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Mathematics Tripos

Part IV

Geometric Aspects of p-adic Hodge Theory

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1.1 Hodge decomposition over $\mathbb{C}$

Let X be a smooth projective variety over $\mathbb{C}$. The *Hodge decomposition* is a direct sum decomposition for all $n \geq 0$

$$H^n_{\text{sing}}(X^{\text{an}}, \mathbb{C}) = \bigoplus_{p+q=n} H^{p,q}$$

where LHS is the singular cohomology of the $\mathbb{C}$-analytic manifold X^{an} (the complex analytification) and on RHS

$$H^{p,q} = H^q(X^{\text{an}}, \Omega^p_{X^{\text{an}}})$$

with $\Omega^p_{X^{\text{an}}}$ denoting the sheaf of holomorphic p-forms. Moreover, complex conjugation acts on

$$H^n_{\text{sing}}(X^{\text{an}}, \mathbb{C}) \cong H^n_{\text{sing}}(X^{\text{an}}, \mathbb{Q}) \otimes \mathbb{C}$$

via its action on $\mathbb{C}$ and $H^{p,q} = \overline{H^{q,p}}$. This is called a *pure structure of weight n*.

These are proven via identifying $H^{p,q}$ with Dolbeault cohomology and using the (very deep) theory of harmonic forms. However, part of the theory can be understood purely algebraically. It is known that $H^n_{\text{sing}}(X^{\text{an}}, \mathbb{C})$ gives the cohomology of the constant sheaf $\mathbb{C}$ on X^{an}. On the other hand, consider the de Rham complex

$$\Omega^\bullet_{X^{\text{an}}} = \mathcal{O}_{X^{\text{an}}} \xrightarrow{\mathrm{d}} \Omega^1_{X^{\text{an}}} \xrightarrow{\mathrm{d}} \Omega^2_{X^{\text{an}}} \to \cdots$$

Here d is the usual derivation and the higher d's are given by

$$\mathrm{d}(\omega_1 \wedge \omega_2) = \mathrm{d}\omega_1 \wedge \omega_2 + (-1)^p \omega_1 \wedge \mathrm{d}\omega_2$$

for $\omega_1 \in \Omega^p_{X^{\text{an}}}, \omega_2 \in \Omega^q_{X^{\text{an}}}$. Taking hypercohomology

$$H^n_{\text{dR}}(X^{\text{an}}) := \mathbb{H}^n(X^{\text{an}}, \Omega^\bullet_{X^{\text{an}}})$$

we get the so-called de Rham cohomology group.

Embedding the constant sheaf $\mathbb{C}$ into $\mathcal{O}_{X^{\text{an}}}$ induces a map $\mathbb{C} \to \Omega^\bullet_{X^{\text{an}}}$ of complexes of sheaves. The (holomorphic) Poincaré lemma states that this map is a quasi-isomorphism of sheaves. More precisely, one can cover X^{an} by open balls and for any open ball $U \subseteq X^{\mathrm{an}}$, the complex

$$0 \to \mathbb{C} \to \mathcal{O}_{X^{\mathrm{an}}}(U) \xrightarrow{\mathrm{d}} \Omega^1_{X^{\mathrm{an}}}(U) \to \cdots$$

is exact: any closed differential form can be integrated on an open ball. Thus

$$H^n_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{C}) \cong H^n_{\mathrm{dR}}(X^{\mathrm{an}}).$$

This is the comparison theorem between singular and de Rham cohomology.

Now the complex $\Omega^\bullet_{X^{\mathrm{an}}}$ has a decreasing filtration of subcomplexes

$$\Omega^{\geq p}_{X^{\mathrm{an}}} := 0 \to \cdots \to 0 \to \Omega^p_{X^{\mathrm{an}}} \xrightarrow{\mathrm{d}} \Omega^{p+1}_{X^{\mathrm{an}}} \xrightarrow{\mathrm{d}} \cdots$$

We have that $\mathrm{gr}^p\,\Omega^\bullet_{X^{\mathrm{an}}} \cong \Omega^p_{X^{\mathrm{an}}}$. It is well-known that there is a convergent spectral sequence associated to $\Omega^\bullet_{X^{\mathrm{an}}}$ with the filtration above, called the *Hodge to de Rham spectral sequence*

$$E_1^{pq} = H^q(X^{\mathrm{an}}, \Omega^p_{X^{\mathrm{an}}}) \Rightarrow H^{p+q}_{\mathrm{dR}}(X^{\mathrm{an}}).$$

The filtration on $H^n_{\mathrm{dR}}(X^{\mathrm{an}})$ given by the spectral sequence is called the *Hodge filtration.*

Fact: the Hodge to de Rham spectral sequence degenerates at E_1. This together with the comparison theorem gives the Hodge decomposition

$$H^n_{\mathrm{sing}}(X^{\mathrm{an}}, \mathbb{C}) = \bigoplus_{p+q=n} H^q(X^{\mathrm{an}}, \Omega^p_{X^{\mathrm{an}}})$$

for all n.

1.2 Algebraisation

On a complex variety X we may consider the algebraic de Rham complex

$$\Omega^\bullet_X := \mathcal{O}_X \xrightarrow{\mathrm{d}} \Omega^1_X \xrightarrow{\mathrm{d}} \cdots$$

For X smooth these are locally free sheaves. The same way as above, we get the algebraic Hodge to de Rham spectral sequence

$$E_1^{pq} = H^q(X, \Omega^p_X) \Rightarrow H^{p+q}_{\mathrm{dR}}(X).$$

Here we use the Zariski topology.

There are two natural maps

$$H^q(X, \Omega^p_X) \to H^q(X^{\mathrm{an}}, \Omega^p_{X^{\mathrm{an}}})$$
$$H^{p+q}_{\mathrm{dR}}(X) \to H^{p+q}_{\mathrm{dR}}(X^{\mathrm{an}})$$

all compatible with the maps in the above spectral sequences. By GAGA the first one is an isomorphism, and by a theorem of Grothendieck the second is also an isomorphism. Hence degeneration of the analytic Hodge to de Rham is equivalent to the degeneration of the algebraic counterpart.

However, there is no algebraic Poincaré lemma, the algebraic de Rham complex is not a resolution of $\mathbb{C}$ and anyway the sheaf cohomology of $\mathbb{C}$ is trivial in the Zariski topology.

1.3 The case of a p-adic base field

Let p be a prime. Recall $\mathbb{C}_p$ is the completion of the algebraic closure $\overline{\mathbb{Q}}_p$ of $\mathbb{Q}_p$. The absolute Galois group $G_{\mathbb{Q}_p} = \mathrm{Gal}(\overline{\mathbb{Q}}_p/\mathbb{Q}_p)$ acts on $\mathbb{C}_p$ by continuity. Let K be a finite extension of $\mathbb{Q}_p$. Similarly we obtain $\mathbb{C}_K$ and an action of G_K. Obviously $\mathbb{C}_K$ is the same as $\mathbb{C}_p$ as a field but it carries the action of a subgroup of $G_{\mathbb{Q}_p}$.

1.3.1 The p-adic cyclotomic character

Let $\mu_{p^n} = \mu_{p^n}(\overline{K})$ denote the group of p^nth roots of unity in $\overline{K}$. Fix a primitive p^nth root ε_n for all $n \geq 0$ such that $\varepsilon_0 = 1$ and $\varepsilon_{n+1}^p = \varepsilon_n$ (compatible system of p^nth root of unity). This is similar to choosing i or $-i$ in $\mathbb{C}$. Put $K_n = K(\varepsilon_n)$ to get a tower

$$K \subseteq K(\varepsilon_1) \subseteq \cdots \subseteq K_\infty = \bigcup_n K_n.$$

For every $\sigma \in G_K$ and $n \geq 0$, there is a unique element $\chi_n(\sigma) \in (\mathbb{Z}/p^n\mathbb{Z})^\times$ such that $\sigma(\varepsilon_n) = \varepsilon_n^{\chi_n(\sigma)}$. By compatibility of $(\varepsilon_n)_n$ we have $\chi_{n+1}(\sigma) = \chi_n(\sigma) \pmod{p^n}$. Thus we have a compatible system $(\chi_n(\sigma))_n$, and hence an element $\chi(\sigma) \in \mathbb{Z}_p^\times$. Clearly this does not depend on the choice of $(\varepsilon_n)_n$ and we get a homomorphism $\chi_{K,p} : G_K \to \mathbb{Z}_p^\times$, called the *p-adic cyclotomic character*. $\ker(\chi \pmod{p^n}) = \mathrm{Gal}(\overline{K}/K(\varepsilon_n))$ which is open kernel so χ is continuous.

1.3.2 Tate twists

Let $X = \mathbb{G}_m$ be the multiplicative group scheme. Consider the Kummer sequence (evaluated at $\overline{K}$)

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \mu_{p^n}(\overline{K}) & \longrightarrow & (\overline{K})^\times & \xrightarrow{a \mapsto a^{p^n}} & (\overline{K})^\times & \longrightarrow & 1 \\ & & \downarrow\cong & & & & & & \\ & & \mathbb{Z}/p^n\mathbb{Z} & & & & & & \end{array}$$

and $a \mapsto a^p$ gives a compatible system $\mu_{p^{n+1}}(\overline{K}) \to \mu_{p^n}(\overline{K})$. Define

$$T_p(\mathbb{G}_m) := \varprojlim_n \mu_{p^n}(\overline{K}),$$

the *p-adic Tate module of* $\mathbb{G}_m$. It is a free $\mathbb{Z}_p$-module of rank 1, with the choice of (ε_n) providing a basis element. In modern literature, we usually write $\mathbb{Z}_p(1)$ for $T_p(\mathbb{G}_m)$. The group G_K acts on $\mathbb{Z}_p(1)$ via $\sigma \cdot \varepsilon = \chi(\sigma) \cdot \varepsilon$.

Define $\mathbb{Q}_p(1) = \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ and more generally for any $\mathbb{Z}_p$-algebra R, define $R(1) := \mathbb{Z}_p(1) \otimes_{\mathbb{Z}_p} R$. We may then define the *Tate twists of* R. First let

$$\mathbb{Z}_p(r) = \mathbb{Z}_p(1)^{\otimes r}$$
$$\mathbb{Z}_p(-r) = \mathrm{Hom}_{\mathbb{Z}_p}(\mathbb{Z}_p(r), \mathbb{Z}_p)$$

with the natural G_K action via tensor power action and dual action $(\sigma \cdot \varphi)(v) = \varphi(\sigma^{-1} \cdot v)$. Then $R(i) = \mathbb{Z}_p(i) \otimes_{\mathbb{Z}_p} R$.

1.4 The Hodge-Tate decomposition

The first proof of the following theorem is given by Faltings:

Theorem 1.1 (Hodge-Tate decomposition)**.** *Let X be a smooth proper variety over K. Then there is a Galois equivariant decomposition*

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K}) \otimes_K \mathbb{C}_K(-j).$$

This isomorphism is functorial in X.

The LHS is étale cohomology

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) := \underbrace{\varprojlim_n H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}/p^n\mathbb{Z})}_{H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)} \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$$

and hence admits a G_K action via the tensor product action coming from the natural action on $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ and on $\mathbb{C}_K$.

The RHS has a G_K-action via its action on $\mathbb{C}_K(-j)$ which is the tensor product action.

Example. Let $A/\mathbb{Q}$ be an abelian variety (i.e. a complete connected group variety, which is automatically projective as a variety and commutative as a group), for example an elliptic curve. Its p-adic Tate module is the following object. Let $A(\overline{\mathbb{Q}})[p^n]$ be the p^n-torsion elements. It is known to be isomorphic to $(\mathbb{Z}/p^n\mathbb{Z})^{2d}$ where $d = \dim A$. Then

$$T_p(A) = \varprojlim_n A(\overline{\mathbb{Q}})[p^n].$$

$G_{\mathbb{Q}}$ acts on $A(\overline{\mathbb{Q}})[p^n]$ via the finite quotient $\mathrm{Gal}(\mathbb{Q}(A(\overline{\mathbb{Q}})[p^n])/\mathbb{Q})$ associated to the field generated by the coordinates of p^n-torsion points. Now we may arrange things so that then it induces an action of $G_{\mathbb{Q}}$ on $T_p(A)$ which gives us a continuous representation $\rho : G_{\mathbb{Q}} \to \mathrm{GL}_{2d}(\mathbb{Z}_p)$. We may restrict ρ to $G_{\mathbb{Q}_p}$ via $G_{\mathbb{Q}_p} \hookrightarrow G_{\mathbb{Q}}$. This representation contains information about ρ at a prime.

A fundamental arithmetic invariant is the $\mathbb{Z}$-rank of the Mordell-Weil group $A(\mathbb{Q})$. This invariant is encoded in the p-adic representation of $G_{\mathbb{Q}}$ associated to A: $T_p(A)$ has a canonical Galois action, $T_p(A) \otimes_{\mathbb{Z}_p} \mathbb{Q}_p$ with induced action by $G_{\mathbb{Q}}$ (or locally $G_{\mathbb{Q}_p}$) via ρ. A theorem says that this representation contains information about all Euler factors of the Hasse-Weil L-function at primes of good reduction away from p.

Remark. In number theory (p-adic Hodge theory) we want to understand Galois representations of G_K arising from geometry, as $T_p(A) \otimes \mathbb{Q}_p$ or more generally $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$ where X is smooth proper. By Hodge-Tate decomposition, these representations are *Hodge-Tate*. We will not talk much about this but for reference, see Brinon, Conrad, p-adic Hodge theory. These representations "look like" pure Hodge structures over $\mathbb{C}$ (of course without complex conjugation), hence the name p-adic Hodge theory.

Theorem 1.2 (Tate). *For $i \neq 0$, we have*

$$\mathbb{C}_K(i)^{G_K} = H^0(G_K, \mathbb{C}_K(i)) = H^1(G_K, \mathbb{C}_K(i)) = 0.$$

For $i = 0$, each of these two cohomology groups is isomorphic to a copy of K. In particular,

$$\mathrm{Hom}_{G_K, \mathbb{C}_K}(\mathbb{C}_K(i), \mathbb{C}_K(j)) = 0$$

if $i \neq j$.

