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Mathematics Tripos

Part III

Complex Dynamics

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0 Introduction

What is complex dynamics? Iteration of holomorphic self-maps of Riemann surfaces

long term behaviour under iteration

origin: iterative root finding algorithm, e.g. Newton's method. When and why does this work? Algebraically, the problem is to ask the convergence of the iteration of the map

$$f(z) = z - \frac{p(z)}{p'(z)} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$$

where $\hat{\mathbb{C}}$ is the Riemann sphere.

Goals:

- equidistribution theorem

> **Theorem 0.1** (Friere-Lopez-Mañe, 1983). *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ with degree $d \geq 2$ holomorphic. Then there exists a unique f-invariant probability measure μ_f supported on the unstable locus of f, such that for almost all $\alpha \in \hat{\mathbb{C}}$, $\frac{1}{d^n} \sum_{f^n(z)=\alpha} \delta_z \to \mu_f$ in weak-* topology.*

- universality of the Mandelbrot set: fix $d \geq 2$, define for $c \in \mathbb{C}$, $f_c(z) : z^d + c$. The d-Mandelbrot set is

$$M_d = \{c \in \mathbb{C} : |f_c^n(0)| \nrightarrow \infty\}.$$

> **Theorem 0.2** (McMullen 1997). *The Mandelbrot set is universal for bifurcations, i.e. in any bifurcation locus we see (slightly distorted) copies of some M_d.*

1 Riemann surfaces

Recall at the end of IID Riemann Surfaces

Theorem 1.1 (uniformisation). *Every Riemann surface R is conformally isomorphic to $\tilde{R}/G$ where $\tilde{R}$ is one of the three simply connected Riemann surfaces ($\hat{\mathbb{C}}, \mathbb{C}, \mathbb{D}$) and $G \subseteq \operatorname{Aut}(\tilde{R})$ acts freely and properly discontinuously.*

Three cases:

1. $\hat{\mathbb{C}}$: as $\operatorname{Aut}(\hat{\mathbb{C}})$ are precisely the Möbius transformations and all Möbius transformations have fixed points, $G = 1$ so $R = \hat{\mathbb{C}}$.
2. $\mathbb{C}$: $\operatorname{Aut}(\mathbb{C}) = \{p(z) = az + b, a \neq 0\}$ so G only contains translations, so can be identified by a subgroup of $\mathbb{C}$. Can show that G is one of $\{0\}, \mathbb{Z}\omega_1$ or $\Lambda = \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2$, a lattice. Then R is one of $\mathbb{C}, \mathbb{C}^*$ or $\mathbb{C}/\Lambda$, a complex torus.
3. $\mathbb{D}$: $\operatorname{Aut}(\mathbb{D}) = \{\lambda \cdot \frac{z-a}{1-\bar{a}z} : \lambda \in S^1, a \in \mathbb{D}\}$, which has a lot of elements with no fixed point. This is called *hyperbolic*.

Recall $\mathbb{D}$ is equipped with metric $ds = \frac{2|dz|}{1-|z|^2}$, i.e. the distance between $x, y \in \mathbb{D}$ is

$$\rho(x,y) = \inf_\gamma \int_a^b \frac{2|\gamma'(t)|}{1-|\gamma(t)|^2} dt$$

where $\gamma : [a,b] \to \mathbb{D}$ is a smooth curve from x to y. One can use the origin to show that

$$\rho(x,y) = \frac{\log(1+R)}{\log(1-R)}$$

where $R = |\frac{y-x}{1-\bar{x}y}|$. Elements of $\operatorname{Aut}(\mathbb{D})$ are isometries of this metric so it descends to a hyperbolic metric on any hyperbolic Riemann surface, such that the covering map $\tilde{R} \to R$ is a local isometry.

Exercise. Let $R = \mathbb{D}/G$, $f : R \to R$ holomorphic. Then f lifts to a holomorphic $\tilde{f} : \mathbb{D} \to \mathbb{D}$, unique up to a composition with an element of G, and induces a group homomorphism $\gamma \mapsto \gamma'$ such that $\tilde{f} \circ \gamma = \gamma' \circ \tilde{f}$.

Theorem 1.2 (Pick). *Let $f : S \to T$ be hyperbolic Riemann surfaces with Poincaré metric ρ_S, ρ_T respectively. Then for all $x, y \in S$,*

$$\rho_T(f(x), f(y)) \leq \rho_S(x,y).$$

Proof. By lifting (of $f \circ \pi_S : \mathbb{D} \to T$) it suffices to show this for $S = T = \mathbb{D}$. By computation this is the same as showing

$$\frac{\log(1+R')}{\log(1-R')} \leq \frac{\log(1+R)}{\log(1-R)}$$

where $R = |\frac{y-x}{1-\bar{y}x}|, R' = |\frac{f(y)-f(x)}{1-\overline{f(y)}f(x)}|$. Note $\frac{\log(1+x)}{\log(1-x)}$ is strictly increasing, so suffices to show $R' \leq R$. Let $\mu_1(z) = \frac{y-z}{1-\bar{y}z}, \mu_2(z) = \frac{f(y)-z}{1-\overline{f(y)}z}$. Let $g = \mu_2 \circ f \circ \mu_1^{-1} : \mathbb{D} \to \mathbb{D}$. By Schwarz's lemma $|g(z)| \leq |z|$, i.e. $|\mu_2(f(w))| \leq |\mu_1(w)|$, so $R' \leq R$. ∎

Remark. It follows from the "strict inequality" bit of Schwarz's lemma that exists x, y such that $\rho_T(f(x), f(y)) = \rho_S(x, y)$ if and only if f lifts to a disk automorphism.

