

UNIVERSITY OF
CAMBRIDGE

MATHEMATICS TRIPOS

Part III

Category Theory

Michaelmas, 2018

Lectures by

P. T. JOHNSTONE

Notes by

QIANGRU KUANG

Contents

| | | |
|----------|--|-----------|
| 1 | Definitions and examples | 2 |
| 2 | The Yoneda lemma | 10 |
| 3 | Adjunctions | 16 |
| 4 | Limits | 23 |
| 5 | Monad | 35 |
| 6 | Cartesian closed categories | 46 |
| 7 | Toposes | 54 |
| | 7.1 Sheaves and local operators* | 61 |
| | Index | 66 |

1 Definitions and examples

Definition (category). A *category* \mathbf{C} consists of

1. a collection $\text{ob } \mathbf{C}$ of *objects* A, B, C, \dots ,
2. a collection $\text{mor } \mathbf{C}$ of *morphisms* f, g, h, \dots ,
3. two operations dom and cod assigning to each $f \in \text{mor } \mathbf{C}$ a pair of objects, its *domain* and *codomain*. We write $A \xrightarrow{f} B$ to mean f is a morphism and $\text{dom } f = A, \text{cod } f = B$,
4. an operation assigning to each $A \in \text{ob } \mathbf{C}$ a morphism $A \xrightarrow{1_A} A$,
5. a partial binary operation $(f, g) \mapsto fg$ on morphisms, such that fg is defined if and only if $\text{dom } f = \text{cod } g$ and let $\text{dom } fg = \text{dom } g, \text{cod } fg = \text{cod } f$ if fg is defined

satisfying

1. $f1_A = f = 1_B f$ for any $A \xrightarrow{f} B$,
2. $(fg)h = f(gh)$ whenever fg and gh are defined.

Remark.

1. This definition is independent of any model of set theory. If we're given a particular model of set theory, we call \mathbf{C} *small* if $\text{ob } \mathbf{C}$ and $\text{mor } \mathbf{C}$ are sets.
2. Some texts say fg means f followed by g (we are not).
3. Note that a morphism f is an identity if and only if $fg = g$ and $hf = h$ whenever the compositions are defined so we could formulate the definitions entirely in terms of morphisms.

Example.

1. The category **Set** has all sets as objects and all functions between sets as morphisms (strictly, morphisms $A \rightarrow B$ are pairs (f, B) where f is a set theoretic function).
2. The category **Gp** where objects are groups, morphisms are group homomorphisms. Similarly **Ring** is the category of rings and **Mod_R** is the category of R -modules.
3. The category **Top** has all topological spaces as objects and continuous functions as morphisms. Similarly **Unif** the category of uniform spaces with uniformly continuous functions and **Mf** the category of manifolds with smooth maps.
4. The category **Htpy** has same objects as **Top** but morphisms are homotopy classes of continuous functions.

More generally, given \mathbf{C} we call an equivalence relation \simeq on $\text{mor } \mathbf{C}$ a *congruence* if $f \simeq g$ implies $\text{dom } f = \text{dom } g, \text{cod } f = \text{cod } g$ and $fh \simeq gh$

and vice versa. Then we have a set \mathbf{C}/\simeq with the same objects as \mathbf{C} but congruence classes as morphisms.

5. Given \mathbf{C} , the *opposite category* \mathbf{C}^{op} has the same objects and morphisms as \mathbf{C} , but domain and codomain interchanged and fg in \mathbf{C}^{op} is gf in \mathbf{C} .

This leads to *duality principle*: if P is a valid statement about categories so is P^* attained by reversing all the arrows.

6. A small category with one object is a *monoid*, i.e. a semigroup with 1. In particular, a group is a small category with one object, in which every morphism is an isomorphism (i.e. for all f there exists g such that fg and gf are identities).

7. A *groupoid* is a category in which every morphism is an isomorphism. For example, for a topological space X , the *fundamental groupoid* $\pi(X)$ has all points of X as objects and morphisms $x \rightarrow y$ as homotopy classes $\text{rel } 0, 1$ of paths $\gamma : [0, 1] \rightarrow X$ with $\gamma(1) = y$.

8. A *discrete* category is one whose only morphisms are identities. A *preorder* is a category \mathbf{C} in which for any pair (A, B) there exists at most one morphism $A \rightarrow B$. A small preorder is a set equipped with a binary relation which is reflexive and transitive. In particular a partially ordered set is a partially ordered set is a small preorder in which the only isomorphisms are identities.

9. The category **Rel** has the same objects as **Set** but morphisms $A \rightarrow B$ are arbitrary relations. Given $R \subseteq A \times B, S \subseteq B \times C$, we define

$$S \circ R = \{(a, c) \in A \times C : \exists b \in B \text{ s.t. } (a, b) \in R, (b, c) \in S\}.$$

The identity $1_A : A \rightarrow A$ is $\{(a, a) : a \in A\}$. Similarly, the category **Part** of sets and partial functions (i.e. relations such that for all $(a, b), (a, b') \in R, b = b'$) can be defined.

10. Let K be a field. The category **Mat** $_K$ has natural numbers as objects and morphisms $n \rightarrow p$ are $(p \times n)$ matrices with entries from K . Composition is matrix multiplication.

Definition (functor). Let \mathbf{C}, \mathbf{D} be categories. A *functor* $F : \mathbf{C} \rightarrow \mathbf{D}$ consists of

1. a mapping $A \mapsto F(A)$ from $\text{ob } \mathbf{C}$ to $\text{ob } \mathbf{D}$,
2. a mapping $f \mapsto F(f)$ from $\text{mor } \mathbf{C}$ to $\text{mor } \mathbf{D}$ such that

$$\begin{aligned} \text{dom}(F(f)) &= F(\text{dom } f), \text{cod}(Ff) = F \text{cod}(f) \\ 1_{F(A)} &= F(1_A), (F(f))(F(g)) = F(fg) \end{aligned}$$

wherever fg is defined.

Example. We write **Cat** for the category whose objects are all small categories and whose morphisms are functors between them.

Example.

1. We have *forgetful functors* $U : \mathbf{Gp} \rightarrow \mathbf{Set}, \mathbf{Ring} \rightarrow \mathbf{Set}, \mathbf{Top} \rightarrow \mathbf{Set} \dots$ and $\mathbf{Ring} \rightarrow \mathbf{AbGp}$ (forget multiplication), $\mathbf{Ring} \rightarrow \mathbf{Mon}$ (forget addition).

2. Given A , the free group $F(A)$ has the property given any group G , any $A \xrightarrow{f} UG$, there is a unique homomorphism $FA \xrightarrow{\tilde{f}} G$ extending f .

F is a functor $\mathbf{Set} \rightarrow \mathbf{Gp}$: given any $A \xrightarrow{f} B$, we define $F(f)$ to be the unique homomorphism extending $A \xrightarrow{f} B \hookrightarrow UFB$. Functoriality follows from uniqueness: given $B \xrightarrow{g} C$, $F(gf)$ and $(Fg)(Ff)$ are both homomorphisms extending

$$A \xrightarrow{f} B \xrightarrow{g} C \hookrightarrow UFC.$$

3. Given a set A , we write PA for the set of all subsets of A . We can make P into a functor $\mathbf{Set} \rightarrow \mathbf{Set}$: given $A \xrightarrow{f} B$, we define

$$Pf(A') = \{f(a) : a \in A'\}$$

for $A' \subseteq A$.

But we also have a functor $P^* : \mathbf{Set} \rightarrow \mathbf{Set}^{\text{op}}$ defined by

$$P^*f(B') = \{a \in A : f(a) \in B'\}$$

for $B' \subseteq B$.

4. By a *contravariant functor* $\mathbf{C} \rightarrow \mathbf{D}$ we mean a functor $\mathbf{C} \rightarrow \mathbf{D}^{\text{op}}$ (or $\mathbf{C}^{\text{op}} \rightarrow \mathbf{D}$). (A *covariant functor* is one that doesn't reverse arrows)

Let K be a field. We have a functor $\cdot^* : \mathbf{Mod}_K \rightarrow \mathbf{Mod}_K^{\text{op}}$ defined by

$$V^* = \{\text{linear maps } V \rightarrow K\}$$

and if $V \xrightarrow{f} W$, $f^*(\theta) = \theta f$.

5. We have a functor $\cdot^{\text{op}} : \mathbf{Cat} \rightarrow \mathbf{Cat}$ which is the identity on morphisms. (note that this is *covariant*)
6. A functor between monoids is a monoid homomorphism.
7. A functor between posets is an order-preserving map.
8. Let G be a group. A functor $F : G \rightarrow \mathbf{Set}$ consists of a set $A = F*$ together with an action of G on A , i.e. permutation representation of G . Similarly a functor $G \rightarrow \mathbf{Mod}_K$ is a K -linear representation of G .
9. The construction of the fundamental group $\pi_1(X, x)$ of a space X with basepoint x is a functor

$$\mathbf{Top}_* \rightarrow \mathbf{Gp}$$

where \mathbf{Top}_* is the category of spaces with a chosen basepoint.

Similarly, the fundamental groupoid is a functor

$$\mathbf{Top} \rightarrow \mathbf{Gpd}$$

where \mathbf{Gpd} is the category of groupoids and functors between them.

Definition (natural transformation). Let \mathbf{C}, \mathbf{D} be categories and $F, G : \mathbf{C} \rightarrow \mathbf{D}$ be two functors. A *natural transformation* $\alpha : F \rightarrow G$ consists of an assignment $A \mapsto \alpha_A$ from $\text{ob } \mathbf{C}$ to $\text{mor } \mathbf{D}$ such that $\text{dom } \alpha_A = FA$, $\text{cod } \alpha_A = GA$ for all A , and for all $A \xrightarrow{f} B$ in \mathbf{C} , the square

$$\begin{array}{ccc} FA & \xrightarrow{Ff} & FB \\ \downarrow \alpha_A & & \downarrow \alpha_B \\ GA & \xrightarrow{Gf} & GB \end{array}$$

commutes, i.e. $\alpha_B(Ff) = (Gf)\alpha_A$.

Example.

1. Given categories \mathbf{C}, \mathbf{D} , we write $[\mathbf{C}, \mathbf{D}]$ for the category whose objects are functors $\mathbf{C} \rightarrow \mathbf{D}$ and whose morphisms are natural transformations.
2. Let K be a field and V a vector space over K . There is a linear map $\alpha_V : V \rightarrow V^{**}$ given by

$$\alpha_V(v)(\theta) = \theta(v)$$

for $\theta \in V^*$. This is the V -component of a natural transformation

$$1_{\mathbf{Mod}_K} \rightarrow \cdot^{**} : \mathbf{Mod}_K \rightarrow \mathbf{Mod}_K.$$

3. For any set A , we have a mapping $\sigma_A : A \rightarrow PA$ sending a to $\{a\}$. If $f : A \rightarrow B$ then $Pf(\{a\}) = \{f(a)\}$. So σ is a natural transformation $1_{\mathbf{Set}} \rightarrow P$.
4. Let $F : \mathbf{Set} \rightarrow \mathbf{Gp}$ be the free group functor and $U : \mathbf{Gp} \rightarrow \mathbf{Set}$ the forgetful functor. The inclusions $A \rightarrow UFA$ is a natural transformation $1_{\mathbf{Set}} \rightarrow UF$.
5. Let G, H be groups and $f, g : G \rightarrow H$ be two homomorphisms. Then a natural transformation $\alpha : f \rightarrow g$ corresponds to an element $h = \alpha_*$ such that $hf(x) = g(x)h$ for all $x \in G$, or equivalently $f(x) = h^{-1}g(x)h$, i.e. f and g are conjugate group homomorphisms.
6. Let A and B be two G -sets, regarded as functors $G \rightarrow \mathbf{Set}$. A natural transformation $A \rightarrow B$ is a function f satisfying $f(g.a) = g.f(a)$ for all $a \in A$, i.e. a G -equivariant map.