The following statement basically allows us to recover algebro-geometric invariants $H^i(X, \Omega^j_{X/K})$ and $H^n_{\mathrm{Hodge}}(X) := \bigoplus_{i+j=n} H^i(X, \Omega^j_{X/K})$ from topological/arithmetic invariants $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p)$:

Corollary 1.3. *With the notations in the Hodge-Tate decomposition theorem, we have*

$$H^i(X, \Omega^j_{X/K}) \cong (H^{i+j}_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K(j))^{G_K}.$$

Proof. Tensor both sides of Hodge-Tate decomposition by $\mathbb{C}_K(j)$ (we replace j by k in the formula),

$$H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Q}_p) \otimes_{\mathbb{Q}_p} \mathbb{C}_K(j) \cong \bigoplus_{i+k=n} H^i(X, \Omega^k_{X/K}) \otimes_K \mathbb{C}_k(j-k)$$

and apply $(-)^{G_K}$. Tate's theorem gives the claim as $\mathbb{C}_K(k-j)^{G_K} = 0$ as $j \neq k$. $\square$

1.5 Integral p-adic Hodge theory

The above is the so-called rational p-adic Hodge theory.

It is important to understand the p-torsion part of $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{Z}_p)$ in terms of the geometry of X. In particular, we want a description of $H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_p)$. It is much less understood than the rational version.

Theorem 1.4 (Bhatt, Morrour, Scholze). *Assume that X is proper smooth over K, and it spreads to a proper smooth $\mathcal{O}_K$-scheme $\mathfrak{X}$ (i.e. X is the geometric fibre of $\mathfrak{X}$). Then*

$$\dim_{\mathbb{F}_p}(H^n_{\text{ét}}(X_{\overline{K}}, \mathbb{F}_p)) \leq \sum_{i+j=n} \dim_k H^i(\mathfrak{X}_k, \Omega^j_{\mathfrak{X}_k/k}).$$

In other words, the mod-p cohomology of $X_{\overline{K}}$ is related to the geometry of $\mathfrak{X}_k$.

Outline of the proof of Hodge-Tate decomposition:

1. Local study of Hodge cohomology via perfectoid spaces. We construct a pro-étale cover $X_\infty \to X$ and study the cohomology of X_∞.
2. Descent: descend the previous understanding of the Hodge cohomology of X_∞ to X. To illustrate this process in practice, we work out the case of abelian varieties with good reduction.

2 The Hodge-Tate decomposition for abelian varieties with good reduction

For an introduction to abelian varieties, see Milne, Abelian varieties.

Let K be as above. Recall the following facts:

1. an *abelian scheme* X over S (base scheme) is a smooth group scheme such that all fibres are varieties. They are commutative and if S is normal then X is projective.

2. we say an abelian variety A over K has *good reduction* if exists an abelian scheme $\mathcal{A}$ over $\mathcal{O}_K$ such that its generic fibre is A.

Theorem 2.1. *There exists a canonical isomorphism*

$$\begin{aligned} H^1_{\text{ét}}(A_{\mathbb{C}_K}, \mathbb{C}_K) &:= H^1_{\text{ét}}(A_{\mathbb{C}_K}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} \mathbb{C}_K \\ &\cong (H^1(A, \mathcal{O}_X) \otimes_K \mathbb{C}_K) \oplus (H^0(A, \Omega^1_{A/K}) \otimes_K \mathbb{C}_K(-1)). \end{aligned}$$

We want to sketch the proof. We shall construct a map

$$\alpha_A : H^1(A, \mathcal{O}_A) \otimes_{\mathbb{Z}_p} \mathbb{C}_K \to H^1_{\text{ét}}(A_{\mathbb{C}_K}, \mathbb{C}_K)$$

using perfectoid theory and a map

$$\beta_A : H^0(A, \Omega^1_{A/K}) \otimes \mathbb{C}_K(-1) \to H^1_{\text{ét}}(A_{\mathbb{C}_K}, \mathbb{C}_K).$$

Then $j_A = \alpha_A \oplus \beta_A$ induces the Hodge-Tate decomposition.

Why is it enough to consider only the first étale cohomology group? Recall the follow theorem (Milne, Theorem 12.1):

1. there is a canonical isomorphism $H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Z}_p) \cong T_p(A)^\vee$. Moreover the usual wedge product defines isomorphisms

   $$\bigwedge^r H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Z}_\ell) \to H^r_{\text{ét}}(A_{\overline{K}}, \mathbb{Z}_\ell)$$

   for all r. As a consequence, the following algebrs are isomorphic:

   (a) the cohomological algebra $H^*_{\text{ét}}(A_{\overline{K}}, \mathbb{Z}_p) = \bigoplus H^i_{\text{ét}}(A_{\overline{K}}, \mathbb{Z}_p)$ with cup product.

   (b) the exterior algebra $\bigwedge^* H^1_{\text{ét}}(A_{\overline{K}}, \mathbb{Z}_p)$ with wedge product.

   (c) the dual of $\bigwedge^* T_p(A)$ with wedge product.

2. There is a canonical identification of algebras

   $$H^*(A, \mathbb{C}_K) \cong H^*(T_p(A), \mathbb{C}_K)$$

   where RHS is continuous group cohomology and with cup products on both sides (hint: use the above and the fact that $H^1(A, \mathbb{C}_K) \cong T_p(A)^\vee \otimes \mathbb{C}_K$).

3. The $\mathcal{O}_K$-module $H^1_{\text{ét}}(\mathcal{A}, \mathcal{O}_{\mathcal{A}})$ is free of rank $d = \dim A$. Moreover the cohomological ring $H^*_{\text{ét}}(\mathcal{A}, \mathcal{O}_{\mathcal{A}})$ is the exterior algebra $\bigwedge^* H^1_{\text{ét}}(\mathcal{A}, \mathcal{O}_{\mathcal{A}})$. In particular all cohomology groups $H^n_{\text{ét}}(\mathcal{A}, \mathcal{O}_{\mathcal{A}})$ are torsion free.

2.1 The perfectoid construction of α_A

We work with $\mathcal{O}_{\mathbb{C}_K}$-scheme $\mathcal{A}_{\mathcal{O}_{\mathbb{C}_K}}$. Write $\mathcal{A}_n := \mathcal{A}_{\mathcal{O}_{\mathbb{C}_K}}$ for $n \geq 0$ and consider a tower

$$\cdots \xrightarrow{[p]} \mathcal{A}_n \xrightarrow{[p]} \cdots \xrightarrow{[p]} \mathcal{A}_1 \xrightarrow{[p]} \mathcal{A}_0 = \mathcal{A}_{\mathcal{O}_{\mathbb{C}_K}}$$

where the maps are multiplication by p. We write $\mathcal{A}_\infty = \varprojlim \mathcal{A}_n$ and let $\pi_n : \mathcal{A}_n \to \mathcal{A}_0, \pi : \mathcal{A}_\infty \to \mathcal{A}_0$ be the natural maps. The limit exists (perfectoid abelian varieites): $[p]$ is a finite map, hence we can apply the following theorem: if every continuous map in the inverse system is affine then the inverse limit exists (Stacks TAG 01YX). As we have qcqs schemes,

$$H^q(\mathcal{A}_\infty, \mathcal{O}_{\mathcal{A}_\infty}) = \varinjlim H^q(\mathcal{A}_n, \mathcal{O}_{\mathcal{A}_n}).$$

Now translation by p^n-torsion points (in general, for a point a, t_a is the composition

$$\begin{aligned} \mathcal{A} &\to \mathcal{A} \times_{\mathcal{O}_K} \mathcal{A} \to \mathcal{A} \\ x &\mapsto (x, a) \mapsto xa \end{aligned}$$

)gives an action of $\mathcal{A}(\mathcal{O}_{\mathbb{C}_K})[p^n] \cong A(\mathbb{C}_K)[p^n]$ (the isomorphism can be checked using the valuation criterion of properness (?)) on $\pi_n : \mathcal{A}_n \to \mathcal{A}_0$. Taking inverse limits, we have an action of $T_p(A)$ on $\pi : \mathcal{A}_\infty \to \mathcal{A}_0$. Therefore taking pullbacks we obtain a map

$$H^*(\mathcal{A}_0, \mathcal{O}_{\mathcal{A}_0}) \to H^*(\mathcal{A}_\infty, \mathcal{O}_{\mathcal{A}_\infty})$$

Due to the presence of the group action, the image of this map is contained in the $T_p(A)$-invariants of the target. Thus we have a map

$$H^*(\mathcal{A}_0, \mathcal{O}_{\mathcal{A}_0}) \to H^*(T_p(A), H^*(\mathcal{A}_\infty, \mathcal{O}_{\mathcal{A}_\infty}))$$

where RHS is continuous group cohomology. We will consider this in the derived sense:

$$\psi : R\Gamma(\mathcal{A}_0, \mathcal{O}_{\mathcal{A}_0}) \to R\Gamma_{\text{cont}}(T_p(A), R\Gamma(\mathcal{A}_\infty, \mathcal{O}_{\mathcal{A}_\infty})).$$

We need the following vanishing theorem.

Proposition 2.2. *The canonical map $\mathcal{O}_{\mathbb{C}_K} \to R\Gamma(\mathcal{A}_\infty, \mathcal{O}_{\mathcal{A}_\infty})$ induces an isomorphism (in the derived category) after p-adic completion.*

Proof. As $\mathcal{A}$ is an abelian scheme, its cohomology is an exterior algebra on $H^1(\mathcal{A}, \mathcal{O}_{\mathcal{A}})$. Moreover multiplication by an integer N on $\mathcal{A}$ induces a multiplication by N on $H^1(\mathcal{A}, \mathcal{O}_{\mathcal{A}})$. Combining these observations with the formula

$$H^q(\mathcal{A}_\infty, \mathcal{O}_{\mathcal{A}_\infty}) = \varinjlim H^q(\mathcal{A}_n, \mathcal{O}_{\mathcal{A}_n}),$$

we see that

$$H^i(\mathcal{A}_\infty, \mathcal{O}_{\mathcal{A}_\infty}) = \begin{cases} \mathcal{O}_{\mathbb{C}_K} & i = 0 \\ H^i(\mathcal{A}_{\mathcal{O}_{\mathbb{C}_K}}, \mathcal{O}_{\mathcal{A}_{\mathbb{C}_K}})[\frac{1}{p}] & i > 1 \end{cases}$$

(localisation is exactly this colimit).

So modulo any power of p, these latter groups vanish, so does taking p-adic completion. The claim then follows. ☐

p-adic completion of ψ gives

$$\hat{\psi} : R\Gamma(\mathcal{A}, \mathcal{O}_{\mathcal{A}}) \to R\Gamma_{\text{cont}}(T_p(A), \mathcal{O}_{\mathbb{C}_K}).$$

(LHS A or A0?) On the other hand, as abelian varieties are $K(\pi, 1)$, we can interpret the above map as a map

$$\hat{\psi} : R\Gamma(\mathcal{A}, \mathcal{O}_{\mathcal{A}}) \to R\Gamma(A_{\mathbb{C}_K}, \mathcal{O}_{\mathbb{C}_K}).$$

In particular, applying H^1 and inverting p, we get a map

$$H^1(A, \mathcal{O}_A) \to H^1_{\text{ét}}(A_{\mathbb{C}_K}, \mathbb{C}_K).$$

After linearising we get

$$\alpha_A : H^1(A, \mathcal{O}_A) \otimes \mathbb{C}_K \to H^1(A_{\mathbb{C}}, \mathbb{C}_K).$$

Remark. A $K(\pi, 1)$ spaces is a qcqs schemes X that has finitely many connected components, and for any geometic point $\overline{x}$ of X and p-torsion lcc abelian sheaf $\mathcal{F}$ (locally constant constructible sheaf, i.e. every geometric point of X has an étale neighbourhood U such that $\mathcal{F}|_U$ is a constant sheaf induced by a finite set). The natural maps

$$H^q(\pi_1^{\text{ét}}(X, \overline{x}), \mathcal{F}_{\overline{x}}) \to H^q_{\text{ét}}(X, \mathcal{F})$$

are all isomorphisms. This uses the finite monodromy correspondence that

$$\{\text{lcc sheaves}\} \longleftrightarrow \{\text{finite cont}\pi_1^{et}(X, \overline{x})\text{-sets}\}\mathcal{F} \mapsto \mathcal{F}_{\overline{x}}$$

This extends to smooth $\mathbb{Z}_p$-schemes (inverse limits of lcc $\mathbb{Z}/p^n\mathbb{Z}$-sheaves). We can show

$$H^1(\pi_1^{\text{ét}}, \mathbb{Z}_p) \cong \text{Hom}(\pi_1^{\text{ét}}, \mathbb{Z}_p).$$

It can also be shown that $T_p(A) \cong \pi_1^{\text{ét}}(X, \overline{x}) \otimes \mathbb{Z}_p$. By the adjunction between extension of scalars, we get RHS isomorphic to

$$\text{Hom}(T_p(A), \mathbb{Z}_p) = T_p(A)^\vee.$$

2.2 Fontaine's construction of β_A

Theorem 2.3 (differential forms on $\mathcal{O}_{\mathbb{C}_K}$)**.** *We write Ω for the Tate module of $\Omega^1_{\mathcal{O}_{\mathbb{C}_K}/\mathcal{O}_K}$. This module is free of rank 1 over $\mathcal{O}_{\mathbb{C}_K}$. Moreover there is a Galois-equivariant isomorphism $\mathbb{C}_K(1) \cong \Omega[\frac{1}{p}]$*[a] *such that a compatible system $(\varepsilon_n)_n$ of p-power roots of unity is mapped to $(\mathrm{d}\log \varepsilon_n)_n \in \Omega$.*

[a] The action on Ω is as follow: $T_p\Omega^1 = \varprojlim \Omega^1_{\mathcal{O}_{\mathbb{C}_K}/\mathcal{O}_K}[p^n]$. $\Omega^1_{\mathcal{O}_{\mathbb{C}_K}/\mathcal{O}_K}$ has a natural G_K-action by functoriality and this induces an action on Ω.

Recall that the p-adic logarithm

$$\begin{aligned} \log : \mathcal{O}_{\mathbb{C}_K}^\times &\to p\mathcal{O}_{\mathbb{C}_K} \\ x &\mapsto \sum_{n \geq 1} (-1)^{n-1} \frac{x^n}{n} \end{aligned}$$

Then ε_n is mapped to a p^n-torsion element via $\mathrm{d}\log$ (this map is actulally $\mathcal{O}_{\mathbb{C}_K}^\times \to \Omega^1_{\mathcal{O}_{\mathbb{C}_K}/\mathcal{O}_K}, f \mapsto \frac{\mathrm{d}f}{f}$). Taking inverse limit $(\varepsilon_n)_n \in \Omega$ we get $\mathbb{Z}_p(1) \to \Omega$. Base changing it to get

$$\mathcal{O}_{\mathbb{C}_K}(1) = \mathbb{Z}_p(1) \otimes \mathcal{O}_{\mathbb{C}_K} \to \Omega.$$

Fontaine proves that this map is injective with torsion cokernel, giving $\mathbb{C}_K(1) \cong \Omega[\frac{1}{p}]$.