Example. Contracting holomorphic maps is a very strong requirement. Compre to, for example, $f(z) = z + 1$ on $\hat{\mathbb{C}}$ (drawing of different behaviour on two hemispheres).

Case of $\hat{\mathbb{C}}$:

Proposition 1.3. *Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic nonconstant map. Then f is a rational map, i.e. exists $a_1, \dots, a_m, b_1, \dots, b_n, c \in \mathbb{C}$ such that*

$$f(z) = c \cdot \frac{(z - a_1) \cdots (z - a_m)}{(z - b_1) \cdots (z - b_n)}.$$

Proof. wlog $f(\infty) \in \mathbb{C}$ (if not, replace with $\frac{1}{f}$). Then exist a finite collection $f^{-1}(\infty) = \{b_1, \dots, b_n\} \subseteq \mathbb{C}$. About any b_i have locally

$$f(z) = \sum_{j=-k}^{\infty} a_{ij}(z - b_i)^j.$$

Set $Q_i = \sum_{j=-k}^{-1} a_{ij}(z - b_i)^j$. Then $g - f - Q_1 - \cdots - Q_n$ has no pole so must be constant. □

Universal cover $\mathbb{C}$: just a remark. Yes there are interesting dynamics. For example $z \mapsto e^z$ belongs to the realm of transcendental dynamics. $\mathbb{C}/\Lambda$ also admits nonconstant holomorphic maps. See example sheet 1.

stable and unstable locus Motivating example: $z \mapsto z^2$ on $\hat{\mathbb{C}}$. We can restrict f to the unit disk and we see $f|_{\mathbb{D}}^n \to 0$ uniformly on compact subsets of $\mathbb{D}$. Similarly on $\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$, $f|_{\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}}^n \to \infty$. On the other hand for $|z_0| = 1$, there is no neighbourhoood $z_0 \in U$ such that $f|_U^n(z)$ converging to a holomorphic function, as such a limit would have a discontinuity.

Definition (locally uniform convergence/divergence)**.** Let S, T be metric spaces, $f_n : S \to T$ a sequence of continuous maps. We say (f_n) *converges locally uniformly* if for all compact $K \subseteq S$, for all $\varepsilon > 0$ exists $N \in \mathbb{N}$ such that for all $m, n > N$, $\sup_{x \in K} d_T(f_m(x), f_n(x)) < \varepsilon$.

We say (f_n) *diverges locally uniformly* if for all compact $K \subseteq S$ and all compact $K' \subseteq T$, exists $N \in \mathbb{N}$ such that for all $n > N$, $f_n(K) \cap K' = \emptyset$.

Recall that if holomorphic f_n's converge locally uniformly on S then it has a holomorphic limit function.

Remark. If T is compact then we never have locally uniform divergence.

Definition (normal family)**.** We say a family $\mathcal{F} = \{f : S \to T\}$ of continuous functions is *normal* if every sequence $(f_n) \subseteq \mathcal{F}$ has a subsequence which either converges locally uniformly or diverges locally uniformly.

Exercise.

1. Show normality depends only on the topology, not the metric, of the spaces.
2. Normality is local: if $\mathcal{F}$ is a family of continuous maps, $S = \bigcup_\alpha U_\alpha$, then if $\mathcal{F}|_{U_\alpha}$ is normal for all α then $\mathcal{F}$ is normal for S.

A word of caution: in some texts (such as Ahlfors) the definition of normality excludes divergence.

Definition (equicontinuity)**.** A family $\mathcal{F}$ of continuous maps on a domain $U \subseteq \mathbb{C}$ with values in a metric space T is *equicontinuous* if for all $\varepsilon > 0$, exists $\delta > 0$ such that for all $z, w \in U$ and for all $f \in \mathcal{F}$, if $|z - w| < \delta$ then $d_T(f(z), f(w)) < \varepsilon$.

Theorem 1.4 (Arzela-Ascoli)**.** *Let $\mathcal{F}$ be a family of continuous maps $U \to T$ with $U \subseteq \mathbb{C}$ a domain and T a metric space. Then $\mathcal{F}$ has the property that any sequence has a locally uniformly convergent subsequence if and only if*

1. *$\mathcal{F}$ is equicontinuous on every compact $K \subseteq U$,*
2. *for every $z \in U$, $\{f(z) : f \in \mathcal{F}\}$ lies in a compact subset of T.*

Corollary 1.5. *If T is compact then $\mathcal{F}$ is normal if and only if it is equicontinuous on compact subsets of U.*

Proof. Only if is easy. For if, we use separability of $\mathbb{C}$. Let $\{z_k\}$ be a countable dense subset of U and fix $\{f_n\} \subseteq \mathcal{F}$. Can find a set of indices $n_{11} < n_{12} < \cdots$ such that $f_{n_{1i}}(z_1)$ converges, and a subsequence of these $n_{21} < n_{22} < \cdots$ such that $f_{n_{2i}}(z_2)$ converges. Let $g_k = f_{n_{kk}}$. Then for all z_i the limit $\lim_{k\to\infty} g_k(z_i)$ exists in T. Now given $K \subseteq U$ compact, by equicontinuity for any $\varepsilon > 0$ exists $\delta > 0$ such that for all $z, w \in U$ with $|z - w| < \delta$, we have for all $f \in \mathcal{F}$, $d_T(f(z), f(w)) < \varepsilon$. Cover K by δ-balls, extract a finite subcover, and choose some z_i in each, say $z_1, \dots, z_\ell$. For each z_i, $1 \le i \le \ell$, exists N_i such that for all $n, m \ge N_i$, $d_T(g_n(z_i), g_m(z_i)) < \varepsilon$. Let $N = \max_i N_i$. Then for all $z \in K$,

$$
\begin{aligned}
d_T(g_n(z), g_m(z)) &\le d_T(g_n(z), g_n(z_i)) + d_T(g_n(z_i), g_m(z_i)) + d_T(g_n(z_i), g_m(z)) \\
&< 3\varepsilon
\end{aligned}
$$

□

Theorem 1.6 (Montel)**.** *Suppose S, T are Riemann surfaces and T is hyperbolic. Then all families of holomorphic maps $S \to T$ are normal.*

Proof. If S is not hyperbolic, lifting plus Liouville's theorem imply that all maps are constant. Given a family $\mathcal{F}$ of constant maps, let $\{f_n\} \subseteq \mathcal{F}$ be a sequence. Then if $\{f_n(S)\}$ lies in a compact set in T, then exists a convergent subsequence; if not, exists a subsequence which leaves any compact set, so diverges locally uniformly.