When we say “natural isomorphism”, it is ambiguous and can formally mean two different things: one could mean there is a natural transformation going the other way which when composed produces identity, or each component is an isomorphism. It turns out they coincide:

Lemma 1.1. *Let $F, G : \mathbf{C} \rightarrow \mathbf{D}$ be two functors and $\alpha : F \rightarrow G$ a natural transformation. Then α is an isomorphism in $[\mathbf{C}, \mathbf{D}]$ if and only if each α_A is an isomorphism in \mathbf{D} .*

Proof. Only if is trivial. For if, suppose each α_A has an inverse β_A . We need to prove the β 's satisfy the naturality condition: give $f : A \rightarrow B$ in \mathbf{C} , we need to show that

$$\begin{array}{ccc} GA & \xrightarrow{Gf} & GB \\ \downarrow \beta_A & & \downarrow \beta_B \\ FA & \xrightarrow{Ff} & FB \end{array}$$

commutes. But

$$(Ff)\beta_A = \beta_B\alpha_B(Ff)\beta_A = \beta_B(GF)\alpha_A\beta_A = \beta_B(Gf)$$

by naturality of α . □

In study of algebraic theories (for example), we are interested in isomorphisms of objects and investigate the properties of objects “up to isomorphism”. However, in category theory a weaker notion of isomorphism is often more useful:

Definition (equivalence). Let \mathbf{C} and \mathbf{D} be categories. By an *equivalence* between \mathbf{C} and \mathbf{D} we mean a pair of functors $F : \mathbf{C} \rightarrow \mathbf{D}, G : \mathbf{D} \rightarrow \mathbf{C}$ together with natural isomorphisms $\alpha : 1_{\mathbf{C}} \rightarrow GF, \beta : FG \rightarrow 1_{\mathbf{D}}$. We write $\mathbf{C} \simeq \mathbf{D}$ if \mathbf{C} and \mathbf{D} are equivalent.

We say a property P of categories is a *category property* if whenever \mathbf{C} has P and $\mathbf{C} \simeq \mathbf{D}$ then \mathbf{D} has P .

For example, being a groupoid or a preorder are categorical properties, but being a group or a partial order are not.

Example.

1. The category **Part** is equivalent to the category **Set*** of pointed sets (and basepoint preserving functions). We define

$$\begin{aligned} F : \mathbf{Set}_* &\rightarrow \mathbf{Part} \\ (A, a) &\mapsto A \setminus \{a\} \end{aligned}$$

and if $f : (A, a) \rightarrow (B, b)$ then $Ff(x) = f(x)$ if $f(x) \neq b$ and undefined otherwise, and

$$\begin{aligned} G : \mathbf{Part} &\rightarrow \mathbf{Set}_* \\ A &\mapsto A^+ = (A \cup \{A\}, A) \end{aligned}$$

and if $f : A \rightarrow B$ is a partial function, we define

$$x \mapsto \begin{cases} f(x) & \text{if } x \in A \text{ and } f(x) \text{ defined} \\ B & \text{otherwise} \end{cases}$$

Then FG is the identity on **Part** but GF is not. However there is an isomorphism

$$(A, a) \rightarrow ((A \setminus \{a\})^+, A \setminus \{a\})$$

sending a to $A \setminus \{a\}$ and everything else to itself. This is natural.

Note that there can be no isomorphism $\mathbf{Set}_* \rightarrow \mathbf{Part}$ since **Part** has a 1-element isomorphism class $\{\emptyset\}$ and **Set*** doesn't.

2. The category \mathbf{fdMod}_K of finite-dimensional vector spaces over K is equivalent to $\mathbf{fdMod}_K^{\text{op}}$: the functors in both directions are \cdot^* and both isomorphisms are the natural transformations given by double dual.
3. \mathbf{fdMod}_K is also equivalent to \mathbf{Mat}_K : we write $F : \mathbf{Mat}_K \rightarrow \mathbf{fdMod}_K$ by $F(n) = K^n$, and $F(A)$ is the map represented by A with respect to the standard basis. To define functor G the other way, choose a basis for each finite-dimensional vector space and define

$$G(V) = \dim V$$

$$G(V \xrightarrow{f} W) = \text{matrix representing } f \text{ w.r.t. chosen bases}$$

GF is the identity, provided we choose the standard bases for the space K^n . $FG \neq 1$ but the chosen bases give isomorphisms $FG(V) = K^{\dim V} \rightarrow V$ for each V , which form a natural isomorphism.

Example 3 illustrates a general principle: when constructing a pair of functors between equivalent categories, usually one is “canonical” and the other requires some choice, and a clever choice results in a particularly simple form for one way of composition. The next theorem abstracts away the “choice” and tells us when a functor is part of an equivalence purely by its properties.

The criterion is stated in term of “bijectivity” of functors, informally. It is generally a bad idea to look at sur/injectivity of functors on objects. Instead the correct way is to look at their behaviour on morphisms.

Definition (faithful, full, essentially surjective). Let $\mathbf{C} \xrightarrow{F} \mathbf{D}$ be a functor.

1. F is *faithful* if given $f, f' \in \text{mor } \mathbf{C}$ with $\text{dom } f = \text{dom } f', \text{cod } f = \text{cod } f'$ and $Ff = Ff'$ then $f = f'$.
2. F is *full* if given $FA \xrightarrow{g} FB$ in \mathbf{D} then there exists $A \xrightarrow{f} B$ in \mathbf{C} with $Ff = g$.
3. F is *essentially surjective* if for every $B \in \text{ob } \mathbf{D}$ there exists $A \in \text{ob } \mathbf{C}$ and an isomorphism $FA \rightarrow B$ in \mathbf{D} .

Definition. A subcategory $\mathbf{C}' \subseteq \mathbf{C}$ is *full* if the inclusion $\mathbf{C}' \rightarrow \mathbf{C}$ is a full functor.

Example. \mathbf{Gp} is a full subcategory of \mathbf{Mon} but \mathbf{Mon} is not a full subcategory of the category \mathbf{SGp} of semigroups.

Lemma 1.2. *Assuming the axiom of choice. A functor $F : \mathbf{C} \rightarrow \mathbf{D}$ is part of an equivalence $\mathbf{C} \simeq \mathbf{D}$ if and only if it's full, faithful and essentially surjective.*

Proof. Suppose given G, α, β as in the definition of equivalence of categories. Then for each $B \in \text{ob } \mathbf{D}$, β_B is an isomorphism $FGB \rightarrow B$ so F is essentially surjective.

Given $A \xrightarrow{f} B$ in \mathbf{C} , we can recover f from Ff as the via conjugation by α :

$$\begin{array}{ccc} GFA & \xrightarrow{GFf} & GFB \\ \alpha_A \uparrow & & \alpha_B \uparrow \\ A & \xrightarrow{f} & B \end{array}$$

Hence if $A \xrightarrow{f'} B$ satisfies $Ff = Ff'$ then $f = f'$.

Similarly there is a natural preimage given a morphism in \mathbf{D} . Given $FA \xrightarrow{g} FB$, define f to be the composite

$$\begin{array}{ccc} GFA & \xrightarrow{Gg} & GFB \\ \alpha_A \uparrow & & \alpha_B \uparrow \\ A & \dashrightarrow & B \end{array}$$

Then $GFf = \alpha_B f \alpha_A^{-1} = Gg$. As G is faithful for the same reasons as F , $Ff = g$.

Conversely, for each $B \in \text{ob } \mathbf{D}$, choose $GB \in \text{ob } \mathbf{C}$ and an isomorphism $\beta_B : FGB \rightarrow B$ in \mathbf{D} . Given $B \xrightarrow{g'} B'$, define $Gg' : GB \rightarrow GB'$ to be the unique morphism whose image under F is the composition

$$\begin{array}{ccc} B & \xrightarrow{g'} & B' \\ \beta_B \uparrow & & \beta_{B'} \uparrow \\ FGB & \dashrightarrow & FGB' \end{array}$$

Faithfulness implies functoriality: given $B' \xrightarrow{g''} B''$, $(Gg'')(Gg')$ and $G(g'g')$ have the same image under F so they are equal.

By construction, β is a natural transformation $FG \rightarrow 1_{\mathbf{D}}$.

Given $A \in \text{ob } \mathbf{C}$, define $\alpha_A : A \rightarrow GFA$ to be the unique morphism whose image under F is

$$FA \xrightarrow{\beta_{FA}^{-1}} FGFA.$$

α_A is an isomorphism since β_{FA} also has a unique preimage under F . Finally α is a natural transformation, since any naturality square for α is mapped by F to a commutative square (corresponding to naturality square for β) and F is faithful. \square

Note that axiom of choice is only used in the if part. The lemma is useful as it saves us from making explicit choices when showing an equivalence by exhibiting inverses. However note that the choice is always required.

Definition (skeleton). By a *skeleton* of a category \mathbf{C} we mean a full subcategory \mathbf{C}_0 containing one object from each isomorphism class. We say \mathbf{C} is *skeletal* if it's a skeleton of itself.

Example. \mathbf{Mat}_K is skeletal and the image of $F : \mathbf{Mat}_K \rightarrow \mathbf{fdMod}_K$ is a skeleton of \mathbf{fdMod}_K (essentially because F is full and faithful).

Remark. Almost any assertion about skeletons is equivalent to the axiom of choice. See example sheet 1 Q2. This is one reason why we should not restrict out attention to skeletal categories.

Definition (monomorphism, epimorphism). Let $A \xrightarrow{f} B$ be a morphism in \mathbf{C} .

1. We say f is a *monomorphism* or *monic* if given any pair $g, h : C \rightarrow A$, $fg = fh$ implies $g = h$.
2. We say f is a *epimorphism* or *epic* if it is a monomorphism in \mathbf{C}^{op} , i.e. given any pair $g, h : B \rightarrow C$, $gf = hf$ implies $g = h$.

We denote monomorphisms by $f : A \rightarrowtail B$ and epimorphisms by $f : A \twoheadrightarrow B$.

Any isomorphism is monic and epic. More generally if f has a left inverse then it's monic. We call such monomorphisms *split*.

Definition (balanced). We say \mathbf{C} is a *balanced* category if any morphism which is both monic and epic is an isomorphism.

Example.

1. In **Set**, monomorphism is precisely an injection (one direction is easy and for the other direction take $C = 1 = \{*\}$) and epimorphism is precisely a surjection (use morphisms $B \rightarrow 2 = \{0, 1\}$). Thus **Set** is balanced.
2. In **Gp**, monomorphism is precisely an injection (use homomorphism from the free group with one generator, i.e. $\mathbb{Z} \rightarrow A$) and epimorphism is precisely a surjection (use free product with amalgamation). Thus **Gp** is balanced.
3. In **Rng**, monomorphism is precisely an injection (similarly to free group) but the inclusion $\mathbb{Z} \rightarrow \mathbb{Q}$ is an epimorphism, since if $f, g : \mathbb{Q} \rightarrow R$ agree on all integers they agree everywhere. So **Rng** is not balanced.
4. In **Top**, monomorphism is precisely an injection and epimorphism is precisely a surjection (same argument as **Set**) but **Top** is not balanced since a continuous bijection need not to have a continuous inverse.

2 The Yoneda lemma

It may seem weird to devote an entire chapter to a lemma. However, although the Yoneda lemma is indeed a lemma, with a simple statement and straightforward proof, it is much more than a normal lemma and underlies the entire category theory.

