by the operations of Proposition 5.13, we obtain the class of *elementary amenable groups*. They are strictly contained in amenable groups, as Grigorchuk group of intermediate growth.

Open question:

1. Is “elementary amenable” a quasi-isometry invariant?
2. Is Thompson’s group F amenable?

More topics in geometric groups theory:

- Martin Bridson: world of finitely presented groups.
- Gromov: Space and Questions.
- word problem. word problem for hyperbolic groups, solved by Dehn.
- small cancellation. It leads to the construction of many interesting examples.
- expander graphs, Kazhdan’s property (T).

To conclude this course we mention the characterisation of non-amenability in terms of Ponzi scheme. Consider a function $\rho : G \rightarrow G$ such that

- exists R such that $d(g, \rho(g)) \leq R$.
- $|\rho^{-1}(g)| \geq 2$.

Imagine each person indexed by G holds £1, and person g passes his money to $\rho(g)$. In this process everyone ends up with strictly more money than they had, and money has moved by a bounded distance. An example is F_2 , where everyone passes the money towards the centre of the Cayley graph. This condition is, in fact, equivalent to non-amenability.

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