Suppose S is hyperbolic, $\{f_n\} \subseteq \mathcal{F}$. If exists $x \in S$ such that $\{f_n(x)\}$ lies in a compact subset of T, the same is true for all $y \in S$. Pick's therom implies equicontinuity so by Arzela-Ascoli $\{f_n\}$ has a convergent subsequence. Otherwise fix $x \in S$ and $y \in T$ and exists a subsequence $\{f_{n_k}\}$ such that $d_T(f_{n_k}(x), y) \to \infty$. Given $K \subseteq S, K' \subseteq T$ compact, by Pick's theorem $d_T(f_{n_k}(z), y) \to \infty$ for all $z \in K$ and so $f_{n_k}(K) \cap K' = \emptyset$ for $k \gg 1$. Thus the sequence diverges locally uniformly. $\square$

Example. If $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational map and $D \subseteq \hat{\mathbb{C}}$, then the family of iterates $\mathcal{F} = \{f^n\}_{n\in\mathbb{N}}$ is normal if $\bigcup_{n\in\mathbb{N}} f^n(D)$ omits 3 or more points of $\hat{\mathbb{C}}$, as any domain of $\hat{\mathbb{C}}$ with complement of cardinality ≥ 3 is hyperbolic.

Definition (proper map)**.** Suppose U, V are open subsets of Riemann surfaces and $f : U \to V$. f is proper if for every $K \subseteq V$ compact, $f^{-1}(K)$ is compact in U.

Proper maps have well-defined degrees and satisfies the Riemann-Hurwitz formula. The proofs are similar to the compact case.

Proposition 1.7. *Suppose U, V are open in $\hat{\mathbb{C}}$. If $f : U \to V$ is proper holomorphic nonconstant then f has a well-defined degree, i.e. for all $x \in V$, $|f^{-1}(x)|$ is independent of x, counting multiplicity.*

Theorem 1.8 (Riemann-Hurwitz)**.** *With same assumptions as above, the Euler characteristic $\chi(U)$, $\chi(V)$ satisfy*

$$\chi(U) = (\deg f)\chi(V) - \sum_{p\in U}(e_p - 1)$$

where e_p is the local degree/ramification index of f at p.

Note that one strategy is to observe that for $U \subseteq \mathbb{C}$, we can cover by δ-grid, let $\delta \to 0$, we obtain a covering of U by open balls with compact closures and smooth boundaries.

Corollary 1.9. *If $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is holomorphic of degree d, then f has $2d - 2$ critical points, counting multiplicity.*

Corollary 1.10. *If $\chi(U) = \chi(V) = 0$ then $f : U \to V$ proper holomorphic nonconstant is unramified.*

Definition (Fatou set, Julia set)**.** Let $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ be a holomorphic nonconstant map. The *Fatou set* of f is

$$F(f) = \{z \in \hat{\mathbb{C}} : \text{ on some nbhd } z \in U, f^n \text{ forms a normal family}\}.$$

The *Julia set* of f is $J(f) = \hat{\mathbb{C}} \setminus F(f)$.

Example. For $z \mapsto z^2$, we have normality on $(\hat{\mathbb{C}} \setminus \overline{\mathbb{D}}) \cup \mathbb{D}$, and normality fails on any open neighbourhood which intersects S^1. Thus $J(z^2) = S^1$. This example might be slightly misleading, as we'll see that the Julia set of a rational map is almost never smooth.

Lemma 1.11. *$F(f)$ and $J(f)$ are totally f-invariant, i.e. $f^{-1}(F(f)) = F(f), f^{-1}(J(f)) = J(f)$.*

Proof. Suffices to show that if $U \subseteq \hat{\mathbb{C}}$ then $\{f^n|_U\}$ is normal on U if and only if $\{f^{n+1}|_{f^{-1}(U)}\}$ is normal on $f^{-1}(U)$. If $K \subseteq f^{-1}(U)$ compact then

$$\sup_{z \in K} d(f^n(z), f^m(z)) = \sup_{w \in f(K)} d(f^{n-1}(z), f^{m-1}(z)).$$

Since f is proper (continuous map from compact space to Hausdorff space is proper) and continuous, compactness is preserved by both f and f^{-1}. □

Lemma 1.12. *$J(f) = J(f^n)$ and $F(f) = F(f^n)$ for all n.*

Proof. Exercise. □

Remark. The Julia set of f is the smallest (?) closed subset of $\hat{\mathbb{C}}$ which is totally f-invariant and contains at least 3 points (for the moment assume $|J(f)| \geq 3$), since the complement of any such set has $\{f^n\}$ normal by Montel.

For the rest of the course, we consider only rational maps with $\deg f \geq 2$. See example sheet 1 for a description of $J(\mu)$ for μ a Möbius transformation.