Here is a little story about Yoneda lemma. It is named after Nobuo Yoneda, who is better known as a computer scientist than a mathematician. The result is likely not due to him and he wasn't the first person to write it down. In fact, he never actually wrote down the lemma, as opposed to some books claiming the lemma was first to be found in one of Yoneda's papers. The story, according to Saunders Mac Lane, is that after a conference he met Yoneda on a train platform. While exchanging conversations Yoneda told him the result, which Mac Lane was not aware of at that moment but immediately recognised its importance. Mac Lane later attributed the lemma to Yoneda and because of his standing in category theory, the name retains. Perhaps this is the first and only result in mathematics to be enunciated on a train platform!

Definition (locally small category). We say a category \mathbf{C} is *locally small* if, for any two objects A, B , the morphisms $A \rightarrow B$ in \mathbf{C} form a set $\mathbf{C}(A, B)$.

If we fix A and let B vary, the assignment $B \mapsto \mathbf{C}(A, B)$ becomes a functor $\mathbf{C}(A, -) : \mathbf{C} \rightarrow \mathbf{Set}$: given $B \xrightarrow{f} C$, $\mathbf{C}(A, f)$ is the mapping $g \mapsto fg$. Similarly, $A \mapsto \mathbf{C}(A, B)$ defines a functor $\mathbf{C}(-, B) : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$.

Lemma 2.1 (Yoneda lemma). *Let \mathbf{C} be a locally small category, $A \in \text{ob } \mathbf{C}$ and $F : \mathbf{C} \rightarrow \mathbf{Set}$ is a functor. Then natural transformations $\mathbf{C}(A, -) \rightarrow F$ are in bijection with elements of FA .*

Moreover, this bijection is natural in both A and F .

Proof. We prove the first part now. The second part follows very easily once we have gained some intuitions so we'll come back to it later. Given $\alpha : \mathbf{C}(A, -) \rightarrow F$, we define

$$\Phi(\alpha) = \alpha_A(1_A) \in FA.$$

Conversely, given $x \in FA$, we define $\Psi(x) : \mathbf{C}(A, -) \rightarrow F$ by

$$\Psi(x)_B(A \xrightarrow{f} B) = Ff(x) \in FB$$

which is natural: given $g : B \rightarrow C$, we have

$$\begin{aligned} \Psi(x)_C \mathbf{C}(A, g)(f) &= \Psi(x)_C(gf) = F(gf)(x) \\ (Fg)\Psi(x)_B(f) &= (Fg)(Ff)(x) = F(gf)(x) \end{aligned}$$

$$\begin{array}{ccc}
 \mathbf{C}(A, B) & \xrightarrow{\mathbf{C}(A, g)} & \mathbf{C}(A, C) \\
 \downarrow \Psi(x)_B & & \downarrow \Psi(x)_C \\
 & \begin{array}{ccc} f & \xrightarrow{\quad} & gf \\ \downarrow & & \downarrow \\ Ff(x) & \xrightarrow{\quad} & F(gf)(x) \end{array} & \\
 FB & \xrightarrow{Fg} & FC
 \end{array}$$

by functoriality of F . Now left to show they are inverses to each other.

$$\begin{aligned}
 \Phi\Psi(x) &= \Psi(x)_A(1_A) = F(1_A)(x) = x \\
 \Psi\Phi(\alpha)_B(f) &= \Psi(\alpha_A(1_A))_B(f) = Ff(\alpha_A(1_A)) = \alpha_B\mathbf{C}(A, f)(1_A) = \alpha_B(f)
 \end{aligned}$$

so $\Psi\Phi(\alpha) = \alpha$. □

Corollary 2.2 (Yoneda embedding). *The assignment $A \mapsto \mathbf{C}(A, -)$ defines a full faithful functor $\mathbf{C}^{\text{op}} \rightarrow [\mathbf{C}, \mathbf{Set}]$.*

Proof. Put $F = \mathbf{C}(B, -)$ in the above proof, we get a bijection between $\mathbf{C}(B, A)$ and morphisms $\mathbf{C}(A, -) \rightarrow \mathbf{C}(B, -)$ in $[\mathbf{C}, \mathbf{Set}]$. We need to verify that this is functorial. But it sends $f : B \rightarrow A$ to the natural transformation $g \mapsto gf$. So functoriality follows from associativity. □

We call this functor (or the functor $\mathbf{C} \rightarrow [\mathbf{C}^{\text{op}}, \mathbf{Set}]$ sending A to $\mathbf{C}(-, A)$) the *Yoneda embedding* of \mathbf{C} and typically denote it by Y . At first glance, $[\mathbf{C}, \mathbf{Set}]$ seems like a much more complicated entity and is much more unwieldy. However, it stands out as being more concrete and thus easier to deal with. It is in analogy with group representation: instead of an abstract group, we consider its action on a set, which is more explicit and concrete. We'll come back to this point in a minute.

Now return to the second part of the lemma. Suppose for the moment that \mathbf{C} is small, so that $[\mathbf{C}, \mathbf{Set}]$ is locally small. Then we have two functors $\mathbf{C} \times [\mathbf{C}, \mathbf{Set}] \rightarrow \mathbf{Set}$: one sends (A, F) to FA , and the other is the composite

$$\mathbf{C} \times [\mathbf{C}, \mathbf{Set}] \xrightarrow{Y \times 1} [\mathbf{C}, \mathbf{Set}]^{\text{op}} \times [\mathbf{C}, \mathbf{Set}] \xrightarrow{[\mathbf{C}, \mathbf{Set}](-, -)} \mathbf{Set}$$

where the last map maps a pair of functors to the set of natural transformations between them, and the naturality in the statement of Yoneda lemma says that these are naturally isomorphic. We can translate this into an elementary statement, making sense even when \mathbf{C} isn't small: given $A \xrightarrow{f} B$ and $F \xrightarrow{\alpha} G$, there are two ways of producing an element of GB from the natural transformation. For example, given $\beta : \mathbf{C}(A, -) \rightarrow F$, the two ways give the same result, namely

$$\alpha_B(Ff)\beta_A(1_A) = (Gf)\alpha_A\beta_A(1_A)$$

which is equal to $\alpha_B \beta_B(f)$.

$$\begin{array}{ccc}
 [\mathbf{C}, \mathbf{Set}](\mathbf{C}(A, -), F) & \xrightarrow{\quad} & [\mathbf{C}, \mathbf{Set}](\mathbf{C}(B, -), F) \\
 \downarrow & \searrow^{\Phi_{A,F}} & \swarrow_{\Phi_{B,F}} \\
 & FA \xrightarrow{Ff} FB & \\
 & \downarrow \alpha_A \quad \downarrow \alpha_B & \\
 & GB \xrightarrow{Gf} GB & \\
 \swarrow_{\Phi_{A,G}} & & \searrow_{\Phi_{B,G}} \\
 [\mathbf{C}, \mathbf{Set}](\mathbf{C}(A, -), G) & \xrightarrow{\quad} & [\mathbf{C}, \mathbf{Set}](\mathbf{C}(B, -), G)
 \end{array}$$

Definition (representable functor, representation). We say a functor $F : \mathbf{C} \rightarrow \mathbf{Set}$ is *representable* if it's isomorphic to $\mathbf{C}(A, -)$ for some A .
 By a *representation* of F , we mean a pair (A, x) where $x \in FA$ is such that $\Psi(x)$ is an isomorphism. We also call x a *universal element* of F .

Corollary 2.3. *If (A, x) and (B, y) are both representations of F , then there is a unique isomorphism $f : A \rightarrow B$ such that $(Ff)(x) = y$.*

Proof. Consider the composite

$$\mathbf{C}(B, -) \xrightarrow{\Psi(y)} F \xrightarrow{\Psi(x)^{-1}} \mathbf{C}(A, -),$$

By Yoneda embedding, this is of the form $Y(f)$ for a unique isomorphism $f : A \rightarrow B$ and the diagram

$$\begin{array}{ccc}
 \mathbf{C}(B, -) & \xrightarrow{Y(f)} & \mathbf{C}(A, -) \\
 \searrow_{\Psi(y)} & & \swarrow_{\Psi(x)} \\
 & F &
 \end{array}$$

commutes if and only if $(Ff)(x) = y$. □

Example.

1. The forgetful functor $\mathbf{Gp} \rightarrow \mathbf{Set}$ is representable by $(\mathbb{Z}, 1)$. Similarly the forgetful functor $\mathbf{Rng} \rightarrow \mathbf{Set}$ is representable by $(\mathbb{Z}[x], x)$. The forgetful functor $\mathbf{Top} \rightarrow \mathbf{Set}$ is representable by $(\{*\}, *)$.
2. The functor $P^* : \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}$ is representable by $(\{0, 1\}, \{1\})$. This is the bijection between subsets and characteristic functions.
3. Let G be a group. The unique (up to isomorphism) representable functor $G(*, -) : G \rightarrow \mathbf{Set}$ is the *Cayley representation* of G , i.e. the set UG with G acting by left multiplication.

4. Let A and B be two objects of a locally small category \mathbf{C} . Then we have a functor $\mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$ sending C to $\mathbf{C}(C, A) \times \mathbf{C}(C, B)$ (note that it is a purely categorical product and require only cartesian product of the morphism sets). A representation of this, if it exists, is called a (categorical) *product* of A and B , and denoted

$$\begin{array}{ccc} & A \times B & \\ \pi_1 \swarrow & & \searrow \pi_2 \\ A & & B \end{array}$$

This pair has the property that, for any pair $(C \xrightarrow{f} A, C \xrightarrow{g} B)$, there is a unique $C \xrightarrow{h} A \times B$ with $\pi_1 h = f$ and $\pi_2 h = g$.

$$\begin{array}{ccccc} & & A \times B & & \\ & \pi_1 \swarrow & \uparrow h & \searrow \pi_2 & \\ A & \xleftarrow{f} & C & \xrightarrow{g} & B \end{array}$$

Products exist in many categories of interest: in **Set**, **Gp**, **Rng**, **Top**, ... they are “just” cartesian products. In posets they are binary meets.

Dually, we have the notion of *coproduct*

$$\begin{array}{ccc} & A + B & \\ \nu_1 \swarrow & & \nwarrow \nu_2 \\ A & & B \end{array}$$

These also exist in many categories of interest.

5. Let $f, g : A \rightarrow B$ be morphisms in a locally small category \mathbf{C} . We have a functor $F : \mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$ defined by

$$F(C) = \{h \in \mathbf{C}(C, A) : fh = gh\},$$

which is a subfunctor of $\mathbf{C}(-, A)$. A representation of F , if it exists, is called an *equaliser* of (f, g) . It consists of an object E and a morphism $E \xrightarrow{e} A$ such that $fe = ge$ and every h with $fh = gh$ factors uniquely through e . In **Set**, we take $E = \{x \in A : f(x) = g(x)\}$ and e to be inclusion. Similar constructions work in **Gp**, **Rng**, **Top**, ...

$$\begin{array}{ccccc} E & \xrightarrow{e} & A & \xrightarrow[f]{g} & B \\ \uparrow \text{---} h & \nearrow & & & \\ C & & & & \end{array}$$

Dually we have the notion of *coequaliser*.

Remark. If e occurs as an equaliser then it is a monomorphism, since any h factors through it in at most one way. We say a monomorphism is *regular* if it occurs as an equaliser.

Split monomorphisms are regular (see example sheet 1 Q6 (i)). Note that a regular mono that is also epic implies isomorphism: if the equaliser e of (f, g) is epic then $f = g$ so $e \cong 1_{\text{cod } e}$.

Definition (separating/detecting family, separator, detector). Let \mathbf{C} be a category, \mathcal{G} a class of objects of \mathbf{C} .

1. We say \mathcal{G} is a *separating family* for \mathbf{C} if, given $f, g : A \rightarrow B$ such that $fh = gh$ for all $G \xrightarrow{h} A$ with $G \in \mathcal{G}$ then $f = g$. (i.e. the functor $\mathbf{C}(G, -)$ where $G \in \mathcal{G}$ are collectively faithful)
 2. We say \mathcal{G} is a *detecting family* for \mathbf{C} if, given $A \xrightarrow{f} B$ such that every $G \xrightarrow{h} B$ with $G \in \mathcal{G}$ factors uniquely through f , then f is an isomorphism.
- If $\mathcal{G} = \{G\}$ then we call G a *separator* or *detector*.