Proof. Omitted. We will see a more general statement later. □

In particular this connects Tate twists (Galois side) to differential forms (de Rham side). Using this, considering $\mathcal{O}_{\mathbb{C}_K}$-rational points $\operatorname{Spec} \mathcal{O}_{\mathbb{C}_K} \to \mathcal{A}$, we get a map

$$H^0(\mathcal{A}, \Omega^1_{\mathcal{A}/\mathcal{O}_{\mathbb{C}_K}}) \to H^0(\operatorname{Spec} \mathcal{O}_{\mathbb{C}_K}, \Omega^1_{\mathcal{O}_{\mathbb{C}_K}/\mathcal{O}_K}) = \Omega^1_{\mathcal{O}_{\mathbb{C}_K}/\mathcal{O}_K}.$$

This induces a pairing

$$H^0(\mathcal{A}, \Omega^1_{\mathcal{A}/\mathcal{O}_{\mathbb{C}_K}}) \otimes \mathcal{A}(\mathcal{O}_{\mathbb{C}_K}) \to \Omega^1_{\mathcal{O}_{\mathbb{C}_K}/\mathcal{O}_K}.$$

Passing to p-adic Tate module

$$H^0(\mathcal{A}, \Omega^1_{\mathcal{A}/\mathcal{O}_K}) \otimes T_p(A) \to \Omega.$$

Use the identification $T_p(A) \cong H^1_{\text{ét}}(A_{\mathbb{C}_K}, \mathbb{Z}_p)^\vee$, we get a map

$$H^0(\mathcal{A}, \Omega^1_{\mathcal{A}/\mathcal{O}_K}) \to H^1(A_{\mathbb{C}_K}, \mathbb{Z}_p) \otimes \Omega.$$

Inverting p and using Fontaine's theorem and linearising, we get

$$\beta_A : H^0(A, \Omega^1_{A/K}) \otimes \mathbb{C}_K(-1) \to H^1(A_{\mathbb{C}_K}, \mathbb{C}_K).$$

Now taking direct sum of α_A and β_A, we get

$$\gamma_A : (H^1(A, \mathcal{O}_A) \otimes \mathbb{C}_K) \oplus (H^0(A, \Omega^1_{A/K})) \otimes \mathbb{C}_K(-1) \to H^1_{\text{ét}}(A_{\mathbb{C}_K}, \mathbb{C}_K).$$

Remark. Each summand on LHS has dimension d and RHS has dimension $2d$ so it suffices to show injectivity. In fact the two term have different Galios action and γ_A is Galois-equivariant, so it is enough to check injectivity termwise. Classically, for β_A it follows from a classical formal group argument. For α_A we will have a more general statement.

3 Hodge-Tate decomposition in general

3.1 Étale cohomology of adic spaces and rigid varieties

Throughout we will mean étale cohomology of adic spaces (and there won't be cohomology for schemes). Recall the results by Huber

1. let $f : X \to Y$ be a morphism of rigid analytic varieties. f is *étale* if for every $x \in X$, $\mathcal{O}_{X,x}$ is flat over $\mathcal{O}_{Y,f(x)}$ and $\mathcal{O}_{X,x}/\mathfrak{m}_x\mathcal{O}_{X,x}$ is a finitely separable field extension of $\mathcal{O}_{Y,f(x)}/\mathfrak{m}_{f(x)}\mathcal{O}_{Y,f(x)}$.

   There is a natural way to get an adic space X^{ad} from an rigid analytic variety X.

2. Let (A, A^+) be an affinoid K-algebra. A map $f : (A, A^+) \to (B, B^+)$ is *finite étale* if

   (a) $A \to B$ is étale (in the sense of algebra),
   (b) B^+ is the integral closure of A^+ in B.

Definition. A morphism $f : X \to Y$ of adic spaces is *étale* if for all $x \in X$ we can find a factorisation

$$\begin{array}{ccc} x \in U & \xhookrightarrow{\text{aff open}} & X \\ \big\downarrow & & \big\downarrow f \\ f(x) \in V & \xhookrightarrow{\text{aff open}} & Y \end{array}$$

such that $U \to V$ is finite étale.

In this way we get the étale site for adic spaces.

Proposition 3.1. *For every rigid analytic variety X, the natural map of toposes*

$$(\widetilde{X^{\mathrm{ad}}})_{\text{ét}} \to \tilde{X}_{\text{ét}}$$

is an equivalence.

Let X be a rigid analytic variety over $\mathbb{C}_K$.

Theorem 3.2 (Hodge-Tate spectral sequence, Scholze)**.** *Assume X is smooth and proper. Then there exists a spectral sequence*

$$E_2^{ij} = H^i(X, \Omega^j_{X/\mathbb{C}_K})(-j) \Rightarrow H^{i+j}_{\text{ét}}(X, \mathbb{C}_K).$$

If X is already defined over some discrete valuation field $L \subseteq \mathbb{C}_K$ then this spectral sequence degenerates at page 2 and we obtain the Hodge-Tate decomposition.

Remark. By the work of Huber (Étale cohomology of rigid analytic varieties and adic spaces) if X arises as the analytification of some algebraic variety Y then for torsion abelian sheaves (in particular constant sheaves $\mathbb{Z}/n\mathbb{Z}$), étale cohomology of X is isomorphic to the étale cohomology of Y. Hence we get back the algebraic version of the above Hodge-Tate decomposition.

Remark. First Scholze considered the completed structure sheaf $\hat{\mathcal{O}}_X$ on the pro-étale site (to be defined later). Roughly the objects are towers $\{U_i\}$ of finite étale covers with $U_0 \to X$ étale and $\hat{\mathcal{O}}_X$ is the sheaf which assigns to such a tower the completion of the direct limit of the rings of analytic functions on the U_i's. In particular this sheaf is a sheaf of $\mathbb{C}_K$-algebras.

Theorem 3.3 (primitive comparison theorem)**.** *The natural inclusion $\mathbb{C}_K \subseteq \hat{\mathcal{O}}_X$ gives an isomorphism*

$$H^*_{\text{ét}}(X, \mathbb{C}_K) \cong H^*_{\text{proét}}(X, \hat{\mathcal{O}}_X).$$

So to prove Hodge-Tate theorem, it is enough to use pro-étale cohomology with $\hat{\mathcal{O}}_X$ which enables us to consider on both sides sheaf cohomology. Have a natural map

$$v : X_{\text{proét}} \to X_{\text{ét}}$$

(recall that the morphism of sites goes in the opposite direction of the functors between the categories. v is induced by the fact that étale morphisms are pro-étale). The spectral sequence in primitive comparison theorem arises using the Leray spectral sequence for v and the following theorem:

Theorem 3.4 (Hodge-Tate filtration, local version)**.** *There is a canonical isomorphism $\Omega^j_{X/\mathbb{C}_K}(-j) \cong R^j v_* \hat{\mathcal{O}}_X$.*

Our aim is to prove this theorem.

4 Cotangent complex and perfectoid rings

Reference: Stacks TAG 08P5

4.1 Simplicial algebra

Denote by Δ the category of whose objects are the finite ordered sets $[n] = \{0 < 1 < \cdots < n\}$ and morphisms are nondecreasing functions. Δ is called the *simplex category*.

Definition ((co)simplicial object)**.** A *simplicial (resp. cosimplicial)* object in a category $\mathbf{C}$ is a contravariant (resp. covariant) functor $X : \Delta \to \mathbf{C}$. Simplicial objects in $\mathbf{C}$ form a category $\mathbf{Simp}(\mathbf{C})$, whose morphisms are morphisms of functors. Fix integer $n \geq 1$ and $0 \leq i \leq n$, define a *face map* $\varepsilon_i : [n-1] \to [n]$ as the unique non-decreasing map whose image does not contain i. The *degeneracy map* $\eta_i : [n] \to [n-1]$ is the unique non-decreasing map that is surjective and maps exactly two elements to i.

Lemma 4.1. *Giving a simplicial object X in $\mathbf{C}$ is equivalent to giving objects X_n for $n \geq 0$ with face maps and degeneracy maps*

$$\partial_i = X(\varepsilon_i) : X_n \to X_{n-1}, \sigma_i = X(\eta_i) : X_{n-1} \to X_n$$

satisfying

$$\begin{aligned} \partial_i \partial_j &= \partial_{j-1} \partial_i \quad i < j \\ \sigma_i \sigma_j &= \sigma_{j+1} \sigma_i \quad i \leq j \\ \partial_i \sigma_j &= \begin{cases} \sigma_{j-1} \partial_i & i < j \\ \mathrm{id} & i = j \text{ or } i = j+1 \\ \sigma_j \partial_{i-1} & i > j+1 \end{cases} \end{aligned}$$

Example. If B is an object in $\mathbf{C}$, the *constant simplicial object* $B_\bullet$ is $B_n = B$ and all maps in the lemma are id_B.

Definition (augmentation)**.** Give $B \in \mathbf{C}$ and $X_\bullet \in \mathbf{Simp}(\mathbf{C})$, we define the *augmentation* $\varepsilon : X_\bullet \to B$ to be a morphism $X_\bullet \to B_\bullet$.

Definition (associated chain complex of a simplicial object)**.** Given $X_\bullet \in \mathbf{Simp}(\mathbf{A})$ where $\mathbf{A}$ is an abelian category, we define its *associated chain complex* $CX_\bullet$ with $CX_n = X_n$ and differentials $d_n : X_n \to X_{n-1}$ given by $d_n = \sum_{i=0}^{n} (-1)^i \partial_i$.

Definition (simplicial resolution)**.** An augmented simplicial object $\varepsilon : X_\bullet \to B$ in an abelian category is a *simplicial resolution* if $\varepsilon_0 : X_0 \to B$ is epic and the associated chain complex $CX_\bullet$ is acyclic except at degree 0 where its homology is B.

4.2 Construction of cotangent complex

Reference for cotangent complex:

- T. Szamuely and G. Zabradi: The p-adic Hodge decomposition according to Beilison
- Stacks: The cotangent complex

This construction is due to Quillen.

For any ring A, we write $A[S]$ for the polynomial algebra over A on a set of variables x_s indexed by $s \in S$. The functor $S \mapsto A[S]$ is left adjoint to the forgetful functor from $\mathbf{Alg}_A \to \mathbf{Set}$. In particular we have a canonical surjection $\eta_B : A[B] \to B$ for any A-algebra B. Repeating this construction, we obtain $A[A[B]] \to A[B]$. Iterating this, we get a simplicial A-algebra $P_{B/A\bullet}$ augmented over B

$$(\cdots A[A[A[B]]] \rightrightarrows A[A[B]] \to A[B]) \to B.$$

This map is a resolution of B in the category of simplicial A-algebras. More concretely, the associated chain complex is a free resolution of B over A.

Definition (cotangent complex)**.** For any map $A \to B$ of commutative rings, we define the *cotangent complex* $L_{B/A}$, which is a complex of B-modules and most of the time viewed as an an object of $\mathbf{D}(B)$, as follows: set

$$L_{B/A} = C(\Omega^1_{P_\bullet/A} \otimes_{P_\bullet} B_\bullet)$$

where $P_\bullet \to B$ is the resolution that we defined earlier (by polynomial A-algebras) and $\Omega^1_{P_\bullet/A}$ is defined by applying the functor $A' \mapsto \Omega^1_{A'/A}$ for any A-algebra A'. The tensor product goes diagonally. In other words

$$(L_{B/A})_n = \Omega^1_{P_\bullet/A} \otimes_{P_n} B.$$

Non-trivial fact (Thm 2.7 in the first reference) Let $Q_\bullet \to B$ be another simplicial resolution by polynomial A-algebras. We then have a quasi-isomorphism

$$L_{B/A} \simeq C(\Omega^1_{Q_\bullet/A} \otimes_{Q_\bullet} B_\bullet)$$

of complexes, i.e. an isomorphism in $\mathbf{D}(B)$.

Properties of $L_{B/A}$ (try to prove these!):

1. polynomial algebras: if B is a polynomial A-algebra then

   $$L_{B/A} \simeq \Omega^1_{B/A}[0],$$

   (concentrated in degree 0). It is because polynomial algebra resolutions are homotopic to each other, so we may use the constant simplicial A-algebra B to compute $L_{B/A}$.

2. Künneth formula: if B, C are flat A-algebras, then

   $$L_{B\otimes_A C/A} \simeq L_{B/A} \otimes_A C \oplus L_{C/A} \otimes_A B.$$

   We can reduce this to the case of polynomial algebras by passing to resolutions and flatness is used to show that if $P_\bullet \to B$ and $Q_\bullet \to C$ are polynomial resolutions then $P_\bullet \otimes_A Q_\bullet \to B \otimes_A C$ is also a polynomial resolution.

3. transitivity triangle: given a composition $A \to B \to C$, we have a canonical exact (distinguished) triangle
$$L_{B/A} \otimes^L_B C \to L_{C/A} \to L_{C/B}$$
in $\mathbf{D}(\mathbf{C})$. To prove this we first settles the case where $A \to B, B \to C$ are polynomial maps. The general case follows by passing to resolutions. (Thm 2.13 in the first reference)

4. base change: given a flat map $A \to C$ and a map $A \to B$, we have
$$L_{B/A} \otimes_A C \simeq L_{B \otimes_A C/C}.$$
Again pass to resolutions and check for polynomial algebras.

5. vanishing for étale maps: if $A \to B$ is étale then $L_{B/A} \simeq 0$. Strategy of proof: for Zariski localisations (i.e. $B \cong \prod_{i=1}^n A[\frac{1}{f_i}]$ for $f_i \in A$) we have that $B \otimes_A B \simeq B$ induced by pushout diagram. Then the Künneth formula shows that
$$L_{B/A} \simeq L_{B/A} \oplus L_{B/A}$$
so $L_{B/A} \simeq 0$. For an étale map $A \to B$, one shows $m : B \otimes_A B \to B$ is a Zariski localisation so $L_{B/B \otimes_A B} \simeq 0$. By the trnasitivity triangle for
$$B \xrightarrow{i_1} B \otimes_A B \to B,$$
this yields that
$$L_{B \otimes_A \otimes B} \otimes_{B \otimes_A B} B \simeq 0$$
By base change $L_{B \otimes_A B/B} \simeq L_{B/A} \otimes B$, so the base change of $L_{B/A}$ along
$$A \to B \to B \otimes_A B \to B$$
vanishes. But this composition is just the original étale map, so $L_{B/A} \otimes_A B \simeq 0$. Then the canonical map $L_{B/A} \to L_{B/A} \otimes_A B$ has a section coming from the action of B on $L_{B/A}$, hence $L_{B/A} \simeq 0$.