Theorem 1.13. *Let $z \in U \subseteq J(f)$ be open. Then the union $V = \bigcup_{n \in \mathbb{N}} f^n(U)$ contains all but at most 2 points of $\hat{\mathbb{C}}$. Any point $w \notin V$ is a critical point of the Fatou set.*

Proof. The first statement follows from Montel. If $w \notin V$, since $f(V) \subseteq V$, then for all $n \in \mathbb{N}$, $f^{-n}(w) \cap V = \emptyset$. Suppose there are two points $\{z_0, z_1\}$. Then examining possible ramification for these two points which are fixed under f^{-1}, the only possiblities are $z_i \mapsto z_i$ with degree d, or $z_1 \mapsto z_2, z_2 \mapsto z_1$ with degree d. The case for a single point is similiar.

Replacing f by f^2 if needed, it suffcies to show that if $f(z) = z$ and $f'(z) = 0$ then $z \in F(f)$. Note that if $\mu \in \mathrm{Aut}(\hat{\mathbb{C}})$, $g = \mu^{-1} \circ f \circ \mu$, then $g^n = \mu^{-1} \circ f^n \circ \mu$, and so $\mu(J(g)) = J(f)$. Thus wlog $f(0) = 0, f'(0) = 0$. Locally we have $f(z) = a_2 z^2 + a_3 z^3 + \cdots = z^2(a_2 + O(z))$ about 0, so $|f(z)| < |z|$ for z sufficiently close to 0, so on a neighbourhood of 0, $f^n \to 0$ so form a normal family. □

The three cases do happen: $z \mapsto z^d, z \mapsto z^{-d}, z \mapsto p(z)$ for p a polynomial with nonzero constatnt term.

Remark. If $f^n(z_0) = z_0$ for some $n \in \mathbb{N}$ and $(f^n)'(z_0) = \prod_{i=0}^{n-1} f'(f^i(z_0)) = 0$ then $z_0 \in F(f)$.

Corollary 1.14. *If $J(f)$ contains an interior point then $J(f) = \hat{\mathbb{C}}$.*

Proof. If U is open in $J(f)$, $V = \bigcup f^n(U)$ contains all but at most 2 points on $\hat{\mathbb{C}}$. Since $J(f)$ is closed by definition, $J(f) = \hat{\mathbb{C}}$. □

This does happen: let $E_t : y^2 = x(x-1)(x-t)$ for $t \in \mathbb{C} \setminus \{0,1\}$ be an elliptic curve.

$$\begin{array}{ccc} E_t & \xrightarrow{[2]} & E_t \\ \downarrow & & \downarrow \\ \hat{\mathbb{C}} & \xrightarrow{f_t} & \hat{\mathbb{C}} \end{array}$$

where the vertical maps are quotient by ?, i.e. $(x, y) \mapsto x$. Then

$$f_t(z) = \frac{(z^2 - t)^2}{4z(z-1)(z-t)}.$$

We can show $J(f_t)$ is dense in $\hat{\mathbb{C}}$ by showing the Julia set of [2] is dense, and thus $J(f_t) = \hat{\mathbb{C}}$.

Definition (period, multiplier)**.** Let $z_0 \in \hat{\mathbb{C}}$. We say z_0 is *periodic* for a rational f if exists $m \in \mathbb{N}$ such that $f^m(z_0) = z_0$. The minimal such m is the *period* of the cycle containing z_0. If $m = 1$ we also call it a *fixed point.*

If z_0 has period m, the *multiplier* of the cycle is

$$(f^m)'(z_0) = \prod_{i=0}^{m-1} f'(f^i(z_0)).$$

Let λ be the multiplier of z_0. We say z_0 is

1. *superattracting* if $\lambda = 0$,
2. *attracting* if $0 \leq |\lambda| < 1$,
3. *indifferent* if $|\lambda| = 1$,
4. *repelling* if $|\lambda| > 1$.

Recall that we might have to use the chart at infinity to compute the derivative. For example if $z_0 = \infty, f(\infty) = \infty$ then

$$\lambda = \lim_{z \to \infty} \frac{1}{f'(z)}.$$

Definition (basin of attraction)**.** Suppose $C = \{z_0, f(z_0), \dots, f^{m-1}(z_0)\}$ is an attracting cycle. The *basin of attraction* for C is

$$A = \{z \in \hat{\mathbb{C}} : \lim_{n \to \infty} f^{nm}(z) = f^i(z_0) \text{ for some } 0 \leq i \leq m-1\}.$$

Theorem 1.15. *If $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ has an attracting cycle then the basin of attraction is in $F(f)$. On the other hand all repelling cycles are contained in $J(f)$.*

Example. The theorem completely describes the Fatou and Julia set of $z \mapsto z^2$. z_0 is periodic if and only if exists n such that $z_0^{2^n} = z_0$, so z_0 is $0, \infty$ or some root of unity (which forms a dense subset of S^1). $A_0 = \mathbb{D}, A_\infty = \hat{\mathbb{C}} \setminus \overline{\mathbb{D}}$. All other cycles are repelling and so $J(f) = S^1$.

Proof. Since $J(f^m) = J(f)$, wlog assume z_0 is a fixed point. Suppose that $\lambda = f'(z_0)$ is such that $|\lambda| < 1$. By Taylor expansion $|f(z) - z_0| \le c|z - z_0|$ for some constant $c < 1$ for z sufficiently close to z_0. So on a neighbourhood of $_0$, $f^n(z)$ converges uniformly on compact subsets to the constant function z_0. So $z_0 \in F(f)$.

On the other hand if z_0 is repelling so $|\lambda| > 1$, suppose for contradiction that $z_0 \in F(f)$, so exists open neighbourhood U of z_0 on which f^n has a subsequence converging to a holomorphic limit. Since $(f^n)'(z_0) = \lambda^n$, absurd. □

Remark. We will classify later when indifferent points are Julia.