Lemma 2.4.

1. If \mathbf{C} is a balanced category then any separating family is detecting.
2. If \mathbf{C} has equalisers (i.e. every pair has an equaliser) then any detecting family is separating.

Proof.

1. Suppose \mathcal{G} is separating and $A \xrightarrow{f} B$ satisfies condition in definition 2. If $g, h : B \rightarrow C$ satisfy $gf = hf$, then $gx = hx$ for every $G \xrightarrow{x} B$, so $g = h$, i.e. f is epic.
Similarly if $k, \ell : D \rightarrow A$ satisfy $fk = f\ell$ then $ky = \ell y$ for any $G \xrightarrow{y} D$, since both are factorisations of $fk y$ through f . So $k = \ell$, i.e. f is monic.
2. Suppose \mathcal{G} is detecting and $f, g : A \rightarrow B$ satisfy definition 1. Then the equaliser $E \xrightarrow{e} A$ of (f, g) is isomorphism so $f = g$.

□

Example.

1. In $[\mathbf{C}, \mathbf{Set}]$ the family $\{\mathbf{C}(A, -) : A \in \text{ob } \mathbf{C}\}$ is both separating and detecting. This is just a restatement of Yoneda lemma.
2. In \mathbf{Set} , $1 = \{*\}$ is both a separator and a detector since it represents the identity functor $\mathbf{Set} \rightarrow \mathbf{Set}$. Similarly \mathbb{Z} is both in \mathbf{Gp} since it represents the forgetful functor $\mathbf{Gp} \rightarrow \mathbf{Set}$.
Dually, $2 = \{0, 1\}$ is a coseparator and a codetector in \mathbf{Set} since it represents $P^* : \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}$.
3. In \mathbf{Top} , $1 = \{*\}$ is a separator since it represents the forgetful functor $\mathbf{Top} \rightarrow \mathbf{Set}$, but not a detector. In fact \mathbf{Top} has no detecting *set* of objects: for any infinite cardinality κ , let X be a discrete space of cardinality κ , and Y the same set with “co- κ ” topology, i.e. $F \subseteq Y$ closed if and only if $F = Y$ or F has cardinality smaller κ . The identity map $X \rightarrow Y$ is continuous but not a homomorphism. So if $\{G_i : i \in I\}$ is any set of spaces, taking κ larger than cardinality of G_i for all i yields an example to show that the set is not detecting.

4. Let \mathbf{C} be the category of pointed connected CW-complexes and homotopy classes of (basepoint-preserving) continuous maps. J. H. C. Whitehead proved that if $X \xrightarrow{f} Y$ in this category induces isomorphisms $\pi_n(X) \rightarrow \pi_n(Y)$ for all n then it is an isomorphism in \mathbf{C} . This says that $\{S^n : n \geq 1\}$ is a detecting set for \mathbf{C} . But P. J. Freyd showed there is no faithful functor $\mathbf{C} \rightarrow \mathbf{Set}$, so no separating set: if $\{G_i : i \in I\}$ were separating then

$$X \mapsto \prod_{i \in I} \mathbf{C}(G_i, X)$$

would be faithful.

Note that any functor of the form $\mathbf{C}(A, -)$ preserves monos, but they don't preserve epis. We give a name to those special functors.

Definition (projective, injective). We say an object P is *projective* if given

$$\begin{array}{ccc} & & P \\ & \swarrow g & \downarrow f \\ A & \xrightarrow{e} & B \end{array}$$

there exists $P \xrightarrow{g} A$ with $eg = f$. If \mathbf{C} is locally small, this says $\mathbf{C}(P, -)$ preserves epimorphisms.

Dually an *injective* object of \mathbf{C} is a projective object of \mathbf{C}^{op} .

Given a class \mathcal{E} of epimorphisms, we say P is \mathcal{E} -*projective* if it satisfies the condition for all $e \in \mathcal{E}$.

Lemma 2.5. *Representable functors are (pointwise) projectives in $[\mathbf{C}, \mathbf{Set}]$.*

Proof. Suppose given

$$\begin{array}{ccc} & & \mathbf{C}(A, -) \\ & \swarrow \gamma & \downarrow \beta \\ F & \xrightarrow{\alpha} & G \end{array}$$

where α is pointwise surjective. By Yoneda, β corresponds to some $y \in GA$ and we can find $x \in FA$ with $\alpha_A(x) = y$. Now if $\gamma : \mathbf{C}(A, -) \rightarrow F$ corresponds to x , then naturality of the Yoneda bijection yields $\alpha\gamma = \beta$. \square

3 Adjunctions

Definition (adjunction). Let \mathbf{C} and \mathbf{D} be two categories and $F : \mathbf{C} \rightarrow \mathbf{D}, G : \mathbf{D} \rightarrow \mathbf{C}$ be two functors. By an *adjunction* between F and G we mean a bijection between morphisms $\hat{f} : FA \rightarrow B$ in \mathbf{D} and $f : A \rightarrow GB$ in \mathbf{C} , which is natural in A and B , i.e. given $A' \xrightarrow{g} A$ and $B \xrightarrow{h} B'$, $h\hat{f}(Fg) = \widehat{(Gh)}fg : FA' \rightarrow B'$

$$\begin{array}{ccc} FA & \xrightarrow{\hat{f}} & B \\ Fg \uparrow & & \downarrow h \\ FA' & \dashrightarrow & B' \end{array} \quad \begin{array}{ccc} A & \xrightarrow{f} & GB \\ g \uparrow & & \downarrow Gh \\ A' & \dashrightarrow & GB' \end{array}$$

We say F is *left adjoint* to G and write $F \dashv G$.

Example.

1. The functor $F : \mathbf{Set} \rightarrow \mathbf{Gp}$ is left adjoint to the forgetful functor $U : \mathbf{Gp} \rightarrow \mathbf{Set}$ since any function $f : A \rightarrow UB$ extends uniquely to a group homomorphism $\hat{f} : FA \rightarrow B$ and any homomorphism induces a set function. Naturality in B is easy and naturality in A follows from the definition of F as a functor. Similar for $\mathbf{Rng}, \mathbf{Mod}_K, \dots$
2. The forgetful functor $U : \mathbf{Top} \rightarrow \mathbf{Set}$ has a left adjoint D which equips any set with the discrete topology and a right adjoint I which equips any set with the indiscrete topology so $D \dashv U \dashv I$.
3. The functor $\text{ob} : \mathbf{Cat} \rightarrow \mathbf{Set}$ (recall that \mathbf{Cat} is the category of small categories) has a left adjoint D sending A to the discrete category with $\text{ob}(DA) = A$ and only identity morphisms, and a right adjoint I sending A to the category with $\text{ob}(IA) = A$ and one morphism $x \rightarrow y$ for each $(x, y) \in A \times A$. In this case D in turn has a left adjoint π_0 sending a small category \mathbf{C} to its set of *connected components*, i.e. the quotient of $\text{ob } \mathbf{C}$ by the smallest equivalence relation identifying $\text{dom } f$ with $\text{cod } f$ for all $f \in \text{mor } \mathbf{C}$. So $\pi_0 \dashv D \dashv \text{ob} \dashv I$.
4. Let M be the monoid $\{1, e\}$ with $e^2 = e$. An object of $[M, \mathbf{Set}]$ is a pair (A, e) where $e : A \rightarrow A$ satisfying $e^2 = e$. We have a functor $G : [M, \mathbf{Set}] \rightarrow \mathbf{Set}$ sending (A, e) to

$$\{x \in A : e(x) = x\} = \{e(x) : x \in A\}$$

and a functor $F : \mathbf{Set} \rightarrow [M, \mathbf{Set}]$ sending A to $(A, 1_A)$. Claim that

$$F \dashv G \dashv F.$$

Given $f : (A, 1_A) \rightarrow (B, e)$, it must take values in $G(B, e)$ and any $g : (B, e) \rightarrow (A, 1_A)$ is determined by its values on the image of e . In some way this is due to the two ways in which the fixed point of e can be written.

5. Let $\mathbf{1}$ be the discrete category with one object $*$. For any \mathbf{C} , there is a unique functor $\mathbf{C} \rightarrow \mathbf{1}$. A left adjoint for this picks out an *initial object*

of \mathbf{C} , i.e. an object I such that there exists a unique $I \rightarrow A$ for each $A \in \text{ob } \mathbf{C}$. Dually a right adjoint for $\mathbf{C} \rightarrow \mathbf{1}$ corresponds to a *terminal object* of \mathbf{C} .

6. Let $A \xrightarrow{f} B$ be a morphism in \mathbf{Set} . We can regard PA and PB as posets, and we have functors $Pf : PA \rightarrow PB$ and $P^*f : PB \rightarrow PA$. Claim $Pf \dashv P^*f$: we have $Pf(A') \subseteq B'$ if and only if $f(x) \in B'$ for all $x \in A'$, if and only if $A' \subseteq P^*f(B')$.
7. Galois connection: suppose given sets A and B and a relation $R \subseteq A \times B$. We define mappings \cdot^ℓ, \cdot^r between PA and PB by

$$S^r = \{y \in B : \forall x \in S, (x, y) \in R\}, S \subseteq A$$

$$T^\ell = \{x \in A : \forall y \in T, (x, y) \in R\}, T \subseteq B$$

These mappings are order-reversing, i.e. contravariant functors, and $T \subseteq S^r$ if and only if $S \times T \subseteq R$, so by symmetry if and only if $S \subseteq T^\ell$. We say \cdot^r and \cdot^ℓ are *adjoint on the right*.

8. The functor $P^* : \mathbf{Set}^{\text{op}} \rightarrow \mathbf{Set}$ is self-adjoint on the right since a function $A \rightarrow PB$ corresponds bijectively to subsets of $A \times B$, and hence by symmetry to functions $B \rightarrow PA$.

Theorem 3.1. *Let $G : \mathbf{D} \rightarrow \mathbf{C}$ be a functor. Then specifying a left adjoint functor for G is equivalent to specifying an initial object of $(A \downarrow G)$ for each $A \in \text{ob } \mathbf{C}$ where $(A \downarrow G)$ has objects pairs (B, f) with $A \xrightarrow{f} GB$ and morphisms $(B, f) \rightarrow (B', f')$ are morphisms $B \xrightarrow{g} B'$ such that the following diagram commutes*

$$\begin{array}{ccc} A & \xrightarrow{f} & GB \\ & \searrow f' & \downarrow Gg \\ & & GB' \end{array}$$

Proof. Suppose given $F \dashv G$. Consider the morphism $\eta_A : A \rightarrow GFA$ corresponding to $FA \xrightarrow{1_A} FA$. Then (FA, η_A) is an object of $(A \downarrow G)$. Moreover, given $g : FA \rightarrow B$ and $f : A \rightarrow GB$, the diagram

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ & \searrow f & \downarrow Gg \\ & & GB \end{array}$$

commutes if and only if

$$\begin{array}{ccc} FA & \xrightarrow{1_A} & FA \\ & \searrow \hat{f} & \downarrow g \\ & & B \end{array}$$

commutes by naturality condition in adjunction, i.e. $g = \hat{f}$. So (FA, η_A) is initial to $(A \downarrow G)$.