6. étale localisation: if $B \to C$ is an étale map of A-algebras then $L_{B/A} \otimes_B C \simeq L_{C/A}$. This follows from 2 and 5 as $L_{C/B} \simeq 0$.

7. relation to Kähler differentials: for any map $A \to B$ we have that $H^0(L_{B/A}) \cong \Omega^1_{B/A}$. This can be directly shown from the definition.

8. smooth algebras: if $A \to B$ is smooth then
$$L_{B/A} \simeq \Omega^1_{B/A}[0].$$
Use that smooth algebras look like polynomial algebras étale locally, and apply 1, 3 and 6.

Example (cotangent complex for a complete intersection)**.** Let R be a ring, $I \subseteq R$ generated by a regular sequence and $S = R/I$. Then $L_{S/R} \simeq I/I^2[1]$. To see this, consider first the case $R = \mathbb{Z}[x_1, \dots, x_n]$ and $I = (x_1, \dots, x_n)$. Then $S = \mathbb{Z}$ and the transitivity triangle for $\mathbb{Z} \to R \to S$ collapses to give
$$L_{S/R} \simeq \Omega^1_{R/\mathbb{Z}} \otimes_R S[1] \simeq I/I^2[1]$$

where the first quasi-isomorphism is induced by $I/I^2 \to \Omega^1_{R/\mathbb{Z}} \otimes_R S, f \mapsto \mathrm{d}f \otimes 1$.

For general R, we choose a regular sequence $f_1, \dots, f_r \in I$. Then we consider the pushout square

$$\begin{array}{ccc} \mathbb{Z}[x_1, \dots, x_r] & \xrightarrow{x_i \mapsto f_i} & R \\ \Big\downarrow{\scriptstyle x_i \mapsto 0} & & \Big\downarrow \\ \mathbb{Z} & \longrightarrow & S \end{array}$$

As f_i's form a regular sequence, the base change map for the cotangent complex implies

$$L_{S/R} \simeq L_{\mathbb{Z}/\mathbb{Z}[x_1, \dots, x_n]} \otimes_{\mathbb{Z}} S \simeq I/I^2[1].$$

The reason we introduced the cotangent complex in the following theorem:

Theorem 4.2 (deformation invariant of the category of finite étale algebras)**.** *For any ring A, denote by $\mathbf{C}_A$ the category of flat A-algebras B such that $L_{B/A} \simeq 0$. Then for any surjective map $A' \to A$ with nilpotent kernel, base change induces an equivalence of categories $C_A \simeq C'_A$. In other words, every $A \to B$ in $\mathbf{C}_A$ lifts uniquely to $A' \to B'$ in $\mathbf{C}_{A'}$.*

Any étale A-algebra B is an onject of $\mathbf{C}_A$. Conversely if B is finitely presented over A and $B \in \mathbf{C}_A$ then can show B is étale over A (TAG 0D12). Hence the above theorem really shows the invariance of the étale sites of such maps (TAG 04DZ).

The above is too restrictive for our purpose, therefore we prove the following statement:

Proposition 4.3. *Assume that A has characteristic p. Let $A \to B$ be a flat map that is relatively perfect, i.e. the relative Frobenius $F_{B/A} : B^{(1)} := B \otimes_{A, F_A} \to B$ is an isomorphism. Then $L_{B/A} \simeq 0$.*

Proof. We show that for any $A \to B$, $F_{B/A}$ induces the zero map $L_{F_{B/A}} : L_{B^{(1)}/A} \to L_{B/A}$. When B is a polynomial A-algebra it is clear (as $\mathrm{d}x^p = 0$) and then we pass to the associated standard resolutions to conclude for the general case. Now if $A \to B$ is relatively perfect then $L_{F_{B/A}}$ is also an isomorphism by functoriality. Then the zero map is also an isomorphism so $L_{B/A} \simeq 0$. □

This leads to the following description of the Witt vector functor:

Witt vectors via deformation theory Let R be a perfect ring of characteristic p. Then R is relatively perfect over $\mathbb{F}_p$ so $L_{R/\mathbb{F}_p} \simeq 0$. Use the theorem about deformation invarinats, which says that R has a unique flat lift to $\mathbb{Z}/p^n\mathbb{Z}$ for any n. This lift is $W_n(R)$ from the canonical construction. Taking inverse limit,

$$W(R) = \varprojlim W_n(R)$$

gives the ring of Witt vectors of R, which can be seen to be the unique p-adically complete p-torisonfree $\mathbb{Z}_p$-algebra lift of R.

Fontaine's $A_{\inf}$ and the map θ Fix a ring A and $A \to B$ in $\mathbf{C}_A$. Analysing the proof of the above theorem (TAG 0D11), one can show the following: if $C' \to C$ is a surjective map of A-algebras with nilpotent kernel, every A-algebra map $B \to C$ has a unique A-algebra lift $B \to C'$. In particular given a p-adically complete $\mathbb{Z}_p$-algebra C, a perfect ring D and a map $D \to C/(p)$, we obtain a unique lift $W_k(D) \to C/(p^n)$. Taking limits we get a unique map $W(D) \to C$ which lifts $D \to C/(p)$. This way we obtain Fontaine's θ map via abstract nonsense:

Proposition 4.4. *Given any p-adically complete ring R, the canonical projection*

$$\overline{\theta} : R^\flat := \varprojlim_{x \mapsto x^p} R/p \to R/p$$

lifts uniquely to a map

$$\theta : A_{\inf}(R) := W(R^\flat) \to R.$$

Exercise. $\overline{\theta}$ is surjective if and only if R/p is semiperfect, i.e. has sujrective Frobenius. In this case by p-adic completeness θ is also surjective.

A short review of integral perfectoid theory.

Definition (integral perfectoid ring)**.** A ring R is *integral perfectoid* if R is π-adically complete for some $\pi \in R$ such that $p \in (\pi^p)$, the ring R/p has surjective Frobenius and the kernel of θ is principal.

Example.

- Ring of integers of a perfectoid field.
- perfect ring of characteristic p.
- if K is a perfectoid field of characteristic 0 then a p-adically complete, p-torsionfree $\mathcal{O}_K$-algebra R is integral perfect if and only if $\mathcal{O}_K/p \to R/p$ is relatively perfect.

Remark (tilting)**.** For an integral perfectoid ring R, the map θ fits into the following commutative diagram

$$\begin{array}{ccc} W(R^\flat) & \xrightarrow{\theta} & R \\ \downarrow & & \downarrow \\ R^\flat & \xrightarrow{\overline{\theta}} & R/p \end{array}$$

Then the above theorem and proposition 4.3 can be used to prove half of the tilting correspondence by Scholze.

4.3 $A_{\inf}$ and differential forms

We sketch the proof of a statement that in particular proves Fontaine's theorem on differential forms on $\mathcal{O}_{\mathbb{C}_K}$ (c.f. theorem 2.3).

Let A be a perfect ring of characteristic p. By proposition 4.3 we have a canonical map $\mathbb{Z}_p \to W(A)$ and its has the property that $L_{A/\mathbb{F}_p} \simeq 0$. Note that in the base change property of cotangent complex, if we relax the flatness of $A \to C$ to $\mathrm{Tor}^A_{>0}(B, C) = 0$, the same argument shows that

$$L_{BN/A} \otimes^L_A C \simeq L_{B\otimes_A C/C}.$$

Thus

$$L_{W(A)/\mathbb{Z}_p} \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p \simeq L_{A/\mathbb{F}_p} \simeq 0.$$

Digression on derived completion Reference TAG 091N, 0BKF.

Definition (derived completeness)**.** Let A be a commutative ring and I be a finitely generated ideal in A. An A-complex $M^\bullet \in \mathbf{D(A)}$ is *derived I-complete* if for each $f \in I$, the derived inverse limit

$$T(M^\bullet, f) := R\lim(\cdots \xrightarrow{f} M \xrightarrow{f} M) \in \mathbf{D(A)}$$

vanishes. This is equivalent to requiring that the natural map

$$M^\bullet \to R\lim(M^\bullet \otimes^L_{\mathbb{Z}[x]} \mathbb{Z}[x]/(x^n))$$

(we treat $M^\bullet$ as a $\mathbb{Z}[x]$-module via $\mathbb{Z}[x] \to A, x \mapsto f$) is a quasi-isomorphism.

An A-module M is *derived I-complete* if $M[0]$ is derived I-complete.

Fact:

1. For any finitely generated idael $I \subseteq A$, M is I-adically complete if and only if $M \in \mathbf{D(A)}$ is derived I-complete and the filtration is separated, i.e. $\bigcap I^n M = 0$.
2. Derived I-completenesss can be checked on any generating set of I.
3. An object $M^\bullet \in \mathbf{D(A)}$ is derived I-complete if and only if $H^i(M^\bullet)$ is so.
4. Derived Nakayama lemma: let $I \subseteq A$ be finitely generated. Then for any $M^\bullet \in \mathbf{D(A)}$ which is derived I-complete, if $M^\bullet \otimes^L_A A/I = 0 \in \mathbf{D(A)}$ then $M^\bullet = 0$ $\mathbf{D(A)}$ as well. See TAG 0GTU.
5. If $I = (f_1, \dots, f_r)$, the collection of all derived I-complete A-complexes form a full triangulated subcategory of $\mathbf{D(A)}$ closed under derived inverse limits.

Now we can define derived completion. For any $M^\bullet \in \mathbf{D(A)}$, the object

$$\hat{M}^\bullet := R\lim(M^\bullet \otimes^L_{\mathbb{Z}[x_1,\dots,x_r]} \mathbb{Z}[x_1^n, \dots, x_r^n])$$

is called the *derived I-completion* of $M^\bullet$.

If I is principal then

$$\hat{M}^\bullet \simeq R\lim(M^\bullet \otimes^L_A (A \xrightarrow{f^n} A)).$$

Now back to the discussion of Witt ring. We have

$$L_{W(A)/\mathbb{Z}_p} \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p \simeq L_{A/\mathbb{F}_p} \simeq 0$$

so taking the derived p-completion and using derived Nakayama and that

$$0 \simeq L_{W(A)/\mathbb{Z}_p} \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p \simeq \widehat{L_{W(A)/\mathbb{Z}_p}} \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p$$

we see that $\widehat{L_{W(A)/\mathbb{Z}_p}} \simeq 0$. By transitivity triangle, for any $W(A)$-algebra R, we have

$$\widehat{L_{R/\mathbb{Z}_p}} \simeq \hat{L}_{R/W(A)}.$$

If R is an integral perfectoid ring and $A = R^\flat$, with $\theta : W(A) \to R$ then $\widehat{L_{R/\mathbb{Z}_p}} \simeq \widehat{L_{R/A_{\text{inf}}}}$. But $\ker\theta$ is principal generated by a non-zero divisor (see Bhatt-Scholze Lemma 3.10 for a proof. See also IV Perfectoid Spaces). Then our example about the cotangent complex for complete intersections shows

$$\hat{L}_{R/\mathbb{Z}_p} \simeq \ker\theta/(\ker\theta)^2[1].$$

In particular, this is a free R-module of rank 1. In the case of $R = \mathcal{O}_{\mathbb{C}_K}$, this recovers Theorem 2.3: LHS is quasi-isomorphic to Ω

$$T_p(\Omega^1_{\mathcal{O}_{\mathbb{C}_p/\mathbb{Z}_p}}) \simeq \hat{L}_{\mathcal{O}_{\mathbb{C}_K/\mathbb{Z}_p}}[-1].$$

RHS is "$\mathcal{O}_{\mathbb{C}_K}(1)$": if we choose a compatible system of p-poer roots of unity and look at its tilt $\varepsilon^\flat \in \mathcal{O}^\flat_{\mathbb{C}_K}$ then $\mu := [\varepsilon^\flat] \in \ker\theta$ and its iamge spans a copy of $\mathcal{O}_{\mathbb{C}_K}(1)$. The quotient $(\ker\theta/(\ker\theta)^2)/\mathcal{O}_{\mathbb{C}_K(1)}$ is torsion and killed by $p^{1/(p-1)}$ (section 3.3 in Bhatt-Morrow-Scholze).

Remark (on the notation A_{inf} (and A_{crys})). Let R be an integral perfectoid ring. Be definition $\overline{\theta} : R^\flat \to R/p$ is the inverse limit of the maps $\phi^n : R/p \to R/p$. Since R is perfectoid (using semiperfectness), these maps are infinitesimal thickenings (i.e. surjections with nilpotent kernels). Therefore $R^\flat$ is the inverse limit of infinitesimal thickenings on R/p. Using that $R^\flat$ is perfect, $\overline{\theta} : R^\flat \to R/p$ is universal in the sense that for any infinitesimal thickening $S \to R/p$ with S an $\mathbb{F}_p$-algebra, we have $R^\flat \to S$ factoring $\overline{\theta}$. One uses the construction in the last lecture to show that $\theta : W(R^\flat) \to R$ is also an inverse limit of infinitesimal thickenings of R and in a universal way. Then by definition $A_{\text{R}} = W(R^\flat)$ is the global sections of the structure sheaf of the infinitesimal site for Spec(R/p). Hence the name A_{R}.

Likewise, adjoining divided powers along the kernels of θ to A_{inf} and p-adically completing produces A_{crysR} and similarly one can show A_{crysR} is the global sections of the structure sheaf on the crystalline site of Spec(R/p).

5 Pro-étale site

Recall from SGA 4 I.8 some abstract nonsense about pro-objects in a category $\mathbf{C}$.

Definition (cofiltered category)**.** A category I is *cofiltered* if for every $i_1, i_2 \in I$, exists $i_3 \in I$ such that there are morphisms $i_3 \to i_1, i_3 \to i_2$, and if $i_1 \rightrightarrows i_2$ are two arrows the exists a morphism $i \to i_1$ such that the compositions $i \to i_1 \rightrightarrows i_2$ are equal.