1.1 Holomorphic Lefschetz fixed point formula

Definition (residue index)**.** Let z_0 be a fixed point of a rational map f. The *residue index* of f at z_0 is

$$i_f(z_0) = \frac{1}{2\pi i} \int_\gamma \frac{dz}{z - f(z)}$$

where γ is a small, positively oriented circle about z_0.

Lemma 1.16. *Let z_0 have multiplier $\neq 1$. Then $i_f(z_0) = \frac{1}{1-\lambda}$.*

Proof. It is an exercise to check the multiplier is coordinate-independent. By definition the resude index is translation/conjugation independent, so wlog $z_0 = 0$. Then on a neighbourhood of 0, $f(z) = \lambda z + a_2 z^2 + \dots$ so

$$\frac{1}{z - f(z)} = \frac{1}{(1-\lambda)z(1 + O(z))} = \frac{1}{(1-\lambda)z} + g(z)$$

with g holomorphic on a neighbourhood of 0. Integrate. □

Theorem 1.17 (holomorphic Lefschetz on $\hat{\mathbb{C}}$)**.** *Say $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ of degree ≥ 2. Then the fixed points of f satisfy*

$$\sum_{z = f(z)} i_f(z) = 1.$$

Proof. Conjugation if necessary (exercise: use the above lemma to show the residue index is coordinate-independent), wlog $f(\infty) \neq \infty$. Choose $R \gg 0$ so that all fixed points of f are in $D(0, R)$. Call the positively oriented boundary C_R. By residue theorem

$$\begin{aligned}\sum_{z=f(z)} i_f(z) &= \frac{1}{2\pi i}\int_{C_R} \frac{dz}{z - f(z)} \\ &= \frac{1}{2\pi i}\int_{-C_{1/R}} \frac{-dw}{w^2(\frac{1}{w} - f(\frac{1}{w}))} \\ &= \frac{1}{2\pi i}\int_{C_{1/R}} \frac{dw}{w(1 - wf(\frac{1}{w}))} \\ &= \mathrm{Res}_{w=0} \frac{1}{w(1 - wf(\frac{1}{w}))} \\ &= 1\end{aligned}$$

□

Corollary 1.18. *Suppose* $\deg f \geq 2$*. Then* $J(f) \neq \emptyset$*.*

Proof. Consider the fixed points of f. Assume first no fixed point multiplier is 1. Then $\lambda \mapsto \frac{1}{1-\lambda}$ sends the unit circle to the line $\mathrm{Re} = \frac{1}{2}$, and $\mathbb{D}$ to $\mathrm{Re} > \frac{1}{2}$. Thus if $|\lambda| \leq 1$ for all fixed point multipliers, and not equal to 1, (there is no multiplicity), there are $d+1 \geq 3$ distinct fixed points (?), so $\mathrm{Re}(\sum_{z=f(z)} i_f(z)) \geq \frac{3}{2}$, absurd. If exists a repelling point then done. So suppose z_0 is fixed with $\lambda = 1$. Then in local coordinates $f(z) = z + a_k z^k + \dots$ where $a_k \neq 0$. Inductively $f(z) = z + na_k z^k + \dots$ so the kth derivative of $f^n(z_0)$ is $k!na_k \to \infty$ as $n \to \infty$, so the iterates cannot form a normal family on a neighbourhood of z_0. □

Remark.

1. Suppose z_0 is a indifferent fixed point, λ a root of unity. If $\lambda^k = 1$ then $(f^k)'(z_0) = \prod_{i=0}^{k-1} f'(f(z_0)) = \lambda^k = 1$. Thus the preceding argument shows that $z_0 \in J(f^k) = J(f)$.
2. It is possible for a rational map to have non repelling fixed point. For example $z \mapsto z^2 + \frac{1}{4}$. The fixed points are ∞ and $\frac{1}{2}$ which is a double fixed point.
3. Any finite *grand orbit* (the set $\{z \in \hat{\mathbb{C}} : f^m(z) = f^n(z_0)$ for some $m, n\}$) is necessarily Fatou (exercise), so $|J(f)| = \infty$.

Recall: suppose $f : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ is a rational map of degree $d \geq 2$.

1. If U is open, $U \cap J(f) \neq \emptyset$ then $\bigcup_{n\geq 1} f^n(U)$ contains all but at most 2 points and contains $J(f)$.
2. $J(f)$ contains all repelling cycles and all indifferent cycles with roots of unity multipliers.
3. $J(f) \neq \emptyset$ and $|J(f)| = \infty$.

Proposition 1.19. *Suppose f has a periodic cycle which is attracting, with attracting basin A. Then $J(f) = \partial A$.*

Proof. Given U an open neighbourhood such that $U \cap J(f) \neq \emptyset$, exists n such that $f^n(U) \cap A = \emptyset$. As A is closed under preimages, $U \cap A \neq \emptyset$. Thus $J(f) \subseteq \overline{A}$. Since $A \subseteq F(f)$, $J(f) \subseteq \partial A$.

Conversely suppose $z_0 \in \partial A$ and U is a neighbourhood of z_0. Suppose $\{f^n\}$ forms a normal family on U. On $U \cap A$, any holomorphic limit g of iterates of f must take finitely many constant values, but g cannot be locally constant as U contains points not in the basin, absurd. Thus $z_0 \in J(f)$. □

Example. Any *polynomial* f has $J(f)$ the boundary of basin at ∞. Note that it might also be the boundary of another basin, for example $z \mapsto z^2, z \mapsto z^2 - 1$.