Conversely, suppose given an initial object (FA, η_A) for each $(A \downarrow G)$. Given $f : A \rightarrow A'$, we define $Ff : FA \rightarrow FA'$ to be the unique morphism making

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & GFA \\ \downarrow f & & \downarrow GFf \\ A' & \xrightarrow{\eta_{A'}} & GFA' \end{array}$$

commutes. Functoriality follows from uniqueness: given $f' : A' \rightarrow A''$, both $F(f'f)$ and $(Ff')(Ff)$ are both morphisms $(FA, \eta_A) \rightarrow (FA'', \eta_{A''}f'f)$ in $(A \downarrow G)$. To show $F \dashv G$: given $A \xrightarrow{f} GB$, we define $\hat{f} : FA \rightarrow B$ to be the unique morphism $(FA, \eta_A) \rightarrow (B, f)$ in $(A \downarrow G)$. This is a bijection with inverse

$$(FA \xrightarrow{g} B) \mapsto (A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB).$$

The latter mapping is natural in B since G is a functor, and in A since by construction η is a natural transformation $1_{\mathbf{C}} \rightarrow GF$. \square

Corollary 3.2. *If F and F' are both left adjoint to $G : \mathbf{D} \rightarrow \mathbf{C}$ then they are naturally isomorphic.*

Proof. It basically follows from the fact that initial object in any category, if exists, is unique up to isomorphism. For any A , (FA, η_A) and $(F'A, \eta'_A)$ are both initial in $(A \downarrow G)$ so there is a unique isomorphism

$$\alpha_A : (FA, \eta_A) \rightarrow (F'A, \eta'_A).$$

In any naturality square for α , the two ways round are both morphisms in $(A \downarrow G)$ whose domain is initial, so they're equal. \square

Lemma 3.3. *Given*

$$\mathbf{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathbf{D} \begin{array}{c} \xrightarrow{H} \\ \xleftarrow{K} \end{array} \mathbf{E}$$

with $F \dashv G$ and $H \dashv K$, we have

$$HF \dashv GK.$$

Proof. We have bijections between morphisms $A \rightarrow GKC$, morphisms $FA \rightarrow KC$ and morphisms $HFA \rightarrow C$, which are both natural in A and C . \square

Corollary 3.4. *Given a commutative square*

$$\begin{array}{ccc} \mathbf{C} & \longrightarrow & \mathbf{D} \\ \downarrow & & \downarrow \\ \mathbf{E} & \longrightarrow & \mathbf{F} \end{array}$$

of categories and functors, if the functors all have left adjoints, then the diagram of left adjoints commutes up to natural isomorphisms.

Proof. By Lemma 3.3, both ways round the diagram of left adjoints are left adjoint to the composite $\mathbf{C} \rightarrow \mathbf{F}$, so by Theorem 3.1 they are isomorphic. \square

Actually, we didn't use the full strength of Lemma 3.3: if we require merely instead that the original commutative diagram is only up to natural isomorphism, then we'll get the same conclusion. In practice, however, the weaker version stated above will usually suffice.

Definition (unit, counit). Given an adjunction $F \dashv G$, the natural transformation $\eta : 1_{\mathbf{C}} \rightarrow GF$ emerging in the proof of Theorem 3.1 is called the *unit* of the adjunction.

Dually we have a natural transformation $\varepsilon : FG \rightarrow 1_{\mathbf{D}}$ such that $\varepsilon_B : FGB \rightarrow B$ corresponds to $GB \xrightarrow{1_{GB}} GB$, is called the *counit*.

Theorem 3.5. Given $F : \mathbf{C} \rightarrow \mathbf{D}, G : \mathbf{D} \rightarrow \mathbf{C}$, specifying an adjunction $F \dashv G$ is equivalent to specifying two natural transformations

$$\begin{aligned} \eta : 1_{\mathbf{C}} &\rightarrow GF \\ \varepsilon : FG &\rightarrow 1_{\mathbf{D}} \end{aligned}$$

satisfying the commutative diagrams

$$\begin{array}{ccc} F & \xrightarrow{F\eta} & FGF \\ & \searrow 1_F & \downarrow \varepsilon_F \\ & & F \end{array} \qquad \begin{array}{ccc} G & \xrightarrow{\eta G} & GFG \\ & \searrow 1_G & \downarrow G\varepsilon \\ & & G \end{array}$$

which are called the triangular identities.

Proof. First suppose given $F \dashv G$. Define η and ε as in Theorem 3.1 and its dual. Now consider the composite

$$FA \xrightarrow{F\eta_A} FGF A \xrightarrow{\varepsilon_{FA}} FA.$$

Under the adjunction this corresponds to

$$A \xrightarrow{\eta_A} GFA \xrightarrow{1_{GFA}} GFA$$

but this also corresponds to 1_{FA} so $\varepsilon_{FA}F\eta_A = 1_{FA}$. The other identity is dual.

Conversely, suppose η and ε satisfying the triangular identities. Given $A \xrightarrow{f} GB$, let $\Phi(f)$ be the composite

$$FA \xrightarrow{Ff} FGB \xrightarrow{\varepsilon_B} B$$

and given $FA \xrightarrow{g} B$, let $\Psi(g)$ be

$$A \xrightarrow{\eta_A} GFA \xrightarrow{Gg} GB.$$

Then both Φ and Ψ are both natural. Need to show that $\Phi\Psi$ and $\Psi\Phi$ are identity mappings. But

$$\begin{aligned} & \Psi\Phi(A \xrightarrow{f} GB) \\ = & A \xrightarrow{\eta_A} GFA \xrightarrow{GFf} GFGB \xrightarrow{G\varepsilon_B} GB \\ = & A \xrightarrow{f} GB \xrightarrow{\eta_{GB}} GFGB \xrightarrow{G\varepsilon_B} GB \\ = & A \xrightarrow{f} GB \end{aligned}$$

where the second equality is naturality of η and the third equality is triangular equation. Dually $\Phi\Psi(g) = g$. \square

Sometimes this is taken to be the definition of adjunction.

Obviously two inverse functors form an adjunction. We have seen before a weaker notion of inverse, namely a pair of functors forming an equivalence of categories. The question is, do they always form an adjunction? The answer is yes, but sometimes we can't see it since we've chosen the wrong isomorphism.

Lemma 3.6. *Given functors $F : \mathbf{C} \rightarrow \mathbf{D}, G : \mathbf{D} \rightarrow \mathbf{C}$ and natural isomorphisms*

$$\begin{aligned} \alpha & : 1_{\mathbf{C}} \rightarrow GF \\ \beta & : FG \rightarrow 1_{\mathbf{D}} \end{aligned}$$

there are isomorphism

$$\begin{aligned} \alpha' & : 1_{\mathbf{C}} \rightarrow GF \\ \beta' & : FG \rightarrow 1_{\mathbf{D}} \end{aligned}$$

which satisfy the triangular identities so $F \dashv G$ and $G \dashv F$.

This is often summarised as “every equivalence is an adjoint equivalence”.

Proof. We fix $\alpha' = \alpha$ and modify β . We have to change the domain and codomain of β by conjugation.

Let β' be the composite

$$FG \xrightarrow{(FG\beta)^{-1}} FGFG \xrightarrow{(F\alpha_G)^{-1}} FG \xrightarrow{\beta} 1_{\mathbf{D}}.$$

Note that $FG\beta = \beta_{FG}$ since

$$\begin{array}{ccc} FGFG & \xrightarrow{FG\beta} & FG \\ \downarrow \beta_{FG} & & \downarrow \beta \\ FG & \xrightarrow{\beta} & 1_{\mathbf{D}} \end{array}$$

commutes by naturality of β and β is monic.

Now $(\beta'_F)(F\alpha')$ is the composite

$$\begin{aligned}
 & F \xrightarrow{F\alpha} FGF \xrightarrow{(\beta_{FGF})^{-1}} FGF GF \xrightarrow{(F\alpha_{GF})^{-1}} FGF \xrightarrow{\beta_F} F \\
 = & F \xrightarrow{(\beta_F)^{-1}} FGF \xrightarrow{FGF\alpha} FGF GF \xrightarrow{(F\alpha_{GF})^{-1}} FGF \xrightarrow{\beta_F} F \\
 = & F \xrightarrow{(\beta_F)^{-1}} FGF \xrightarrow{\beta_F} F \\
 = & 1_F
 \end{aligned}$$

since $GF\alpha = \alpha_{GF}$. Similarly $(G\beta')(\alpha'_G)$ is

$$\begin{aligned}
 & G \xrightarrow{\alpha_G} GFG \xrightarrow{(GFG\beta)^{-1}} GFG GF \xrightarrow{(GF\alpha_G)^{-1}} GFG \xrightarrow{G\beta} G \\
 = & G \xrightarrow{(G\beta)^{-1}} GFG \xrightarrow{\alpha_{GFG}} GFG GF \xrightarrow{(GF\alpha_G)^{-1}} GFG \xrightarrow{G\beta} G \\
 = & G \xrightarrow{(G\beta)^{-1}} GFG \xrightarrow{G\beta} G \\
 = & 1_G
 \end{aligned}$$

□

Lemma 3.7. *Suppose $G : \mathbf{D} \rightarrow \mathbf{C}$ has a left adjoint F with counit $\varepsilon : FG \rightarrow 1_{\mathbf{D}}$, then*

1. G is faithful if and only if ε is pointwise epic,
2. G is full and faithful if and only if ε is an isomorphism.

Proof.

1. Given $B \xrightarrow{g} B'$, Gg corresponds under the adjunction to the composite

$$FGB \xrightarrow{\varepsilon_B} B \xrightarrow{g} B'.$$

Hence the mapping $g \mapsto Gg$ is injective on morphisms with domain B (and specified codomain) if and only if $g \mapsto g\varepsilon_B$ is injective, if and only if ε_B is epic.

2. Similarly, G is full and faithful if and only if $g \mapsto g\varepsilon_B$ is bijective. If $\alpha : B \rightarrow FGB$ is such that $\alpha\varepsilon_B = 1_{FGB}$, i.e. α is left inverse of ε_B , then

$$\varepsilon_B\alpha\varepsilon_B = \varepsilon_B,$$

whence $\varepsilon_B\alpha = 1_B$. So ε_B is an isomorphism. Thus ε is an isomorphism.

□

Definition (reflection, reflective subcategory). By a *reflection* we mean an adjunction in which the right adjoint is full and faithful (equivalently the counit is an isomorphism).

We say a full subcategory $\mathbf{C}' \subseteq \mathbf{C}$ is *reflective* if the inclusion $\mathbf{C}' \rightarrow \mathbf{C}$ has a left adjoint.

Example.

1. The category **AbGp** of abelian groups is reflective in **Gp**: the left adjoint sends a group G to its *abelianization* G/G' , where G' is the subgroup generated by all commutators $[x, y] = xyx^{-1}y^{-1}$ for $x, y \in G$. (The unit of the adjunction is the quotient map $G \rightarrow G/G'$)

This is where the name “reflecting subcategory” comes from: mirror is 2D so reflection in a mirror cannot create a copy of a 3D object. However, it keeps every 2D detail fully and faithfully. Similarly abelianization does not tell you everything about G but as much as an abelian group can. c.f. universal property.

2. Given an abelian group, let A_T denote the torsion subgroup, i.e. the subgroup of elements of finite orders. The assignment $A \mapsto A/A_T$ gives a left adjoint to the inclusion $\mathbf{tfAbGp} \rightarrow \mathbf{AbGp}$, where \mathbf{tfAbGp} is the full subcategory of torsion-free abelian groups.

On the other hand $A \mapsto A_T$ is the right adjoint to the inclusion $\mathbf{tAbGp} \rightarrow \mathbf{AbGp}$ from torsion abelian groups to abelian groups, so this subcategory is coreflective. There are many many examples in algebra involving (co)reflective subcategories, and the constructions all give rise to important universal properties.