Definition (pro-object)**.** Let $\mathbf{C}$ be a category. A *pro-object* of $\mathbf{C}$ is a functor $F : I \to \mathbf{C}$ where I is a small cofiltered category.

The category pro-$\mathbf{C}$ is the category whose objects are pro-objects of $\mathbf{C}$, and if $F : I \to \mathbf{C}, G : J \to \mathbf{C}$ are pro-objects, $\mathrm{Hom}_{\mathbf{pro-C}}(F, G)$ is the limit of the functor $I^{\mathrm{op}} \times J \to \mathbf{Set}$ given by $\mathrm{Hom}_{\mathbf{C}}(F(-), G(-))$. Thus the homset in $\mathbf{pro} - \mathbf{C}$ is

$$\lim_J \operatorname*{colim}_I \mathrm{Hom}_{\mathbf{C}}(F(i), G(j)).$$

Equivalent point of view: take $\hat{\mathbf{C}} = \mathbf{Set}^{\mathbf{C}^{\mathrm{op}}}$ and consider $\mathbf{pro} - \mathbf{C}$ to be the full subcategory given by functors that are small cofiltered limits of representatable objects. Note that by Yoneda we have a natural embedding $\mathbf{C} \to \hat{\mathbf{C}}$, called the *Yoneda embedding*. To see the equivalence, if we have a functor $F : I \to \mathbf{C}$, compose it with the Yoneda-embedding to get $\tilde{F} : I \to \hat{\mathbf{C}}$. Take limit $\lim \tilde{F} \in \hat{\mathbf{C}}$. This yields a functor from $\mathbf{pro} - \mathbf{C}$ in the first definition to that in the second definition. This functor is fully faithful and essentially surjective.

Since we can combine double inverse systems to single inverse systems, the category $\hat{\mathbf{C}}$ has arbitrary cofiltered inverse limits.

Now let X be a locally noetherian adic space, i.e. it is locally of the form $\mathrm{Spa}(A, A^+)$ where either A is noetherian or A has a noetherian ring of definition. We may look at pro-$X_{\text{ét}}$. Note that every $U \in$ pro-$X_{\text{ét}}$ has an underlying topological space $|U| = \varprojlim |U_i|$ in the category $\mathbf{Top}$. We may thus speak of topological properties such as open, quasicompact etc.

Definition. The *pro-finite étale site* X_{proft} has as underlying category the category of pro-$X_{\text{fét}}$. Coverings are given by open morphisms $\{f_i : U_i \to U\}$ such that $|U| = \bigcup f_i(|U_i|)$.

For a profinite group G, let $G-\mathbf{fset}$ be the site whose underlying category is the category of finite sets S with continuous G-action and whose coverings are given by families of G-equivariant maps $\{f_i : S_i \to S\}$ such that $S = \bigcup f_i(S_i)$.

Let $G-\mathbf{pfset}$ be the site with objects profinite sets S with continuous G-actions and coverings $\{f_i : S_i \to S\}$ open G-equivariant maps such that $S = \bigcup f_i(S_i)$.

Proposition 5.1. *Let X be a connected locally noetherian adic space. Then there is a canonical equivalence of sites*

$$X_{\text{profét}} \cong \pi_1(X, \overline{x})-\mathbf{pfset}.$$

Sketch proof. The functor sends $U \in X_{\text{profét}}$ to $\varprojlim s_i$ where s_i are the stalks of U_i at $\overline{x}$. Note that by definition $G{-}\mathbf{pfset} \cong \mathbf{pro}{-}(G-\mathbf{fset})$ and we use the well-known fact that

$$X_{\text{fét}} \cong \pi_1(X, \overline{x}){-}\mathbf{fset}.$$

One may check that coverings are also identified. □

Now we define the pro-étale site. We take the definitions in Scholze: p-adic Hodge theory for rigid analytic vareities.

Definition. A morphism $U \to V$ in $\mathbf{pro}{-}X_{\text{ét}}$ is called *étale* (resp. *finite étale*) if there exists a morphism $U_0 \to V_0$ in $X_{\text{ét}}$ (resp. $X_{\text{fét}}$) such that $U = V \times_{U_0} V_0$ via some morphism $V \to V_0$.

A morphism $U \to V$ in $\mathbf{pro}{-}X_{\text{ét}}$ is called *pro-étale* if it can be written as a cofiltered inverse limit $U = \varprojlim U_i$ of objects $U_i \to V$ in $\mathbf{pro}{-}X_{\text{ét}}$ that are étale in $\mathbf{pro}{-}X_{\text{ét}}$, such that $U_i \to U_j$ is finite étale and surjective for large $i > j$. Such a presentation is called *pro-étale* presentation.

Definition (pro-étale site)**.** The *pro-étale site* $X_{\text{proét}}$ has as underlying category space the full subcategory of $\mathbf{pro}{-}X_{\text{ét}}$ of objects that are pro-étale over X. A cover in $X_{\text{proét}}$ is given by a family $\{f_i : U_i \to U\}$ such that f_i's are pro-étale and $|U| = \bigcup f_i(|U_i|)$.

Verifying that $X_{\text{proét}}$ is a site amounts to prove the following proposition, which we state only:

Proposition 5.2.

1. *If $U, V, W \in \mathbf{pro}{-}X_{\text{ét}}$ and $U \to V$ is an étale morphism (resp. finite étale, resp. pro-étale) and $W \to V$ is any morphism then $U \times_V W$ exists in $\mathbf{pro}{-}X_{\text{ét}}$ and $U \times_V W \to W$ is étale (resp. finite étale, resp. pro-étale), and the map $|U \times_V W| \to |U| \times_{|V|} |W|$ is surjective.*

2. *Composition of étale (resp. finite étale) morphisms is étale (resp. finite étale).*

3. *Any pro-étale map $U \to V$ is open.*

4. *Let $U \to V, V \to W$ be pro-étale morphism in $\mathbf{pro}{-}X_{\text{ét}}$ and $W \in X_{\text{proét}}$. Then $U, V \in X_{\text{proét}}$ and the composition $U \to W$ is pro-étale.*

There is a fully fiathful embedding of categories $X_{\text{profét}} \subseteq X_{\text{proét}}$. The coverings coincide so we get morphism of sites $X_{\text{proét}} \to X_{\text{profét}}$. There is a canonical map $v : X_{\text{proét}} \to X_{\text{ét}}$ as any étale map $U \to V$ in $X_{\text{ét}}$ is by definition étale in $X_{\text{proét}}$.

Theorem 5.3.

- *An object $U \in X_{\text{proét}}$ is* quasicompact *(every covering can be refined to a finite subcovering) if and only if $|U|$ is quasicompact.*

- *An object $U \in X_{\text{proét}}$ is* quasiseparated *(for all pair $V \to U \leftarrow W$ where V, W are quasicompact, $V \times_U W$ is quasicompact) if and only* *if $|U|$ is quasiseparated*

- *A morphism $f : U \to V$ is quasicompact (resp. quasiseparated) if and only if $|f| : |U| \to |V|$ is quasicompact (resp. quasiseparated).*

Theorem 5.4 (comparison theorem between étale and pro-étale cohomology)**.** *For any $\mathcal{F} \in \tilde{X}_{\text{ét}}$, the adjunction morphism $\mathcal{F} \to Rv_*v^*\mathcal{F}$ is an isomorphism (in the derived category). Consequently v^* gives a fully faithful embedding $\tilde{X}_{\text{ét}} \to \tilde{X}_{\text{proét}}$.*

Proof. We know that for $i \geq 0$, $R^i v_* v^* \mathcal{F}$ is the sheaf associated to the presheaf $U \mapsto H^i(U, v^*\mathcal{F})$ where U is considered as an object in $X_{\text{proét}}$. As X is quasiseparated, one can just work with $X_{\text{proétqc}} \subseteq X_{\text{proét}}$. Recall that we have the following theorem

Theorem 5.5. *Let $\mathcal{F} \in \tilde{X}_{\text{ét}}$ and $U \in X_{\text{proét}}$ with pro-étale presentation $U = \varprojlim U_i$ such that U is qcqs. Then for any $j \geq 0$*

$$H^j(U, v^*\mathcal{F}) = \varinjlim H^j(U_i, \mathcal{F}).$$

By this theorem for $j = 0$, $H^0(U, v^*\mathcal{F}) = H^0(U, \mathcal{F})$. Moreover in $j > 0$, $H^j(U, v^*\mathcal{F}) = H^j(u, \mathcal{F})$. But any section vanishes locally on the étale topology, so the associated sheaf is trivial, so we get a quasiisomorphism $\mathcal{F} \to Rv_*v^*\mathcal{F}$. □

5.1 Structure sheaf on pro-étale site

Definition. Let X be a locally noetherian adic space over $\text{Spa}(\mathbb{Q}_p, \mathbb{Z}_p)$.

1. The *uncompleted structure sheaf* $\mathcal{O}_X := v^*\mathcal{O}_{X_{\text{ét}}}$ with subsheaf of integral elements $\mathcal{O}_X^+ := v_*\mathcal{O}_{X_{\text{ét}}}^+$.
2. The *integral completed structure sheaf* $\hat{\mathcal{O}}_X^+ := \varprojlim \mathcal{O}_X^+/p^n$ and the *completed structure sheaf* $\hat{\mathcal{O}}_X := \hat{\mathcal{O}}_X^+[\frac{1}{p}]$.

Remark. One can really see that for any $x \in |U|$ for any object $U \in X_{\text{proét}}$, we have a natural conrinuous valuation $f \mapsto |f(x)|$ on $\mathcal{O}_X$ and

$$\mathcal{O}_X^+(U) = \{f \in \mathcal{O}_X(U) : |f(x)| \leq 1 \text{ for all } x \in |U|\}.$$

Moreover the natural map of sheaves $\mathcal{O}_X^+/p^n \to \hat{\mathcal{O}}_X^+/p^n$ is an isomorphism and $\hat{\mathcal{O}_X}^+(U)$ is flat over $\mathbb{Z}/p$ and p-adically complete. The valuation $f \mapsto |f(x)|$ extends to a continuous valuation on $\hat{\mathcal{O}}_X^+(U)$ and

$$\hat{\mathcal{O}}_X^+(U) = \{f \in \hat{\mathcal{O}}_X(U) : |f(x)| \leq 1 \text{ for all } x \in |U|\}.$$

In particular $\hat{\mathcal{O}}_X^+(U) \subseteq \hat{\mathcal{O}}_X(U)$ is integrally closed.

How to construct a valuation on $\mathcal{O}_X(U)$? This can be checked locally so we may assume that U is qcqs. Also any point $x \in |U| = \varprojlim |U_i|$ is given by a compatible system $x_i \in |U_i|$ which in turn corresponds to continuous valuations $\tilde{x}_i$ on $\mathcal{O}_{X\text{ét}}(U_i) = \mathcal{O}_X(U_i)$. But

$$\mathcal{O}_X(U) = (v^*\mathcal{O}_{X_t})(U) = \varinjlim \mathcal{O}_X(U_i)$$

so $\tilde{x}_i$ combine into a continuous valuation on $\mathcal{O}_X(U)$.

Now we turn to determine a basis for the pro-étale topology. Assume that K is a perfectoid field and X is locally noetherian over $\mathrm{Spa}(K, K^+)$ (for example $\mathrm{Spa}(\mathbb{C}_K, \mathcal{O}_{\mathbb{C}_K})$).

Definition ((affinoid) perfectoid object of pro-étale topology)**.** An object $U = \varprojlim U_i \in X_{\text{proét}}$ is called *affinoid perfectoid* if it satisfies the following:

1. each $U_i = \mathrm{Spa}(R_i, R_i^+)$ is affinoid,
2. setting $R^+ = \widehat{\operatorname{colim} R_i^+}$ (p-adic completion) and $R := R^+[\frac{1}{p}]$, the pair (R, R^+) is a perfectoid K-algebra ($\mathrm{Spa}(R, R^+)$ is affinoid perfectoid over $\mathrm{Spa}(K, K^+)$). For such an object, write $\hat{U} = \mathrm{Spa}(R, R^+)$.

We say that U is *perfectoid* if it has an open cover by affinoid perctoids.

Example (perfectoid torus)**.** If

$$X = \mathbb{T}^n = \mathrm{Spa}(K\langle T_1^{\pm 1}, \dots, T_k^{\pm 1}\rangle, K^+\langle T_1^{\pm 1}, \dots, T_n^{\pm 1}\rangle)$$

then the inverse limit $\tilde{\mathbb{T}}^n \in X_{\text{proét}}$ of

$$\mathrm{Spa}(K\langle T_1^{\pm 1/p^m}, \dots, T_k^{\pm 1/p^m}\rangle, K^+\langle T_1^{\pm 1/p^m}, \dots, T_n^{\pm 1/p^m}\rangle)$$

is affinoid perfectoid.

Remark. Affinoid perfectoid objects are important because they provide a basis for the pro-étale topology. If X is smooth then locally it admits an étale map to $\mathbb{T}^n$

$$\begin{array}{ccc} x \in U & \xrightarrow{\text{étale}} & \mathbb{T}^n \\ & \searrow & \downarrow \\ & & \mathrm{Spa}(\mathbb{Q}_p, \mathbb{Z}_p) \end{array}$$

(See Huber, Étale cohomology I.6.10).

Proof. If $X = \mathbb{T}^n$ then $\tilde{\mathbb{T}}^n \to \mathbb{T}^n$ is an explicity cover. If $X \to \tilde{\mathbb{T}}^n$ is pro-étale then X is also perfectoid: we may factor $X \to \tilde{\mathbb{T}}^n$ as the composition $X \xrightarrow{f} X_0 \xrightarrow{g} \tilde{\mathbb{T}}^n$ where f is an inverse limit of finite étale surjective maps $X_i \to X_0$ and $X = \varprojlim X_i$, and g is an étale map.

Since étale maps can be written locally as a composition of a rational subset of a finite étale cover. As rational subsets of affinoid perfectoid spaces are affinoid perfectoid, we need to check for finite étale maps. By almost purity, i.e. tilting induces an equivalence of categories between finite étale R-algebras and finite étale $R^\flat$-algebras where R is any perfectoid Tate algebra, and the fact that for perfectoid $\mathbb{F}_p$-algebras finite étale induces perfectness, we get what we want.

For $X \to X_0$, we use that the completion of direct limit of affinoid perfectoid is affinoid perfectoid and again the same reasoning as above for finite étale.