Corollary 1.20. *Fix $z_0 \in J(f)$. Then the full preimage $\{z : f^n(z) = z_0$ for some $n \geq 0\}$ forms a dense subset of $J(f)$.*

Proof. Fix $z_1 \in J(f)$ and a neighbourhood $U \ni z_1$. If it contains no preimage of z_0 then $\bigcup f^n(U) \not\ni z_0$, absurd. □

Topological preimage equidistribution

1.2 Attracting (and repelling) cycles

Definition (topologically attracting)**.** A fixed point p of f is *topologically attracting* if there exists a neighbourhood $U \ni p$ such that $\{f^n\}$ converges locally uniformly to p on U.

Lemma 1.21. *A fixed point p of f is attracting if and only if it is topologically attracting.*

Proof. Exercise. Taylor's theorem in one direction, and Schwarz lemma in the other. □

Theorem 1.22. *Suppose f has a fixed point p with multiplier λ, $|\lambda| \neq 0, 1$. Then exists local holomorphic change of coordinates ϕ such that $\phi(p) = 0$ and $\phi \circ f \circ f^{-1}(w) = \lambda w$. Thus coordinate is unique up to multiplication by a constant. ϕ is known as the* Kaenig linearising map.

Proof. wlog $p = 0$ and first suppose $0 < |\lambda| < 1$. Choose a constant c such that $c^2 < |\lambda| < c$. Find $r > 0$ such that for all $z \in D(0, r)$, $|f(z)| \leq c|z|$, so $|f^n(z)| \leq c^n r$. We can find $B > 0$ such that for all $z \in D(0, r)$, $|f(z) - \lambda z| \leq B|z|^2$. Thus for all $z \in D(0, r)$,

$$|f^{n+1}(z) - \lambda f^n(z)| \leq B|f^n(z)|^2 \leq Br^2c^{2n}.$$

Let $w_n = \frac{f^n(z)}{\lambda^n}$. Then

$$|w_{n+1}(z) - w_n(z)| = \left| \frac{f^{n+1}(z)}{\lambda^{n+1}} - \frac{f^n(z)}{\lambda^n} \right| \leq \frac{1}{|\lambda|^{n+1}} Br^2 c^{2n} = \frac{Br^2}{\lambda} \left| \frac{c^2}{\lambda} \right|^n$$

so w_n converges locally uniformly on $D(0,r)$. Set $\phi(z) = \lim w_n(z)$. As $z \mapsto w_n(z)$ has derivative 1 at 0, so does ϕ so it has a holomorphic inverse.

For uniqueness suppose ψ is another such coordinate, then for $w \in \psi(U)$ have $\lambda\phi(\psi^{-1}(w)) = \phi(\psi^{-1}(\lambda w))$. Done by comparing local power series.

For $|\lambda| > 1$ apply the same argument to a branch of f^{-1}. □

Corollary 1.23. *Suppose p is an attracting fixed point of f with multiplier $\lambda \neq 0$ and basin A. Then exists a holomorphic $\phi : A \to \mathbb{C}$ such that the following diagram commutes*

$$\begin{array}{ccc} A & \xrightarrow{f} & A \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ \mathbb{C} & \xrightarrow{\lambda} & \mathbb{C} \end{array}$$

Proof. Define $\phi(z) = \lim_{n\to\infty} \frac{\phi_0(f^n(z))}{\lambda^n}$ where ϕ_0 is the linearlising coordinates on a neighbourhood of p. Check the details. □

Definition (immediate basin)**.** The *immediate basin* of an attracting cycle is the union of the Fatou components containing the cycle elements.

...

$\frac{z^2-1}{z^2-c}$ attracting 5-cycle $\infty \mapsto 1 \mapsto 0 \mapsto \frac{1}{c}$

Proposition 1.24. *Let f be a rational map with $f(p) = p$ an attracting fixed point. Then the immediate basin of p contains a critical point of f.*

Proof. wlog $p = 0$. The component U of $F(f)$ is hyperbolic as $|J(f)| = \infty$. Thus we have

$$\begin{array}{ccc} \mathbb{D} & \xrightarrow{F} & \mathbb{D} \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ U & \xrightarrow{f} & U \end{array}$$

If f has no critical points in U then $f \circ \pi$ is a covering map $\mathbb{D} \to U$ so exists $G : \mathbb{D} \to \mathbb{D}$ covering it. If π, F, G fix 0, G is inverse to F. Thus $F \in \mathrm{Aut}(\mathbb{D})$ so F, f are hyperbolic local isometries, contradicting 0 an attracting fixed point. □

Corollary 1.25. *f has at most $2d - 2$ attracting cycles.*

Corollary 1.26. *f has at most $4d - 4$ non-repelling cycles.*

Proof. Holomorphic perturbation. Let $f_t(z) = (1-t)f(z) + tz^d$. Note $f_0 = f(z), f_1 = z^d$. Suppose $f^n(\alpha) = \alpha$ with multiplier $\lambda \in S^1$. If α is not a repeated root of $f^n(z) - z$ there is a neighbourhood of 0 and holomorphic $t \mapsto \alpha(t)$ such that $\alpha(0) = \alpha$ and $f^n(\alpha(t)) = \alpha(t)$ for all t, i.e. if $\lambda \neq 1$ (?). But if $\lambda = 1$ we can base change $t \mapsto t^k$. We then have $t \mapsto \lambda(t)$ homomorphic in t, $\lambda(0) = \lambda$ and $(f^n)'(\alpha(t)) = \lambda(t)$. Either $\lambda(t)$ is the constant 1 (more argument needed), or another constant λ, or nonconstant. The first two cases contradict z^d having no indifferent cycles at $t = 1$.