3. Let $\mathbf{KHaus} \subseteq \mathbf{Top}$ be the full subcategory of compact Hausdorff spaces. The inclusion $\mathbf{KHaus} \rightarrow \mathbf{Top}$ has a left adjoint β , the *Stone-Ćech compactification*. It is a reflective subcategory. We’ll revisit this example later in the course.
4. Let X be a topological space. We say $A \subseteq X$ is *sequentially closed* if $x_n \rightarrow x_\infty$ and $x_n \in A$ for all n implies $x_\infty \in A$. Note that closed implies sequentially closed but not vice versa. We say X is *sequential* if all sequentially closed sets are closed, e.g. a metric space. Given a non-sequential space X , let X_s be the same set with topology given by the sequentially open sets (complements of sequentially closed sets) in X . Certainly the identity $X_s \rightarrow X$ is continuous, and defines the counit of an adjunction between the inclusion $\mathbf{Seq} \rightarrow \mathbf{Top}$ and its right adjoint $X \mapsto X_s$.
5. If X is a topological space, the poset CX of closed subsets of X is reflective in PX with reflector given by closure and the poset OX of open subsets is coreflective with coreflector given by interior.

4 Limits

Definition (diagram, cone, limit).

1. Let \mathbf{J} be a category (almost always small, often finite). By a *diagram of shape \mathbf{J}* in \mathbf{C} we mean a functor $D : \mathbf{J} \rightarrow \mathbf{C}$. The objects $D(j)$ for $j \in \text{ob } \mathbf{J}$ are called *vertices* of the diagram and the morphisms $D(\alpha)$ $\alpha \in \text{mor } \mathbf{J}$ are called *edges* of D .
2. Given $D : \mathbf{J} \rightarrow \mathbf{C}$, a *cone* over D consists of an object A of \mathbf{C} , called the *apex* of the cone, together with morphisms $A \xrightarrow{\lambda_j} D(j)$ for each $j \in \text{ob } \mathbf{J}$, called the *legs* of the cone, such that

$$\begin{array}{ccc} & A & \\ \lambda_j \swarrow & & \searrow \lambda_{j'} \\ D(j) & \xrightarrow{D(\alpha)} & D(j') \end{array}$$

commutes for all $j \xrightarrow{\alpha} j'$ in $\text{mor } \mathbf{J}$.

Given cones $(A, (\lambda_j)_{j \in \text{ob } \mathbf{J}})$ and $(B, (\mu_j)_{j \in \text{ob } \mathbf{J}})$, a *morphism* of cones between them is a morphism $A \xrightarrow{f} B$ such that

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \lambda_j \searrow & & \swarrow \mu_j \\ & D(j) & \end{array}$$

commutes for all j .

We write $\mathbf{Cone}(D)$ for the category of all cones over D .

3. A *limit* for D is a terminal object of $\mathbf{Cone}(D)$, if this exists.

Dually we have the notion of cone *under* a diagram (sometimes called *cocone*) and of *colimit* (i.e. initial cone under D).

For example, if \mathbf{J} is the category

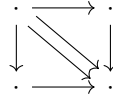
$$\begin{array}{ccc} \cdot & \longrightarrow & \cdot \\ \downarrow & \searrow & \downarrow \\ \cdot & \longrightarrow & \cdot \end{array}$$

with 4 objects and 5 non-identity morphisms, a diagram of shape \mathbf{J} is a commutative square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow g & & \downarrow h \\ C & \xrightarrow{k} & D \end{array}$$

On the other hand, to express a not-necessarily-commutative square, we use

the shape



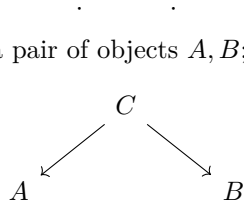
An alternative way to understand limits is that, if \mathbf{C} is locally small and \mathbf{J} is small, we have a functor $\mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$ sending A to the set of cones with apex A . A limit for D is a representation of this functor.

A third way to visualise cones: if ΔA denotes the constant diagram of shape \mathbf{J} with all vertices A and all edges 1_A , then a cone over D with apex A is the same thing as a natural transformation $\Delta A \rightarrow D$. Δ is a functor $\mathbf{C} \rightarrow [\mathbf{J}, \mathbf{C}]$ and $\mathbf{Cone}(D)$ is the arrow category $(\Delta \downarrow D)$. So to say that every diagram of shape \mathbf{J} in \mathbf{C} has a limit is equivalent to saying that Δ has a right adjoint. (We say \mathbf{C} has limits of shape \mathbf{J})

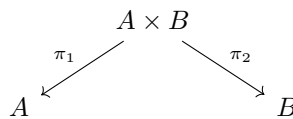
Dually \mathbf{C} has colimits of shape \mathbf{J} if and only if $\Delta : \mathbf{C} \rightarrow [\mathbf{J}, \mathbf{C}]$ has a left adjoint.

Example.

1. Suppose $J = \emptyset$. There is a unique diagram of shape \mathbf{J} in \mathbf{C} ; a cone over it is just an object, and a morphism of cones is a morphism of \mathbf{C} . So a limit for the empty diagram is a terminal object of \mathbf{C} . We defined limits in terms of terminal object but now the terminal object is also a special limit. Dually a colimit for it is an initial object.
2. Let \mathbf{J} be the category



A diagram of shape \mathbf{J} is a pair of objects A, B ; a cone over it is a span



and a limit for it is a *product*

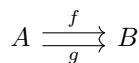
as defined in example 4 on page 12. Dually a colimit for it is a *coproduct*.

More generally, if \mathbf{J} is a small discrete category, a diagram of shape \mathbf{J} is an indexed family $(A_j : j \in \mathbf{J})$, and a limit for it is a product $(\prod_{j \in \mathbf{J}} A_j \xrightarrow{\pi_j} A_j : j \in \mathbf{J})$. Dually, $(A_j \xrightarrow{\nu_j} \sum_{j \in \mathbf{J}} A_j : j \in \mathbf{J})$, sometimes also written as $\coprod_{j \in \mathbf{J}} A_j$.

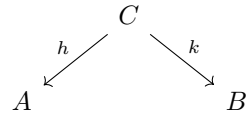
3. Let \mathbf{J} be the category



A diagram of shape \mathbf{J} is a parallel pair

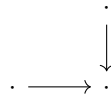


a cone over it is

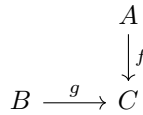


satisfying $fh = k = gh$, or equivalently a morphism $h : C \rightarrow A$ satisfying $fh = gh$. A (co)limit for the diagram is a (co)equaliser as defined in example 5 on page 12.

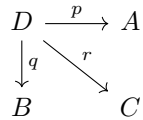
4. Let \mathbf{J} be the category



A diagram of shape \mathbf{J} is a cospan



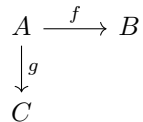
a cone over it is



satisfying $fp = r = gq$, or equivalently a span (p, q) completing the diagram to a commutative square. A limit for the diagram is called a *pullback* of (f, g) . In \mathbf{Set} , the apex of the pullback is the “fibre product”

$$A \times_C B = \{(x, y) : A \times B : f(x) = g(y)\}.$$

Dually, colimits of shape \mathbf{J}^{op} are *pushouts*. Given



we “push g along f ” to get the RHS of the colimit square.

5. Let \mathbf{J} be the poset of natural numbers. A diagram of shape \mathbf{J} is a *directed system*

$$A_0 \xrightarrow{f_0} A_1 \xrightarrow{f_1} A_2 \xrightarrow{f_2} A_3 \xrightarrow{f_3} \dots$$

A colimit for this is called a *direct limit*: it consists of A_∞ equipped with morphisms $A_n \xrightarrow{g_n} A_\infty$ satisfying $g_n = g_{n+1}f_n$ for all n and universal among such. Dually we have *inverse system* and *inverse limit*.

Theorem 4.1.

1. Suppose \mathbf{C} has equalisers and all finite (respectively small) products. Then \mathbf{C} has all finite (respectively small) limits.

2. Suppose \mathbf{C} has pullbacks and a terminal object, then \mathbf{C} has all finite limits.

Proof.

1. Suppose given $D : \mathbf{J} \rightarrow \mathbf{C}$. Form the industrial strength products

$$P = \prod_{j \in \text{ob } \mathbf{J}} D(j), Q = \prod_{\alpha \in \text{mor } \mathbf{J}} D(\text{cod } \alpha).$$

We have morphisms $f, g : P \rightarrow Q$ defined by

$$\pi_\alpha f = \pi_{\text{cod } \alpha}, \pi_\alpha g = D(\alpha) \pi_{\text{dom } \alpha}$$

for all α . Let $e : E \rightarrow P$ be an equaliser of (f, g) . The composites

$$\lambda_j = \pi_j e : E \rightarrow D(j)$$

form a cone over D : given $\alpha : j \rightarrow j'$ in \mathbf{J} ,

$$D(\alpha) \lambda_j = D(\alpha) \pi_j e = \pi_{\alpha} g e = \pi_{\alpha} f e = \pi_{j'} e = \lambda_{j'}.$$

Given any cone $(A, (\mu_j : j \in \text{ob } \mathbf{J}))$ over D , there is a unique $\mu : A \rightarrow P$ with $\pi_j \mu = \mu_j$ for each j and

$$\pi_\alpha f \mu = \mu_{\text{cod } \alpha} = D(\alpha) \mu_{\text{dom } \alpha} = \pi_\alpha g \mu$$

for all α , and hence $f \mu = g \mu$, so exists unique $\nu : A \rightarrow E$ with $e \nu = \mu$. So $(E, (\lambda_j : j \in \text{ob } \mathbf{J}))$ is a limit cone.

2. It is enough to construct finite products and equalisers. But if 1 is the terminal object, then a pullback for

$$\begin{array}{ccc} & & A \\ & & \downarrow \\ B & \longrightarrow & 1 \end{array}$$

has the universal property of a product $A \times B$ and we can form $\prod_{i=1}^n A_i$ inductively as

$$A_1 \times (A_2 \times (A_3 \times \cdots (A_{n-1} \times A_n) \cdots)).$$

Now to form the equalisers of $f, g : A \rightarrow B$, consider the cospan

$$\begin{array}{ccc} & & A \\ & & \downarrow (1_A, f) \\ A & \xrightarrow{(1_A, g)} & A \times B \end{array}$$

A cone over this consists of

$$\begin{array}{ccc} P & \xrightarrow{h} & A \\ \downarrow k & & \\ A & & \end{array}$$

satisfying $(1_A, f)h = (1_A, g)k$ or equivalently $1_A h = 1_A k$ and $fh = gk$, or equivalently a morphism $h : P \rightarrow A$ satisfying $fh = gh$. So a pullback for $(1_A, f)$ and $(1_A, g)$ is an equaliser of (f, g) .

□

Definition (complete, cocomplete). We say a category \mathbf{C} is *complete* if it has all small limits. Dually, \mathbf{C} is *cocomplete* if it has all small colimits.

For example, \mathbf{Set} is both complete and cocomplete: products are cartesian products and coproducts are disjoint unions. Similarly \mathbf{Gp} , \mathbf{AbGp} , \mathbf{Rng} , \mathbf{Mod}_K are all complete and cocomplete.¹ \mathbf{Top} is also complete and cocomplete, with both product and coproduct given by the underlying set.

Definition (preserve limit, reflect limit, create limit). Let $F : \mathbf{C} \rightarrow \mathbf{D}$ be a functor.

1. We say F *preserves limits* of shape \mathbf{J} if, given $D : \mathbf{J} \rightarrow \mathbf{C}$ and a limit cone $(L, (\lambda_j : j \in \text{ob } \mathbf{J}))$ in \mathbf{C} , $(FL, (F\lambda_j : j \in \text{ob } \mathbf{J}))$ is a limit for FD .
2. We say F *reflects limits* of shape \mathbf{J} if, given $D : \mathbf{J} \rightarrow \mathbf{C}$ and a cone $(L, (\lambda_j : j \in \text{ob } \mathbf{J}))$ such that $(FL, (F\lambda_j : j \in \text{ob } \mathbf{J}))$ is a limit for FD then $(L, (\lambda_j : j \in \text{ob } \mathbf{J}))$ for D .
3. We say F *creates limits* of shape \mathbf{J} if, given $D : \mathbf{J} \rightarrow \mathbf{C}$ and a limit $(M, (\mu_j : j \in \text{ob } \mathbf{J}))$ for FD , there exists a cone $(L, (\lambda_j : j \in \text{ob } \mathbf{J}))$ over D whose image under F is isomorphic to the limit cone, and any such cone is a limit for D .