In general X is smooth so it factorises locally as

$$\begin{array}{ccc} x \in U & \xrightarrow{\text{étale}} & \mathbb{T}^n \\ & \searrow & \downarrow \\ & & \mathrm{Spa}(K, K^+) \end{array}$$

and for any such $U \in X_{\text{proét}}$, $U \times_{\mathbb{T}^n} \tilde{\mathbb{T}}^n \to U$ is a covering such that $U \times_{\mathbb{T}^n} \tilde{\mathbb{T}}^n \to \tilde{\mathbb{T}}^n$ is pro-étale. □

5.2 Vanishing theorems on $X_{\text{proét}}$

By Lemma 4.1 in "Scholze: p-adic Hodge...", one may see that $\hat{\mathcal{O}}_X$ behaves as expected: for any affinoid perfectoid $U \in X_{\text{proét}}$ with $\hat{U} = \mathrm{Spa}(R, R^+)$, $\hat{\mathcal{O}}_X(U) \cong R$ and $\hat{\mathcal{O}}_X^+(U) \cong R^+$ and $\hat{\mathcal{O}}_X^+$ is really the p-adic completion of $\mathcal{O}_X^+(U)$. We may consider the relative version of $L_{B/A}$, i.e. for $\hat{\mathcal{O}}_X^+$, $L_{\hat{\mathcal{O}}_X^+/\mathcal{O}_{\mathbb{C}_K}}$ is the sheafification of

$$U \mapsto L_{\hat{\mathcal{O}}_X^+(U)/\mathcal{O}_{\mathbb{C}_K}}$$

for $U \in X_{\text{pro-ét}}$.

Proposition 5.6. *The cotangent complex $L_{\hat{\mathcal{O}}_X^+/\mathcal{O}_{\mathbb{C}_K}}$ vanishes mod p on $X_{\text{pro-ét}}$. Hence the p-adic derived completion $\widehat{L_{\hat{\mathcal{O}}_X^+/\mathcal{O}_{\mathbb{C}_K}}} \simeq 0$.*

Proof. It is enough to check for affinoid perfectoid objects that the presheaf

$$U \mapsto L_{\hat{\mathcal{O}}_X^+(U)/\mathcal{O}_{\mathbb{C}_K}} \otimes^L_{\mathbb{Z}_p} \mathbb{F}_p$$

vanishes. By the remark at the beginning of the section, $\hat{\mathcal{O}}_X^+(U) = R^+$ is integrally perfectoid. Therefore as $\mathcal{O}_{\mathbb{C}_K}/p \to R^+/p$ is relatively perfect (by semi-perfectness), we may use proposition 1 in lecture 8 (?) to conclude that $L_{R^+/\mathcal{O}_{\mathbb{C}_K}} \simeq 0$. □

In other words there is no differential geometric information when we work on the ringed site $(X_{\text{pro-ét}}, \hat{\mathcal{O}}_X)$.

Theorem 5.7 (acyclicity of $\hat{\mathcal{O}}_X$ on affinoid perfectoids)**.** *Let $U \in X_{\text{pro-ét}}$ be affinoid perfectoid. Then $H^i(U, \hat{\mathcal{O}}_X) = 0$ for $i > 0$.*

Proof. Scholze p-adic 4.11. The proof goes through almost mathematics by Faltings. One must show that $H^i(U, \hat{\mathcal{O}}_X^+)$ almost vanishes, in the sense that it is killed by (p^{1/p^∞}). □

Remark. We may use this to compute $H^i(U, \hat{\mathcal{O}}_X)$ for $U \in X_{\text{pro-ét}}$ affinoid object. If we choose a pro-étale cover $V \to U$ with V affinoid perfectoid. Then by Leray acyclicity and the above theorem we may use the Čech complex

$$\hat{\mathcal{O}}_X(V) \longrightarrow \hat{\mathcal{O}}_X(V \times_U V) \longrightarrow \hat{\mathcal{O}}_X(V \times_U V \times_U V) \longrightarrow \cdots$$

to compute $H^i(U, \hat{\mathcal{O}}_X)$. Note that all fibre products $V \times_U \cdots \times_U V$ are affinoid perfectoid (Scholze 6.18). Moreover if G is a profinite group and $V \to U$ is a G-torsor, i.e. it is a homogeneous space and G acts freely, then one can show

$$V \times_U V \cong V \times \underline{G}$$

where $\underline{G} : Y \mapsto \mathrm{Hom}_{\mathrm{cont}}(|Y|, G)$, and hence

$$H^i(U, \hat{\mathcal{O}}_X) \cong H^i_{\mathrm{cont}}(G, \hat{\mathcal{O}}_X(V))$$

(this is analogous to the derivation of Hilbert theorem 90 from faithfully flat descent). In other words we may compute pro-étale cohomology of $\hat{\mathcal{O}}_X$ in terms of continuous group cohomology.

Lemma 5.8. *The $\mathcal{O}_X$-mdoule $R^1 v_* \hat{\mathcal{O}}_X$ is locally free of rank n, the dimension of X and taking cup products gives an isomorphism*

$$\bigwedge^i R^1 v_* \hat{\mathcal{O}}_X \simeq R^i v_* \hat{\mathcal{O}}_X.$$

Proof. This is a local statement so we may assume that X is affinoid and there is an étale morphism $X \to \mathbb{T}^n$ such that it factors as the composition of a rational subset of a finite étale cover. Let $\tilde{X} = X \times_{\mathbb{T}^n} \tilde{\mathbb{T}}^n \to X$ be a pro-étale cover such that $\tilde{X}$ is affinoid perfectoid. $\tilde{T}^n$ is a $\mathbb{Z}_p(1)$-torsor:

$$\begin{aligned} \widehat{(\tilde{\mathbb{T}}^n)} &= \mathrm{Spa}(R, R^+) \\ &:= \mathrm{Spa}(\mathbb{C}_K\langle T_i^{\pm 1/p^\infty}, \rangle, \mathcal{O}_{\mathbb{C}_K}\langle T_i^{\pm 1/p^\infty}\rangle) \\ &= \varprojlim_n \mathrm{Spa}(\mathbb{C}_K\langle T_i^{\pm 1/p^n}\rangle, \mathcal{O}_{\mathbb{C}_K}\langle T_i^{\pm 1/p^\infty}\rangle). \end{aligned}$$

Each map in the inverse system is canonically a $\mu_p(\mathbb{C}_K)$-torsor (for example for $n = 1$, $T^{a/p^m} \mapsto \varepsilon_m^a \cdot T^{a/p^m}$) in a compatible way so we get a $\mathbb{Z}_p(1)$-torsor.

By remark above

$$H^i(X_{\text{pro-ét}}, \hat{\mathcal{O}}_X) \cong H^i_{\mathrm{cont}}(\mathbb{Z}_p(1), \hat{\mathcal{O}}_X(\tilde{X})).$$

By Lemma 4.5 and 5.5 in “Scholze: p-adic...”,

$$\hat{\mathcal{O}}_X(\tilde{X}) = \hat{\hat{\mathcal{O}}}_X(X) \otimes_{\mathbb{C}_K\langle T_i^{\pm 1}, \dots, T_n^{\pm 1}\rangle} R$$

and

$$H^i_{\mathrm{cont}}(\mathbb{Z}_p(1), \hat{\mathcal{O}}_X(\tilde{X})) \cong \hat{\mathcal{O}}_X(X) \otimes_{\mathbb{C}_K\langle T_i^{\pm 1}, \dots, T_n^{\pm 1}\rangle} H^i_{\mathrm{cont}}(\mathbb{Z}_p(1), R)$$

so it is enough to do every thing for $X = \mathbb{T}^n$.

We prove for $n = 1$ and the whole argument works for general n. We have a canonical presentation

$$\mathbb{C}_K\langle T^{\pm 1/p^\infty}\rangle \cong \widehat{\bigotimes}_{i \in \mathbb{Z}[\frac{1}{p}]} \mathbb{C}_K \cdot T^i$$

and by the above $\mathbb{Z}_p(1)$-action, if $\varepsilon = (\varepsilon_n) \in \mathbb{Z}_p(1)$ then by standard facts about continuous group cohomology of pro-cyclic groups

$$R\Gamma(X_{\text{pro-ét}}, \hat{\mathcal{O}}_X) \cong \widehat{\bigotimes}_{i\in\mathbb{Z}[\frac{1}{p}]}(\mathbb{C}_K T^i \xrightarrow{T^i \mapsto (\varepsilon^i - 1)\cdot T^i} \mathbb{C}_K T^i)$$

with the convention that if $i = \frac{a}{p^m}$ then $\varepsilon^i = \varepsilon_m^a$.

To see this, recall that, in general, if M is a topological $\mathbb{Z}_p^k$-module such that $M = \varprojlim M/p^m$ then continuous $\mathbb{Z}_p^k$-cohomology with values in M can be computed by the Koszul complex

$$0 \longrightarrow M \longrightarrow M^k \longrightarrow \cdots \longrightarrow \bigwedge^q M^k \longrightarrow \cdots \longrightarrow M^k \xrightarrow{d} M \longrightarrow 0$$

where $d : M^k \to M$ is $(\gamma_1 - 1, \ldots, \gamma_k - 1)$ where the γ_i's provide a basis for $\mathbb{Z}_p^k$. To check this, the Iwasawa algebra $\Lambda = \mathbb{Z}_p[[\mathbb{Z}_p^n]] \cong \mathbb{Z}_p[[x_1, \ldots, x_k]]$ where x_i corresponds to $\gamma_i - 1$, use the Koszul complex

$$0 \longrightarrow \Lambda \longrightarrow \Lambda^k \longrightarrow \cdots \longrightarrow \bigwedge^q \Lambda^k \longrightarrow \cdots \longrightarrow \Lambda^k \xrightarrow{(x_1,\ldots,x_k)} \Lambda \longrightarrow 0$$

is a resolution of $\mathbb{Z}_p$. Now take $\mathrm{Hom}_{\mathrm{cont}}(-, M)$, this gives a resolution of M is a topological $\mathbb{Z}_p^k$-module. Then taking cohomology gives the result.)

So the Koszul complex above assigned to $T^i \mapsto (\varepsilon^i - 1) \cdot T^i$ computes the continuous group cohomology and we get the result.

If $i \in \mathbb{Z}$ then $\varepsilon^i = 1$ so d is 0. If $i \notin \mathbb{Z}$ then $\varepsilon^i \neq 1$ so $\varepsilon^i - 1 \neq 0$ so d is an isomorphism. Thus up to quasiisomorphism we may ignore $i \notin \mathbb{Z}$ powers to get

$$R\Gamma(X_{\text{pro-ét}}, \hat{\mathcal{O}}_X) \cong \widehat{\bigoplus}_{i\in\mathbb{Z}}(\mathbb{C}_K T^i \xrightarrow{0} \mathbb{C}_K T^i).$$

Now it is easy to see that H^1 is free of rank 1 and H^i's are exterior products of H^1. □

5.3 Construction of the map ϕ^i

Recall that we reduce the degeneracy of Hodge-Tate spactral sequence to the statement that we have isomorphism

$$\phi^i : \Omega^i_{X/\mathbb{C}_K} \to R^i v_* \hat{\mathcal{O}}_X.$$

We choose a formal model $\mathfrak{X}$ over $\mathcal{O}_{\mathbb{C}_K}$ of X (a formal model is a formal scheme $\mathfrak{X}$ topologically of finite type over $\mathcal{O}_{\mathbb{C}_K}$ such that its general fibre is X. By a theorem of Raynaud we may always choose one since X is proper). Write $\mathfrak{X}_{\mathrm{aff}}$ for the category of affine opens in $\mathfrak{X}$ with the indiscrete topology (i.e. the only coverings are isomophisms). It is easy to check that then all presheaves are sheaves. Consider the morphisms of ringed sites

$$\mu : (X_{\text{pro-ét}}, \hat{\mathcal{O}}_X) \xrightarrow{v} (X_{\text{ét}}, \mathcal{O}_X) \xrightarrow{\pi} (\mathfrak{X}_{\mathrm{aff}}, \mathcal{O}_{\mathfrak{X}})$$

We construct a map

$$\varphi : \Omega^1_{\mathfrak{X}/\mathcal{O}_{C_K}} \to R^1 \mu_* \hat{\mathcal{O}}_X(1) \tag{1}$$

($\Omega^1_{\mathfrak{X}/\mathcal{O}_{\mathbb{C}_K}}$ denotes the Kähler differentials on the formal scheme $\mathfrak{X}$: if $\mathfrak{X} = \mathrm{Spf}(R)$ of topologically of finite type over R then consider the continuous Kähler differentials on R. This module is the p-adic completion of $\Omega^1_{R/\mathcal{O}_{\mathbb{C}_K}}$ of algebraic differentials).

In general if F is a right exact functor then $R^i(F \circ G) \cong F \circ R^i G$ for some other functors (this can be seen using δ-functors). Hence as pullbacks are right exact, if we construct (1) and take π^*, we have a map $\pi^*(\varphi)$

$$\pi^* \Omega^1_{\mathfrak{X}/\mathcal{O}_{\mathbb{C}_K}} \to R^1 v_* \hat{\mathcal{O}}_X(1).$$

As $\pi^* \Omega^1_{\mathfrak{X}/\mathcal{O}_{\mathbb{C}_K}} = \Omega^1_{X/\mathbb{C}_K}$, twisting $\pi^*(\varphi)$ gives ϕ^1. Taking exterior products and use the lemma above we get all ϕ^i's.