By conformality of holomorphic maps, the measure

$$\mu(\{\theta \in S^1 : |\lambda(\varepsilon e^{i\theta})|\}) \to \frac{1}{2}$$

as $\varepsilon \to 0$. Repeating this process for all indifferent cycles, exists a direction θ such that perturbation in the θ-direction makes half of these cycles attracting. For sufficiently small choice of $\varepsilon e^{i\theta}$, attracting cycles remain attracting. Let N be the number of indifferent cycles of f, M the number of attracting cycles of f, then the number of non-repelling cycles of f is $N + M = 2(M/2 + N/2) \leq 2(2d - 2)$. □

Remark. $f^n(z) = z$ has $z^n + 1$ roots counting multiplicity, so must have a repelling cycle.

Note we can be more precise, see example sheet.

Theorem 1.27. *If $f(0) = 0$ is attracting with multiplier $\lambda \neq 0$. Let ϕ be a linearising coordinate with local inverse $\psi : \mathbb{D}(0, \varepsilon) \to A_0$, where A_0 is the immediate basin of 0. ψ extends to a holomorphic map on a disk $\mathbb{D}(0, r)$ of some maximal radius r, extending homeomorphically to $\partial\mathbb{D}(0, r)$ and $\psi(\partial\mathbb{D}(0, r))$ contains a critical point of f.*

Remark. Actually detecting whether f has an attractor is harder. Open problem: does $z \mapsto z^2 - \frac{3}{2}$ has an attractor?

Caution: linearising map need not continuously extend to $J(f)$.

Theorem 1.28. *If f rational has $J(f)$ disconnected then $J(f)$ has uncountably many connected components.*

Proof. If $J(f) = J_0 \cup J_1$ where J_0, J_1 are disjoint compact nonempty. Given $z \in J$, define a seuqnce $\beta(z) = (\beta_n(z))$ where $\beta_n(z) = i$ if $f^n(z) \in J_i$. If z, w are in the same connected component of $J(f)$ then $\beta(z) = \beta(w)$. It suffices to show that for any initial $\beta_1(z), \ldots, \beta_k(z)$, exists $n > k$ such that exists $z' \in J(f)$ such that $\beta_i(z') = \beta_i(z)$ for all $1 \leq i \leq k$ but $\beta_n(z') \neq \beta_n(z)$. Define

$$U_{z,k} = \{w \in \hat{\mathbb{C}} : f^i(w) \notin J_{1-\beta_i(z)} \text{ for all } 1 \leq i \leq k\}.$$

This is open and contains $F(f)$. Some subsequence $(\beta_{n_j}(z))$ is constant, say the constant 0. If $\beta_i(z') = \beta_i(z)$ for all $1 \leq i \leq k$ then $\beta_i(z') = \beta_i(z)$ for all i, then

$$f^{n_j}(U_{z,k}) \subseteq \mathbb{C} \setminus J_1.$$

The maps $f^{nj} : U_{z,k} \to \mathbb{C} \setminus J_1$ form a normal family, contradiction. Thus $\{\beta(z) : z \in J(f)\}$ is uncountable. □

Superattractor

Theorem 1.29. *Suppose $f(0) = 0$ is with local expansion $f(z) = a_m z^m + a_{m+1} z^{m+1} + \dots, m \geq 2$. Then there exists a holomorphic change of coordinates ϕ on a neighbourhood of 0 such that $\phi(0) = 0$, $\phi(f(z)) = \phi(z)^m$. ϕ is unique up to multiplication by an $(m-1)$th root of unity. ϕ is called the* Böttcher coordinate*.*

Proof. We sketch the proof only. The details are the same as Kaenig's. Write locally $f(z) = z^m(1 + h(z))$, where $h(z) \to 0$ as $z \to 0$ is holomorphic, where $a_m = 1$ (otherwise conjugate by $\alpha f(z/\alpha)$). Write $1 + h(z) = \exp(k(z))$ for some holomorphic $h(z)$ on a neighbourhood of 0. Then there exists holomorphic $k_n(z)$ on this neighbourhood so that $f^n(z) = z^{m^n} \exp(k_n(z))$. Choose the branch $\phi_n(z)$ of the m^nth root of $f^n(z)$ such that $\phi_n(z) = z(1+O(z))$. Then ϕ_n converges uniformly to some holomorphic ϕ on this neighbourhood which satisfies the statement. Uniqueness follows from identification of Taylor expansion, a la Kaenig. ☐

Corollary 1.30. *Let $f(0) = 0$ be superattracting, with basin A and Böttcher coordinate ϕ on a neighbourhood of 0. Then $z \mapsto |\phi(z)|$ extends to a continuous map $|\phi| : A \to [0, 1)$ satisfying $|\phi(f(z))| = |\phi(z)|^m$ for $z \in A$.*

Proof. Given $z \in A$, set $|\phi|(z) = |\phi(f^n(z))|^{1/m^n}$, where $n \gg 1$ such that $f^n(z)$ is in the neighbourhood domain of ϕ. The desired equality is immediate. ☐

2 Polynomial dynamics

Definition (filled Julia set)**.** Let $p(z) = a_d z^d + a_{d-1} z^{d-1} + \cdots + a_0, d \geq 2, a_d \neq 0$. The *filled Julia set* of p is

$$K(p) = \{z \in \mathbb{C} : |f^n(z)| \nrightarrow \infty \text{ as } n \to \infty\}.$$

Note this is the complement of the basin of infinity.

From our results on boundaries of basins, $\partial K(p) = J(p)$. We know we have a Böttcher coordinate on a neighbourhood of ∞: choose this (i.e. $\phi(1/z)^{-1}$) such that $\phi(\infty) = \infty$.