Remark.

1. If \mathbf{C} has limits of shape \mathbf{J} and $F : \mathbf{C} \rightarrow \mathbf{D}$ preserves them and reflects isomorphisms then F reflects limits of shape \mathbf{J} .
2. F reflects limits of shape 1 if and only if F reflects isomorphism.
3. If \mathbf{D} has limits of shape \mathbf{J} and $F : \mathbf{C} \rightarrow \mathbf{D}$ creates them, then F both preserves and reflects them.
4. In any of the statement of Theorem 4.1, we may replace both instances of “ \mathbf{C} has” by either “ \mathbf{C} has and $F : \mathbf{C} \rightarrow \mathbf{D}$ preserves” or “ \mathbf{D} has and $F : \mathbf{C} \rightarrow \mathbf{D}$ creates”.

Example.

1. $U : \mathbf{Gp} \rightarrow \mathbf{Set}$ creates all small limits: given a family $(G_i : i \in I)$ of groups, there is a unique group structure on $\prod_{i \in I} UG_i$ making the projections homomorphisms, and this makes it a product in \mathbf{Gp} . Similarly for equalisers.

But U doesn't preserve coproducts: $U(G * H) \not\cong UG \amalg UH$.

2. $U : \mathbf{Top} \rightarrow \mathbf{Set}$ preserves all small limits and colimits but doesn't reflect them: if L is a limit for $D : \mathbf{J} \rightarrow \mathbf{Top}$ and L is not discrete, there is another cone with apex L_d , which is the same underlying space as L with discrete topology, mapped to the same limit in \mathbf{Set} .

¹Note that the products have underlying set the cartesian products of those of each component, but coproducts tend to be different. We'll discuss this later.

3. The inclusion functor $I : \mathbf{AbGp} \rightarrow \mathbf{Gp}$ reflects coproducts, but doesn't preserve them. The direct sum $A \oplus B$ (coproduct in \mathbf{AbGp}) is not normally isomorphic to the free product $A * B$, which is not abelian unless either A or B is trivial, but if $A \cong \{e\}$ then $A \times B \cong A \oplus B \cong B$.

The following lemma tells us how to construct limit in functor categories:

Lemma 4.2. *If \mathbf{D} has limits of shape \mathbf{J} then so does the functor category $[\mathbf{C}, \mathbf{D}]$ for any \mathbf{C} , and the forgetful functor $[\mathbf{C}, \mathbf{D}] \rightarrow \mathbf{D}^{\text{ob } \mathbf{C}}$ creates them.*

Proof. Suppose given a diagram of shape \mathbf{J} in $[\mathbf{C}, \mathbf{D}]$. Think of it as a functor $D : \mathbf{J} \times \mathbf{C} \rightarrow \mathbf{D}$. For each $A \in \text{ob } \mathbf{C}$, let $(LA, (\lambda_{j,A} : j \in \text{ob } \mathbf{J}))$ be a limit cone for the diagram $D(-, A) : \mathbf{J} \rightarrow \mathbf{D}$.

Given $A \xrightarrow{f} B$ in \mathbf{C} , the composition

$$LA \xrightarrow{\lambda_{j,A}} D(j, A) \xrightarrow{D(j,f)} D(j, B)$$

form a cone over $D(-, B)$, since the square

$$\begin{array}{ccc} D(j, A) & \xrightarrow{D(j,f)} & D(j, B) \\ \downarrow D(\alpha, A) & & \downarrow D(\alpha, B) \\ D(j', A) & \xrightarrow{D(j',f)} & D(j', B) \end{array}$$

commutes. So there is a unique $Lf : LA \rightarrow LB$ making

$$\begin{array}{ccc} LA & \xrightarrow{\lambda_{j,A}} & D(j, A) \\ \downarrow Lf & & \downarrow D(j,f) \\ LB & \xrightarrow{\lambda_{j,B}} & D(j, B) \end{array}$$

commute for all j . Uniqueness follows from functoriality: given $g : B \rightarrow C$, $L(gf)$ and $L(g)L(f)$ are factorisations of the same cone through the limit LC . And this is the unique functor structure on $A \mapsto LA$ making the $\lambda_{j,-}$ into natural transformations.

The cone $(L, (\lambda_{j,-} : j \in \text{ob } \mathbf{J}))$ is a limit: suppose given another cone $(M, (\mu_{j,-} : j \in \text{ob } \mathbf{J}))$, then for each A , $(MA, (\mu_{j,A} : j \in \text{ob } \mathbf{J}))$ is a cone over $D(-, A)$, so induces a unique $\alpha_A : MA \rightarrow LA$. Naturality of α follows from uniqueness of factorisation through a limit. So $(M, (\mu_{j,-}))$ factors uniquely through $(L, (\lambda_{j,-}))$. (This is creation in the strict sense, i.e. equality instead of isomorphism) \square

Remark. In any category, a morphism $A \xrightarrow{f} B$ is monic if and only if

$$\begin{array}{ccc} A & \xrightarrow{1_A} & A \\ \downarrow 1_A & & \downarrow f \\ A & \xrightarrow{f} & B \end{array}$$

is a pullback. Hence any functor which preserves pullbacks preserves monomorphisms. In particular if \mathbf{D} has pullbacks monomorphisms in $[\mathbf{C}, \mathbf{D}]$ are just pointwise monos. See example sheet 1 for a counterexample for the necessity of the pullback condition. Now we can delete the word ‘‘pointwise’’ in the statement of Lemma 2.5.

Theorem 4.3 (right adjoint preserves limits). *Suppose $G : \mathbf{D} \rightarrow \mathbf{C}$ has a left adjoint. Then G preserves all limits which exist in \mathbf{D} .*

Proof 1 with additional assumption. Suppose \mathbf{C} and \mathbf{D} both have limits of shape \mathbf{J} . We have a commutative diagram

$$\begin{array}{ccc} \mathbf{C} & \xrightarrow{F} & \mathbf{D} \\ \downarrow \Delta & & \downarrow \Delta \\ [\mathbf{J}, \mathbf{C}] & \xrightarrow{[\mathbf{J}, F]} & [\mathbf{J}, \mathbf{D}] \end{array}$$

where Δ sends an object to its constant diagram and $[\mathbf{J}, F]$ is composition with F . All functors in it have right adjoints (in particular, $[\mathbf{J}, F] \dashv [\mathbf{J}, G]$). So by Corollary 3.4 the diagram of right adjoints

$$\begin{array}{ccc} \mathbf{D} & \xrightarrow{G} & \mathbf{C} \\ \lim_{\mathbf{J}} \uparrow & & \lim_{\mathbf{J}} \uparrow \\ [\mathbf{J}, \mathbf{D}] & \xrightarrow{[\mathbf{J}, G]} & [\mathbf{J}, \mathbf{C}] \end{array}$$

commutes up to isomorphism, i.e. G preserves limits of shape \mathbf{J} . \square

Proof 2. Suppose given $D : \mathbf{J} \rightarrow \mathbf{D}$ and a limit cone $(L, (\lambda_j : L \rightarrow D(j) : j \in \text{ob } \mathbf{J}))$. Given a cone $(A, (\alpha_j : A \rightarrow GD(j) : j \in \text{ob } \mathbf{J}))$ over GD , the morphisms $FA \xrightarrow{\hat{\alpha}_j} D(j)$ form a cone over D , so they induce a unique $FA \xrightarrow{\hat{\beta}} L$ such that $\lambda_j \hat{\beta} = \hat{\alpha}_j$ for all j . Then $A \xrightarrow{\beta} GL$ is the unique morphism satisfying $(G\lambda_j)\beta = \alpha_j$ for all j . So $(GL, (G\lambda_j : j \in \text{ob } \mathbf{J}))$ is a limit cone in \mathbf{C} . \square

The last major theorem in this chapter is adjoint functor theorem. It says that morally the converse of the above theorem is also true, i.e. a functor preserving all limits ought to have a left adjoint. It may only fail so if some limits do not exist. The “primeval” adjoint functor theorem is exactly this: if \mathbf{D} has and $G : \mathbf{D} \rightarrow \mathbf{C}$ preserves *all* limits, then G has a left adjoint. However, this is too strong a condition as the categories having all limits can be shown to be preorders. Thus there are two more versions cut down on the all limits requirement and use some set theory to replace part of it.

Lemma 4.4. *Suppose \mathbf{D} has and $G : \mathbf{D} \rightarrow \mathbf{C}$ preserves limits of shape \mathbf{J} . Then for any $A \in \text{ob } \mathbf{C}$ the arrow category $(A \downarrow G)$ has limits of shape \mathbf{J} , and the forgetful functor $U : (A \downarrow G) \rightarrow \mathbf{D}$ creates them.*

Proof. Suppose given $D : \mathbf{J} \rightarrow (A \downarrow G)$. Write $D(j)$ as $(UD(j), f_j)$. Let $(L, (\lambda_j : L \rightarrow UD(j))_{j \in \text{ob } \mathbf{J}})$ be a limit for UD . Then $(GL, (\lambda_j)_{j \in \text{ob } \mathbf{J}})$ is a limit for GUD . Since the edges of UD are morphisms in $(A \downarrow G)$, the f_j form a cone over GUD so there is a unique $h : A \rightarrow GL$ such that $(G\lambda_j)h = f_j$ for all j , i.e. there’s a unique h such that the λ_j ’s are all morphisms in $(L, h) \rightarrow (UD(j), f_j)$ in $(A \downarrow G)$. We need to show that $((L, h), (\lambda_j)_{j \in \text{ob } \mathbf{J}})$ is a limit cone in $(A \downarrow G)$. If $(C, (\mu_j)_{j \in \text{ob } \mathbf{J}})$ is any cone over \mathbf{D} then $(C, (\mu_j)_{j \in \text{ob } \mathbf{J}})$ is a cone over UD so there is a unique $\ell : C \rightarrow L$ with $\lambda_j \ell = \mu_j$ for all j . We need to show $(G\ell)k = h$. But

$$(G\lambda_j)(G\ell)k = (G\mu_j)k = f_j = (G\lambda_j)h$$

for all j so $(G\ell)k = h$ by uniqueness of factorisations through limits. \square

Recall that we have seen a limit for the empty diagram is a terminal object. We can also consider dually the diagram of “maximal size”, namely that of a category over itself, to realise initial object as a limit. Think for example posets, in which limit for an empty diagram is maximum while limit for the

Lemma 4.5. *A category \mathbf{C} has an initial object if and only if $1_{\mathbf{C}} : \mathbf{C} \rightarrow \mathbf{C}$, regarded as a diagram of shape \mathbf{C} in \mathbf{C} , has a limit.*

Note that this is an exception to \mathbf{J} being small.

Proof. First suppose \mathbf{C} has an initial object I . Then the unique morphisms $(I \rightarrow A : A \in \text{ob } \mathbf{C})$ form a cone over $1_{\mathbf{C}}$ and given any cone $(\lambda_A : C \rightarrow A : A \in \text{ob } \mathbf{C})$, for any A the triangle

$$\begin{array}{ccc} C & \xrightarrow{\lambda_I} & I \\ & \searrow \lambda_A & \downarrow \\ & & A \end{array}$$

commutes so λ_I is the unique factorisation of $(\lambda_A : A \in \text{ob } \mathbf{C})$ through $(I \rightarrow A : A \in \text{ob } \mathbf{C})$.