Before we construct φ, we remark Bhatt, Morrow, Scholze, integral p-adic Hodge theory 8.2 contains all the details so we will omit some coimputations. Consider morphisms of sheaves of rings on $X_{\text{pro-ét}}$

$$\mathbb{Z}_p \to \mathcal{O}_{\mathbb{C}_K} \to \hat{\mathcal{O}}_X^+.$$

We have the transitivity triangle

$$L_{\mathcal{O}_{\mathbb{C}_K}/\mathbb{Z}_p} \otimes_{\mathcal{O}_{\mathbb{C}_K}} \hat{\mathcal{O}}_X^+ \to L_{\hat{\mathcal{O}}_X^+/\mathbb{Z}_p} \to L_{\hat{\mathcal{O}}_X^+/\mathcal{O}_{\mathbb{C}_K}}.$$

After taking derived p-adic completion, RHS vanishes (Proposition from lecture 12). Therefore

$$L_{\mathcal{O}_{\mathbb{C}_K}/\mathbb{Z}_p} \widehat{\otimes}_{\mathcal{O}_{\mathbb{C}_K}} \hat{\mathcal{O}}_X^+ \simeq \widehat{L_{\hat{\mathcal{O}}_X^+/\mathbb{Z}_p}}.$$

By Fontaine's theorem, especially the proof, after inverting p

$$\Omega \otimes_{\mathcal{O}_{\mathbb{C}_K}} \hat{\mathcal{O}}_X^+[1][\frac{1}{p}] \simeq \hat{\mathcal{O}}_X(1)[1] \simeq \widehat{L_{\hat{\mathcal{O}}_X^+/\mathbb{Z}_p}}[\frac{1}{p}].$$

Taking pullback with respect to μ,

$$\widehat{L_{\mathcal{X}/\mathbb{Z}_p}} \to R\mu_* \widehat{L_{\hat{\mathcal{O}}_X^+/\mathbb{Z}_p}} \to R\mu^* \widehat{L_{\hat{\mathcal{O}}_X^+/\mathbb{Z}_p}}[\frac{1}{p}] \simeq R\mu_* \hat{\mathcal{O}}_X(1)[1].$$

Apply H^0 to get

$$H^0(\widehat{L_{\mathfrak{X}/\mathbb{Z}_p}}) \to R^1 \mu_* \hat{\mathcal{O}}_X(1).$$

If we manage to show $\Omega^1_{\mathfrak{X}/\mathcal{O}_{\mathbb{C}_K}} \cong H^0(\widehat{L_{\mathfrak{X}/\mathbb{Z}_p}})$ then we get the desired φ.

To show the isomorphism take

$$\mathbb{Z}_p \to \mathcal{O}_{\mathbb{C}_K} \to \mathcal{O}_{\mathfrak{X}}.$$

The transitivity triangle provides

$$L_{\mathcal{O}_{\mathbb{C}_K}/\mathbb{Z}_p} \otimes_{\mathcal{O}_{\mathbb{C}_K}} \mathcal{O}_{\mathfrak{X}} \to L_{\mathfrak{X}/\mathbb{Z}_p} \to L_{\mathfrak{X}/\mathcal{O}_{\mathbb{C}_K}}.$$

Applying derived p-adic completion and noting that LHS vanishes (by derived Nakayama) we get

$$H^0(\widehat{L_{\mathfrak{X}/\mathbb{Z}_p}}) \cong H^o(\widehat{L_{\mathfrak{X}/\mathcal{O}_{\mathbb{C}_K}}})$$

and as $H^0(L_{B/A}) \simeq \Omega^1_{B/A}$ and we take "p-adic completion", we know also that $\Omega^1_{\mathfrak{X}/\mathcal{O}_{\mathbb{C}_K}}$ is the p-adic completion of algebraic Kähler differentials. Thus the desired isomorphism follows. More details can of this step can be found in Gaber-Ramero: Almost ring theory.

Now we show that the maps ϕ^i's are all isomorphisms. Using the key lemma from lecture 12, it is enough to prove for $\phi^1 : \Omega^1_{X/\mathbb{C}_K}(-1) \to R^1 v^1_* \hat{\mathcal{O}}_X$. We note that both sides are coherent sheaves on $X_{\text{ét}}$ (properness and the fact that we have a rigid variety X). By smoothness $X \to \mathrm{Spa}(\mathbb{C}_K, \mathcal{O}_K)$ can be written as a composition of a rational subset of a finite étale cover of $\mathbb{T}^n = \mathrm{Spa}(\mathbb{C}_K\langle T_1^{\pm 1}, \ldots, T_n^{\pm 1}\rangle, \mathcal{O}_{\mathbb{C}_K}\langle T_1^{\pm 1}, \ldots, T_n^{\pm 1}\rangle)$. Using standard facts about higher direct images (composition and vanishing) and Grothendieck spectral sequence provides that we may assume that $X = \mathbb{T}^n$ and also we may take global sections. We may assume that $n = 1$ as both sides are compatible with taking fibre products of adic spaces. In other words, we have reduced the problem to showing

$$\phi^1(X) : \Omega^1_{X/\mathbb{C}_K}(-1) \to H^1(X_{\text{pro-ét}}, \hat{\mathcal{O}}_X)$$

is an isomorphism where $X = \mathrm{Spa}(\mathbb{C}_K\langle T^{\pm 1}\rangle, \mathcal{O}_{\mathbb{C}_K}\langle T^{\pm 1}\rangle)$.

By the key lemma in lecture 12 we know that both sides are free of rank 1 as $\mathcal{O}_X(X)$-modules. By Fontaine's result $\mathrm{d}\log T \in \Omega^1_{X/\mathbb{C}_K}$ is a generator. Then direct computation shows $\phi^1(\mathrm{d}\log T)$ is a generator for RHS, so ϕ^1 is an isomorphism.

6 Integral p-adic Hodge theory

Let $\mathfrak{X}$ be a smooth and proper formal scheme over $\mathcal{O}_{\mathbb{C}_K}$. Let X be the generic fibre and $\mathfrak{X}_k$ the special fibre ($k = \mathcal{O}_{\mathbb{C}_K}/\mathfrak{m}$). We have a degenerate Hodge-Tate spectral sequence

$$H^i(X, \Omega^j_{X/\mathbb{C}_K})(-j) \Rightarrow H^{i+j}_{\text{ét}}(X, \mathbb{C}_K)$$

which leads to

$$\dim_{\mathbb{Q}_p} H^n_{\text{ét}}(X, \mathbb{Q}_p) = \dim_{\mathbb{C}_K} H^n(X, \mathbb{C}_K) = \sum_{i+j=n} \dim_{\mathbb{C}_K} H^i(X, \Omega^j_{X/\mathbb{C}_K}).$$

This relates étale and Hodge cohomology for the generic fibre.

We may talk about the cohomology of the special fibre $H^i(\mathfrak{X}_k, \Omega^j_{\mathfrak{X}_k/k})$ and $H^n_{\text{ét}}(X, \mathbb{F}_p)$ and we may ask if the above has a good modulo p variant.

Theorem 6.1 (Bhatt, Morrow, Scholze, integral...). *One has inequalities*

$$\dim_{\mathbb{F}_p} H^n_{\text{ét}}(X, \mathbb{F}_p) \leq \dim_{i+j=n} \dim_k H^i(\mathfrak{X}_k, \Omega^j_{\mathfrak{X}_k/k}).$$

Example. This can be a strict inequality. Assume $p = 2$. Let S over $\mathcal{O}_{\mathbb{C}_K}$ be a proper smooth scheme with

$$\pi_1(S_{\mathbb{C}_K}) \xrightarrow{\cong} \pi_1(S) \xleftarrow{\cong} \pi_1(S_k) \cong \mathbb{Z}/2.$$

One may construct an Enriques surface with these properties: take the Enriques surface over $\mathbb{F}_2$ with the desired property. One may lift this to $\mathbb{Z}_2$ by a theorem of Ogus and Lang.

Let E be an elliptic curve over $\mathcal{O}_{\mathbb{C}_K}$ with ordinary reduction. Hence exists an injection $\mu_2 \hookrightarrow E$. Choosing the element -1, $\mu_2(\mathcal{O}_{\mathbb{C}_K})$ defines a morphism $\alpha : \mathbb{Z}/2\mathbb{Z} \to \mu_2 \subseteq E$ of group schemes over $\mathcal{O}_{\mathbb{C}_K}$. There is a $\mathbb{Z}/2\mathbb{Z}$-cover $\tilde{S} \to S$ given by the intersection of three quadratics in $\mathbb{P}^5_{\mathbb{F}_2}$ which admits a free $\mathbb{Z}/2\mathbb{Z}$-action, then $S \cong \tilde{S}/\mathbb{Z}/2\mathbb{Z}$.

Consider the E-torsor (along α) $Y = (\tilde{S} \times_S E)/\mathbb{Z}/2\mathbb{Z}$ (there is a canonical $\mathbb{Z}/2\mathbb{Z}$-action $(j \cdot s, \alpha(j) \cdot e)$. Take the quotient scheme Y, this admits an E-action). $Y_k \to S_k$ is the split torsor $E_k \times_k \tilde{S}_k \to S_k$. Passing to the special fibre, α_k is the zero map over k. One may show that $\dim_{\mathbb{F}_p} H^1_{\text{ét}}(Y_{\mathbb{C}_K}, \mathbb{F}_2) = 2$.

For the Hodge side, by Künneth formula and the observation above for $Y_k \to S_k$

$$h^{0,1}(Y_k) \cong h^{0,1}(S_k) + h^{0,1}(E_k)$$
$$h^{1,0}(Y_k) \cong h^{1,0}(S_k) + h^{1,0}(E_k)$$

Standard facts about elliptic curves show $h^{1,0}(E_k) = h^{0,1}(E_k) = 1$. Alao as $\pi_1(S_k) \cong \mathbb{F}_2$, Artin-Schreier sequence shows $H^1_{\text{ét}}(S_k, \mathcal{O}_{\S_k}) \neq 0$ so RHS has dimension at least 3.

Before we explain the proof of the theorem, we do a quick recap on crystalline cohomology. It is well-known that (in characteristic p) ℓ-adic cohomology ($\ell \neq p$) is a good cohomology theory (it is a Weil cohomology theory) and if X_0 over

$\mathbb{F}_p$ comes from a smooth proper scheme X over R where $R \subseteq \mathbb{C}$, there is a comparison theorem

$$H^*_{\text{ét}}(X_{\overline{k}}, \mathbb{Z}_\ell) = H^*_{\text{sing}}(X^{\text{an}}, \mathbb{Z}) \otimes \mathbb{Z}_\ell.$$

Moreover this does not kill ℓ-torsion of RHS so ℓ-torsion can be recovered from étale cohomology.

However when $\ell = p$ almost none of the above is true so we need a good p-adic cohomology theory. For X smooth proper over $\mathbb{C}$ there exists an isomorphism

$$H^*_{\text{sing}}(X^{\text{an}}, \mathbb{C}) = H^*(X, \Omega^\bullet_{X/\mathbb{C}})$$

by Poincaré lemma and GAGA. The idea know is for X over k of characteristic p, we lift it to $\mathbb{Z}_p$ and consider de Rham cohomology. Questions abound in this process: does X admits a lift $\tilde{X}$ (in general no)? If so is the cohomology independent of the lift? Can the cohomology be defined in a canonical way without referring to $\tilde{X}$?

Inspiration: in characteristic 0, i.e. for X over $\mathbb{C}$, Grothendieck's idea was to consider all possible infinitesimal local liftings.

Definition. For X over $\mathbb{C}$, let $X/\mathbb{C}_{\text{inf}}$ be the site whose underlying category has as objects pairs (U, T) where $U \subseteq X$ open subset and $U \to T$ is a closed nilpotent immersion (i.e. the ideal sheaf is nilpotent). The coverings are $\{(U_i, T_i)\}$ such that $\{T_i\}$ cover T.

A sheaf is a collection $\{\mathcal{F}_T\}$ of Zariski sheaves for each (U, T). Given $f : (U, T) \to (U', T')$, if $f^*\mathcal{F}_T \to \mathcal{F}_{T'}$ is an isomorphism we call $\{\mathcal{F}_T\}$ a *crystal*.

Theorem 6.2. $H^*_{inf}(X) \cong H^*_{sing}(X, \mathbb{C}) \cong H^*(X, \Omega^\bullet_{X/\mathbb{C}})$.

More generally when $X \to Y$ is a closed immersion such that Y is smooth, we have

$$H^*_{\text{inf}}(X) \cong H^*(\hat{Y}, \Omega^\bullet_{\hat{Y}/\mathbb{C}})$$

where $\hat{Y}$ is the formal completion of Y along X.

Applying this to characteristic p we see the following.

Example (example where Poincaré lemma fails)**.** Let $X = \operatorname{Spec} k$. If $k = \mathbb{C}$ and we consider the closed immersion $X \to \mathbb{A}^1_{\mathbb{C}} = Y, \mathbb{C}[x] \to \mathbb{C}[x]/(x)$. By vanishing of higher cohomology for affine schemes, we have that de Rham cohomology of the formal completion can be computed by the complex

$$0 \longrightarrow \mathbb{C}[[x]] \xrightarrow{\mathrm{d}} \mathbb{C}[[x]]\mathrm{d}x \longrightarrow 0$$

Since each $x^n \mathrm{d}x$ can be integrated to $\frac{x^{n-1}}{n}$, we know that $H^0_{\text{dR}}(\hat{Y}) = \mathbb{C}$ and higher cohomology vanishes. This is exactly the same as $H^*_{\text{dR}}(\hat{Y}/\mathbb{C})$ where $\hat{Y}/\mathbb{C}$ is the trivial closed immersion $X \to Y = \operatorname{Spec} \mathbb{C}$.

Now in characteristic p, e.g. $k = \mathbb{F}_p$, the formal completion of $Y = \mathbb{A}^1_{\mathbb{Z}_p}$ along $X \to Y$ is $\operatorname{Spf}(\mathbb{Z}_p\langle t \rangle)$ and de Rham cohomology is computed by

$$0 \longrightarrow \mathbb{Z}_p\langle t \rangle \xrightarrow{\mathrm{d}} \mathbb{Z}_p\langle t \rangle \mathrm{d}t \longrightarrow 0$$

However there is no element in $\mathbb{Z}_p\langle t\rangle$ whose derivative is $t^{p-1}\mathrm{d}t$. So we need eleemnts of the form $\frac{t^n}{n!}$ (n-th infinitesimal lift of t). We add those ot $\mathbb{Z}_p\langle t\rangle$. Then d becomes surjective. So we added elements $\frac{a^n}{n!}$ where a comes from the defining ideal of an infinitesimal object. Looking at object with a PD (power divided) structure with respect to the definie ideal. The same idea give crystalline site and crystalline cohomology.

Breul-Kissing twist We have defined p-adic Tate module $\Omega = T_p(\Omega^1_{\mathcal{O}_{\mathbb{C}_K}/\mathbb{Z}_p})$ and showed that $\widehat{L_{\mathcal{O}_{\mathbb{C}_K}/\mathbb{Z}_p}}[-1] \cong \Omega$. For a $\mathcal{O}_{\mathbb{C}_K}$-module M, define $M\{i\} := M \otimes_{\mathcal{O}_{\mathbb{C}_K}} \Omega^{\otimes i}$. It is related to the Tate twists via an inclusion $M(i) \subseteq M\{i\}$ with torsion cokernel. More generally we may replace $\mathcal{O}_{\mathbb{C}_K}$ by any integral perfectoid ring (by "formal étale nature of A_{inf} and differential forms" section).