Definition (Green's function)**.** Suppose $p(z)$ is a degree d polynomial. The *Green's function* associated to p is

$$G_p(z) = \lim_{n \to \infty} \frac{\log^+ |p^n(z)|}{d^n}$$

where $\log^+ x = \max\{\log x, 0\}$ for $x \geq 0$.

Lemma 2.1. *$G_p(z)$ satisfying the following:*

1. *G_p is continuous everywhere and harmonic on $\mathbb{C} \setminus K(p)$.*
2. *$G_p(z) = \log |z| + O(1)$ as $|z| \to \infty$.*
3. *$G_p(z) \to 0$ as $z \to K(p)$.*
4. *$G_p(p(z)) = dG_p(z)$.*

1, 2, 4 uniquely characterises G_p, and $G_p(z) = \log |\phi_p(z)|$, where ϕ_p is a Böttcher coordinate at ∞ on $\hat{\mathbb{C}} \setminus K(p)$.

Remark.

1. This is how pictures of filled Julia sets are drawn.
2. The Green's function depends only on $K(p)$.

Proof.

1. Consider the function $\log^+ |p(z)| - d \log^+ |z|$ on $\hat{\mathbb{C}}$. It is continuous and takes real values, so is bounded by some $C \in \mathbb{R}$. Then for all n,

$$\left| \frac{\log^+ |p^n(z)|}{d^n} - \frac{\log^+ |p^{n-1}(z)|}{d^{n-1}} \right| \leq \frac{C}{d^n}$$

so for $m \leq n$,

$$\left| \frac{\log^+ |p^n(z)|}{d^n} - \frac{\log^+ |p^m(z)|}{d^m} \right| \leq \sum_{k=m+1}^{n} \frac{C}{d^k} \leq \frac{C}{d^m(d-1)}$$

so G_p is a uniform limit of continuous function so continuous.

Locally, a function is harmonic if and only if it is the real part of a holomophic function, if and only if it equals to $\log|f|$ for some holomorphic f that does not vanish anywhere (since on a simply connected domain we can take logarithm). Given $z \notin K(p)$, find a small disk $D \ni p$ such that $\overline{D} \cap K(p) = \emptyset$. There exists $N \gg 1$ such that $p^n(\overline{D}) \cap \mathbb{D} = \emptyset$ for $n \geq N$. Then $\frac{\log^+ |p^n(z)|}{d^n}$ is harmonic on $\overline{D}$. Since a uniform limit of harmonics is harmonic, we have $G_p(z)$ is harmonic on D as well. Note if $K(p)^{\text{int}}(p) \neq \emptyset$ then $G_p(z) = 0$ there so is harmonic as well. In other words, $G_p(z)$ fails to be harmonic precisely on the Julia set (for more rigorous argument see later).

2. Set $m = 0$, then the bound in 1 gives

$$\left| \frac{\log^+ |p^n(z)|}{d^n} - \log^+ |z| \right| \leq \frac{C}{d-1}$$

so as $n \to \infty$, $|G_p(z) - \log|z|| \leq \frac{C}{d-1}$ for $|z| \gg 0$.

3. $G_p(z) = 0$ on $K(p)$.

4. Definition.

Suppose $H(z)$ is a function satisfying 1, 2 and 4 and consider $G(z) = G_p(z) - H(z)$. By 1 and 2 it is continuous and bounded on $\hat{\mathbb{C}}$. By 4, as $n \to \infty$, $G(p^n(z)) = d^n G(z) \to \infty$ unless $G(z) = 0$. We thus have $G_p(z) = H(z)$. For $G_p(z) = \log|\phi_p(z)|$, check continuity, growth at ∞ and transformation. ☐

Example. For $z \mapsto z^d$,

$$G_p(z) = \lim \frac{\log^+ |z^{d^n}|}{d^n} = \log^+ |z|.$$

$K(p) = \overline{D}$, where $\log^+ |z| = 0$. The basin of infinity is $\hat{\mathbb{C}} \setminus \mathbb{D}$.

Remark. $G_p(z)$ is also known as the *potential function* associated to $K(p)$.

Now back to superattractors.

Theorem 2.2. *Suppose $f(0) = 0$ is superattracting, with Böttcher coordinate ϕ for f at 0. There there exists a unique open disk $\mathbb{D}(0, r)$ of maximal radius $0 < r \leq 1$ such that the inverse ψ of ϕ extends holomorphically to $\psi : \mathbb{D}(0, r) \to A_0$, the immediate basin of attraction of 0. If $r = 1$ then $\psi : D(0, t) \cong A_0$ and 0 is the only critical points of f in A_0. On the other hand if $r < 1$ there exists a nonzero critical point in A_0, which lies on $\psi(\mathbb{D}(0, r))$.*

Proof. Guided on example sheet 2. Non-examinable. ☐

Example. $f(z) = z^2 + \frac{1}{2}$. ϕ sends a neighbourhood of ∞ to the complement of a large closed disk in $\hat{\mathbb{C}}$ isomorphically. ψ can be extended until it hits the image of a critical point.

In the case $f_c(z) = z^2 + c$, there are two critical points $\infty, 0$. ∞ is mapped to itself with multiplicity 2. We have

Corollary 2.3. *Suppose* $0 \notin K(f_c)$*, i.e.* $f_c^n(0) \to \infty$ *as* $n \to \infty$*. Then the Böttcher coordinate* ϕ_c *of* f_c *at* ∞ *extends to a conformal isomorphism on a neighbourhood of* ∞ *which contains* c*.*

Proof. 0 is the only critical point that can move around and $f(0) = c$. Now use extension of Böttcher coordinate. □

Proposition 2.4. *A closed subset of the sphere is connected if and only if the connected components of its complement are simply connected.*

Proof. Beardon, Iteration of Rational Functions and Ahlfors. □

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