Recall that $A_{\mathrm{inf}} := A_{\mathrm{inf}}(\mathcal{O}_{\mathbb{C}_K}) = W(\mathcal{O}^\flat_{\mathbb{C}_K})$ with $\theta : A_{\mathrm{inf}} \to \mathcal{O}_{\mathbb{C}_K}$.

- We have an automorphism $\phi : A_{\mathrm{inf}} \to A_{\mathrm{inf}}$ called the Frobenius on $\mathcal{O}^\flat_{\mathbb{C}_K}$. Write $\tilde{\theta} = \theta \circ \phi^{-1} : A_{\mathrm{inf}} \to \mathcal{O}_{\mathbb{C}_K}$.
- By functoriality $\mathcal{O}^\flat_{\mathbb{C}_K} \to \mathbb{C}^\flat_K$ (fraction field) induces a map $A_{\mathrm{inf}} \to W(\mathbb{C}^\flat_K)$.
- Also $\mathcal{O}^\flat_{\mathbb{C}_K} \to k$ induces $A_{\mathrm{inf}} \to W(k)$.
- We have a canonical map $A_{\mathrm{inf}} \to \mathcal{O}^\flat_{\mathbb{C}_K}$.

Theorem 6.3. *There exists a perfect A_{inf}-complex (i.e. quasi-isomorphic to a bounded complex of finite projective A_{inf}-modules) in a functorial way, denoted by $R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})$, where $\mathfrak{X}$ is a smooth proper smooth scheme over $\mathcal{O}_{\mathbb{C}_K}$, together with ϕ-semilinear endomorphism φ, that is an isomorphism outside the divisor $\tilde{\theta} : \mathrm{Spec}(\mathcal{O}_{\mathbb{C}_K}) \to \mathrm{Spec}(A_{\mathrm{inf}})$. Moreover we have the following comparisons:*

1. *étale cohomology: there exists a canonical isomorphism*
$$R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes_{A_{\mathrm{inf}}} W(\mathbb{C}^\flat_K) \simeq R\Gamma(X_{\text{ét}}, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} W(\mathbb{C}^\flat_K)$$
that is ϕ-equivariant.

   In fact, considering $\mu \in A_{\mathrm{inf}}$ where $\mu = [\varepsilon] - 1 \in \ker\theta$, one may show that this quasi-isomorphism exists over $A_{\mathrm{inf}}[\frac{1}{\mu}]$.
2. *de Rham cohomology: we have a canonical isomorphism*
$$R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X}) \otimes^L_{A_{\mathrm{inf}},\theta} \mathcal{O}_{\mathbb{C}_K} \simeq R\Gamma_{\mathrm{dR}}(\mathfrak{X}/\mathcal{O}_{\mathcal{O}_{\mathbb{C}_K}}).$$
3. *Hodge-Tate spectral sequence: there exists a spectral sequence*
$$H^i(\mathfrak{X}, \Omega^i_{\mathfrak{X}/\mathcal{O}_{\mathbb{C}_K}})\{-j\} \Rightarrow H^{i+j}(\tilde{\theta}^* R\Gamma_{A_{\mathrm{inf}}}(\mathfrak{X})).$$

4. *crystalline cohomology of the special fibre: there exists a ϕ-equivariant isomorphism*

$$R\Gamma_{A_{\inf}}(\mathfrak{X}) \otimes^{L}_{A_{\inf}} W(k) \simeq R\Gamma_{\mathrm{crys}}(\mathfrak{X}_k/W(k)).$$

Remark. Properness of $\mathfrak{X}$ is only used in statement 1.

Consequences of the theorem:

1. We can recover the Hodge-Tate spectral sequence from lecture 1. Use 3 and 1 toghether with the first statement so we may base change along $A_{\inf} \xrightarrow{\tilde{\theta}} \mathcal{O}_{\mathbb{C}_K} \subseteq \mathbb{C}_K$.
2. We can recover the inequality at the beginning of this chapter. Consider the perfect complex

$$K := R\Gamma_{A_{\inf}}(\mathfrak{X}) \otimes_{A_{\inf}} \mathcal{O}^{\flat}_{\mathbb{C}_K}.$$

By 1 we have

$$K \otimes \mathbb{C}^{\flat}_K \simeq R\Gamma(X_{\text{ét}}, \mathbb{F}_p) \otimes \mathbb{C}^{\flat}_K.$$

By 2 we have

$$K \otimes k \simeq R\Gamma_{\mathrm{dR}}(\mathfrak{X}_k/k).$$

By upper-semicontinuity of the ranks of cohomology groups of perfect complex, we have that

$$\dim_{\mathbb{F}_p} H^n_{\text{ét}}(X, \mathbb{F}_p) \leq \dim_k H^n_{\mathrm{dR}}(\mathfrak{X}_k/k).$$

Using the Hodge-to-de Rham spectral sequence,

$$\dim_k H^n_{\mathrm{dR}}(\mathfrak{X}_k/K) \leq \sum_{i+j=n} \dim_k H^i(\mathfrak{X}_k, \Omega^j_{\mathfrak{X}_k/k}).$$

3. We can recover crystalline cohomology. Assume that $H^i_{\mathrm{crys}}(\mathfrak{X}_k), H^{i+1}_{\mathrm{crys}}(\mathfrak{X}_k)$ are p-torsion free. Then $H^i_{\mathrm{crys}}(\mathfrak{X}_k)$ can be recovered functorially from the generic fibre X. More precisely the $\mathbb{Z}_p$-module $H^i_{\text{ét}}(X, \mathbb{Z}_p)$ equipped with the de Rham comparison theorem functorially recovers $H^i_{\mathrm{crys}}(\mathfrak{X}_k)$.

Let B^+_{dR} denote the ring which is given by the completion of $A_{\inf}[\frac{1}{p}]$ by $\xi = \frac{[\varepsilon]-1}{[\varepsilon]^{1/p}-1} = \phi^{-1}(\mu)$. One may show that ξ is a non-zero divisor and generates $\ker \theta$. Fact: B^+_{dR} is a complete DVR with residue field $\mathbb{C}_K$ and uniformiser ξ. Take the fraction field $B_{\mathrm{dR}} = B^+_{\mathrm{dR}}[\frac{1}{\xi}]$.

Theorem 6.4. *There is a canonical identification*

$$H^i_{\mathrm{dR}}(X) \otimes_K B_{\mathrm{dR}} \cong H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes_{\mathbb{Z}_p} B_{\mathrm{dR}}.$$

Using the assumptions on the cohomology groups one can show that $H^i_{\text{ét}}(X, \mathbb{Z}_p)$ is finite free and $H^i_{\mathrm{dR}} \otimes B^+_{\mathrm{dR}}$ is a B^+_{dR}-lattice in $H^i_{\text{ét}}(X, \mathbb{Z}_p) \otimes B_{\mathrm{dR}}$. By a theorem of Fargue, we have an equivalence of categories

$$\{\text{Breul-Kissing-Fargue modules}\} \leftrightarrow \{(T, \Sigma) : T \text{finite free} \mathbb{Z}_p\text{-module}, \Sigma \subseteq T \otimes B_{\mathrm{dR}} \text{ is a } B^+_{\mathrm{dR}}\text{-lattice}\}$$

where a Breul-Kissing-Fargue module is a finitely presented A_{inf}-module M with a ϕ-equivariant isomorphism $\varphi : M[\frac{1}{\varepsilon}] \cong M[\frac{1}{\phi(\varepsilon)}$ such that $M[\frac{1}{p}]$ is finite free over $A_{\mathrm{inf}}[\frac{1}{p}]$.

One can show that $H^i_{A_{\mathrm{inf}}}(\mathfrak{X})$ is a BKF-module. By 1 and 4 if we base change (M, φ) (coming from $(H^i_{\text{ét}}(X, \mathbb{Z}_p, H^i_{\mathrm{dR}}(X) \otimes B^+_{\mathrm{dR}})$ by Fargue) by $W(k)$ we get $(H^i_{\mathrm{crys}}(\mathfrak{X}_k), \varphi)$.

A Review of derived categories

Let $\mathbf{A}$ be an abelian category.

Definition (homotopy category)**.** The *homotopy category* $\mathbf{K}(\mathbf{A})$ is the category whose objects are complexes in $\mathbf{A}$ and whose morphisms are

$$\mathrm{Hom}_{\mathbf{Ch}(\mathbf{A})}(X^\bullet, Y^\bullet)/\sim$$

where $\sim$ is homotopy of complexes. Similarly we define $\mathbf{K}^+(\mathbf{A}), \mathbf{K}^-(\mathbf{A}), \mathbf{K}^b(\mathbf{A})$ to be the full subcategories of bounded below, bounded above and bounded complexes.

Recall that $f : X^\bullet \to Y^\bullet$ is a *quasi-isomorphism* if the induced maps $H^i(f) : H^i(X^\bullet) \to H^i(Y^\bullet)$ are isomorphisms. The *derived category* $\mathbf{D}(\mathbf{A})$ is obtained by localising $\mathbf{K}(\mathbf{A})$ at quasi-isomorphism, i.e. formally inverses all quasi-isomorphisms. Similar for $\mathbf{D}^+(\mathbf{A})$ etc.

The objects of $\mathbf{D}(\mathbf{A})$ are the same as in $\mathbf{K}(\mathbf{A})$. The morphisms are equivalence classes of diagrams

$$\begin{array}{ccccc} & & C^\bullet & & \\ & \swarrow^{s} & & \searrow^{f} & \\ A^\bullet & & & & B^\bullet \end{array}$$

where s is a quai-isomorphism and f is a morphism, written fs^{-1}. The equivalence relation is for C and C' exists a dominant morphisms $C \xleftarrow{u} D \to C'$ such that $s \circ u$ is a quasi-isomorphism.

$\mathbf{D}(\mathbf{A})$ has an analogue of short exact sequences. For $f : X^\bullet \to Y^\bullet$ a morphism of complexes, its *mapping cone* is the complex $M_f^\bullet$ where

$$M_f^n = X^{n+1} \oplus Y^n, d^n : (x^{n+1}, y^n) \mapsto (-d^{k+1}(x^{n+1}), f(x^{n+1}) + d_n(y^n)).$$

Definition (distinguished triangle)**.** A *distinguished triangle* in $\mathbf{K}(\mathbf{A})$ (resp. $\mathbf{D}(\mathbf{A})$) is a sequence of morphisms (the top row in the diagram below)

$$\begin{array}{ccccccc} X^\bullet & \longrightarrow & Y^\bullet & \longrightarrow & Z^\bullet & \longrightarrow & X^\bullet[1] \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ X'^\bullet & \longrightarrow & Y'^\bullet & \longrightarrow & M_f^\bullet & \longrightarrow & X'^\bullet[1] \end{array}$$

which is isomorphic to a triangle of the form as the lower triangle. In this case we have a long exact sequence

$$\cdots \longrightarrow H^i(X) \longrightarrow H^i(Y) \longrightarrow H^i(Z) \longrightarrow H^{i+1}(X) \longrightarrow \cdots$$

Given a functor $F : \mathbf{A} \to \mathbf{B}$, how can we get an induced functor between the derived categories?

- If F is exact then F naturally lifts to a functor $F : \mathbf{D}(\mathbf{A}) \to \mathbf{D}(\mathbf{B})$ by applying F to each element of a complex. This preserves distinguished triangles.

- If F is only right-exact and $\mathbf{A}$ has enough projectives then any $X^\bullet \in \mathbf{D}^-(\mathbf{A})$ is isomorphic to a complex of projectives $P^\bullet$ in $\mathbf{D}(\mathbf{A})$ and we check that
$$\mathbb{L}F(X^\bullet) = \mathbb{L}F(P^\bullet) = F(P^\bullet)$$
is well-defined. This gives the *left derived functor*
$$\mathbb{L}F : \mathbf{D}^-(\mathbf{A}) \to \mathbf{D}^-(\mathbf{B}).$$

- If F is left-exact, we do the same with injective resolutions to get $\mathbb{R}F : \mathbf{D}^+(\mathbf{A}) \to \mathbf{D}^+(\mathbf{B})$, the *right derived functor*.

B Review of simplicial homotopy

Let $\mathbf{C}$ be a category with finite coproducts. Given $X_\bullet \in \mathbf{Simp}(\mathbf{C})$ and a simplicial object $U_\bullet$ in the category of non-empty finite sets, we define the product $X_\bullet \times U_\bullet$ as a simplicial object in $\mathbf{C}$ with terms

$$(X_\bullet \times U_\bullet)_n = \coprod_{u \in U_n} X_n$$

and for a map $j : [m] \to [n]$, the morphism $(X_\bullet \times U_\bullet)(j)$ maps the component X_n indexed by $u \in U_n$ to the component X_m indexed by $U_\bullet(j)(u) \in U_m$ via the morphism $X_\bullet(j)$.

In particular we may talk about $X_\bullet \times \Delta[n]_\bullet$ where $\Delta[n]_\bullet$ is the simplicial set such that $\Delta[n]_m = \mathrm{Hom}_\Delta([m], [n])$ and the simplicial structure is induced by the contravariant property of the Hom functor. Note that $\varepsilon_0, \varepsilon_1 : [0] \to [1]$ induce morphisms $e_0, e_1 : X_\bullet \cong X_\bullet \times \Delta[0]_\bullet \to X_\bullet \times \Delta[1]_\bullet$.

Definition (simplicial homotopy)**.** Assume that $\mathbf{C}$ is as above and consider $f_\bullet, g_\bullet : X_\bullet \to Y_\bullet$ in $\mathbf{Simp}(\mathbf{C})$. A *simplicial homotopy* from $f_\bullet$ to $g_\bullet$ is a morphism $h_\bullet : X_\bullet \times \Delta[1]_\bullet \to Y_\bullet$ satisfying $f_\bullet = h_\bullet \circ e_0, g_\bullet = h_\bullet \circ e_1$. If such a $h_\bullet$ exists we say $f_\bullet$ is *homotopic* to $g_\bullet$.

Given $f_\bullet : X_\bullet \to Y_\bullet, g_\bullet : Y_\bullet \to X_\bullet$ such that $f_\bullet \circ g_\bullet \simeq \mathrm{id}_{Y_\bullet}, g_\bullet \circ f_\bullet \simeq \mathrm{id}_{X_\bullet}$, we say that $X_\bullet$ is *homotopy equivalent* to $Y_\bullet$.

It is a fact that homotopy equivalence induces quasi-isomorphisms between the associated chain complexes (part of Dold-Kan correspondence).

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