Conversely, suppose $(I, (\lambda_A : I \rightarrow A)_{A \in \text{ob } \mathbf{C}})$ is a limit. Then for any $f : I \rightarrow A$ the diagram

$$\begin{array}{ccc} I & \xrightarrow{\lambda_I} & I \\ & \searrow \lambda_A & \downarrow f \\ & & A \end{array}$$

commutes. I is weakly initial as we don't know if it is unique, i.e. if $\lambda_I = 1_I$. In particular, putting $f = \lambda_A$, we see that λ_I is a factorisation of the limit cone through itself so $\lambda_I = 1_I$. Hence every $f : I \rightarrow A$ satisfies $f = \lambda_A$. \square

The primeval adjoint functor theorem follows immediately from the previous two lemmas and 3.3. However, it only applies to functors between preorders. See example sheet 2 Q6.

Theorem 4.6 (general adjoint functor theorem). *Suppose \mathbf{D} is locally small and complete. Then $G : \mathbf{D} \rightarrow \mathbf{C}$ has a left adjoint if and only if G preserves all small limits and satisfies the solution set condition, which says that for each $A \in \text{ob } \mathbf{C}$, there exists a set of morphisms $\{f_i : A \rightarrow GB_i : i \in I\}$ such that every $h : A \rightarrow GC$ factors as*

$$A \xrightarrow{f_i} GB_i \xrightarrow{Gg} GC$$

for some i and some $g : B_i \rightarrow C$.

Proof. If $F \dashv G$ then G preserves limits. To obtain the solution set, note that $\{\eta_A : A \rightarrow GFA\}$ is a singleton solution set by 3.3 since it is initial.

Conversely, by 4.10 $(A \downarrow G)$ is complete and it inherits local smallness from \mathbf{D} . We need to show that if \mathcal{A} is complete and locally small and has a weakly

initial set of objects $\{B_i : i \in I\}$ then \mathcal{A} has an initial object. First form $P = \prod_{i \in I} B_i$; then P is weakly initial. Now form the limit of

$$\begin{array}{ccc} P & \xrightarrow{\quad} & P \\ & \vdots & \\ & & P \end{array}$$

where edges are all the endomorphisms of P . Denote it $i : I \rightarrow P$. I is also weakly initial in \mathcal{A} . Suppose given $f, g : I \rightarrow C$. Form the equaliser $e : E \rightarrow I$ of (f, g) . Then there exists $h : P \rightarrow E$ since P is weakly initial. $ieh : P \rightarrow P$ and 1_P are edges of the diagram so $i = iehi$. But i is monic so $ehi = 1_I$ so e is split epic so $f = g$. Thus I is initial. \square

Example.

1. This example comes from Mac Lane. He asked the question that, if we are not given the free group functor, how can we to construct the left adjoint of forgetful functor? Consider the forgetful functor $U : \mathbf{Gp} \rightarrow \mathbf{Set}$. By 4.6a U creates all small limits so \mathbf{Gp} has them and U preserves them. \mathbf{Gp} is locally smal, given a set A , any $f : A \rightarrow UG$ factors as

$$A \rightarrow UG' \rightarrow UG$$

where G' is the subgroup generated by $\{f(x) : x \in A\}$ and $\text{card } G' \leq \max\{\aleph_0, \text{card } A\}$. Let B be a set of this cardinality. Consider all subsets $B' \subseteq B$, all group structures on B' and all mappings $A \rightarrow B'$. These give us a solution set at A .

2. Consider the category \mathbf{CLat} of complete lattices (posets with arbitrary meets and joints). Again the forgetful functor $U : \mathbf{CLat} \rightarrow \mathbf{Set}$ creates all small limits. But A. W. Hales showed in 1964 that for any cardinal κ there exists complete lattices of cardinality $\geq \kappa$ generated by three elements. So the solution set condition fails at $A = \{x, y, z\}$ as we cannot bound the cartinality of the solution set, and U doesn't have a left adjoint.

The general adjoint functor theorem is general in the sense that it applies to all categories, although it imposes solution set condition, which is a rather strong condition on the functor. Special adjoint functor theorem aims to get rid of the condition on the functor.

Definition (subobject, quotient object). By a *subobject* of an object A of \mathbf{C} we mean a monomorphism $A' \rightarrow A$. The subobjects of A are preordered by $A'' \leq A'$ if there is a factorisation

$$\begin{array}{ccc} A'' & \longrightarrow & A' \\ & \searrow & \downarrow \\ & & A \end{array}$$

Dually we have *quotient objects*.

Definition (well-powered). We say \mathbf{C} is *well-powered* if each $A \in \text{ob } \mathbf{C}$ has a set of subobjects $\{A_i \twoheadrightarrow A : i \in I\}$ such that every subobject of A is isomorphic to some A_i .

Dually if \mathbf{C}^{op} is well-powered, we say \mathbf{C} is *well-copowered*. (not cowell-powered, as it implies badly powered. It also sounds like something being powered by Simon Cowell, which is not quite what we study in this course)

For example in **Set** we can take inclusions $\{A' \hookrightarrow A : A' \in PA\}$. This is also where the name “well-powered” comes from as in the **Set** case it simply means power set exists.

Before stating and proving the special functor theorem we point out a simple yet powerful observation about pullback square.

Lemma 4.7. *Given a pullback square*

$$\begin{array}{ccc} P & \xrightarrow{h} & A \\ \downarrow k & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$$

with f monic. Then k is monic.

Proof. Suppose $x, y : D \rightarrow P$ satisfy $kx = ky$. Then

$$fhx = gkx = gky = fhy.$$

By f is monic so $hx = hy$. So x, y are factorisations of the same cone through the limit cone (h, k) . \square

Theorem 4.8 (special adjoint functor theorem). *Suppose \mathbf{C} and \mathbf{D} are both locally small, and that \mathbf{D} is complete and well-powered and has a coseparating set. Then a functor $G : \mathbf{D} \rightarrow \mathbf{C}$ has a left adjoint if and only if it preserves all small limits.*

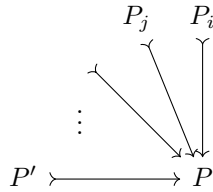
Proof. The only if is given by right adjoint preserves limits. For the other direction, for any $A \in \text{ob } \mathbf{C}$, $(A \downarrow G)$ is locally complete by 4.10, locally small, and well-powered since the subobjects of (B, f) in $(A \downarrow G)$ are just those subobjects $B' \twoheadrightarrow B$ in \mathbf{D} for which f factors through $GB' \twoheadrightarrow GB$. Also if $\{S_i : i \in I\}$ is a coseparating set for \mathbf{D} then the set

$$\{(S_i, f) : i \in I, f \in \mathbf{C}(A, GS_i)\}$$

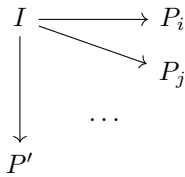
is coseparating in $(A \downarrow G)$: given $g, h : (B, f) \rightarrow (B', f')$ in $(A \downarrow G)$ with $g \neq h$, there exists $k : B' \rightarrow S_i$ for some i with $kg \neq kh$, and then k is also a morphism $(B', f') \rightarrow (S_i, (Gk)f')$ in $(A \downarrow G)$.

So we need to show that if \mathcal{A} is complete, locally small and well-powered and has a coseparating set $\{S_i : i \in I\}$ then \mathcal{A} has an initial object. Form the product $P = \prod_i S_i$. We have the industrial strength product and now we need

the industrial strength pullback. Consider the diagram

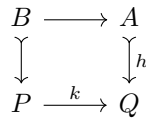


whose edges are representative set of subobjects of P and form its limit

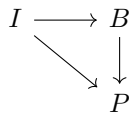


By the argument in the previous lemma, the legs of the cones are all monic; in particular $I \rightarrow P$ is monic, and it's a least subobject of P . Hence I has no proper subobjects. So given $f, g : I \rightarrow A$ their equaliser is an isomorphism and hence $f = g$.

We get uniqueness for free but need to work harder to show existence. Now let A be any object of \mathcal{A} . Form the product $Q = \prod_{i \in I, f \in \mathcal{A}(A, S_i)} S_i$. There is an obvious $h : A \rightarrow Q$ defined by $\pi_{i,f} h = f$; and h is monic, since the S_i 's are a coseparating set. We also have a morphism $k : P \rightarrow Q$ defined by $\pi_{i,f} k = \pi_i$. Now form the pullback



by lemma P is monic so B is a subobject of P . Hence there exists



and hence a morphism $I \rightarrow B \rightarrow A$. □

This result is due to Freyd, who first published it in a book as an exercise to the readers (!).

Example. Consider the inclusion $I : \mathbf{KHaus} \rightarrow \mathbf{Top}$, where \mathbf{KHaus} is the full subcategory of compact Hausdorff spaces. \mathbf{KHaus} has and I preserves small products (by Tychonoff's theorem) and equalisers (since equalisers of pairs $f, g : X \rightarrow Y$ with Y Hausdorff are closed subspaces). Both categories are locally small and \mathbf{KHaus} is well-powered (subobjects of X are isomorphic to closed subspaces). The closed interval $[0, 1]$ is a coseparator in \mathbf{KHaus} by Urysohn's lemma. So by special adjoint functor theorem I has a left adjoint β , the Stone-Ćech compactification.

Remark.

1. Čech's construction of β is as follow: given X , form

$$P = \prod_{f: X \rightarrow [0,1]} [0,1]$$

and define $h : X \rightarrow P$ by $\pi_f h = f$. Define βX to be the closure of the image of h

Čech's proof that this works is essentially the same as SAFT.

2. We could have used GAFT to construct β by a cardinality argument: we get a solution set at X by considering all continuous $f : X \rightarrow Y$ with Y compact Hausdorff and $f(X)$ dense in Y and such Y have cardinality $\leq 2^{\text{card } X}$.

5 Monad

The idea of a monad is what is left in an adjunction when you cannot see one of the categories. Suppose given $f : \mathbf{C} \rightarrow \mathbf{D}, g : \mathbf{D} \rightarrow \mathbf{C}$ with $F \dashv G$. How much of this structure can we describe without mentioning \mathbf{D} ? We have

1. the functor $T = GF : \mathbf{C} \rightarrow \mathbf{C}$,
2. the unit $\eta : 1_{\mathbf{C}} \rightarrow T = GF$,
3. and “shadow” of counit as a natural transformation $\mu = G\varepsilon_F : TT = GF GF \rightarrow GF = T$

satisfying the commutative diagrams 1, 2

$$\begin{array}{ccc} T & \xrightarrow{T\eta} & TT & \xleftarrow{\eta_T} & T \\ & \searrow & \downarrow \mu & \swarrow & \\ & 1_T & & & 1_T \\ & & T & & \end{array}$$

by the triangle inequalities, and 3

$$\begin{array}{ccc} TTT & \xrightarrow{T\mu} & TT \\ \downarrow \mu_T & & \downarrow \mu \\ TT & \xrightarrow{\mu} & T \end{array}$$

by naturality of ε .

Definition (monad). A *monad* $\mathbb{T} = (T, \eta, \mu)$ on a category \mathbf{C} consists of a functor $T : \mathbf{C} \rightarrow \mathbf{C}$ and natural transformations $\eta : 1_{\mathbf{C}} \rightarrow T, \mu : TT \rightarrow T$ satisfying commutative diagrams 1 - 3.

η and μ are called the *unit* and *multiplication* of \mathbb{T} .

The name “monad” was used because of the similarity of axioms of monad with those of monoid. Before that, although being well-known to mathematicians, monad didn’t really have an identifier. It goes by the name “standard construction”, then “triple”, thereby the letter \mathbb{T} . But both are confessions of failure to come up with a meaningful name! Someone invented the name “monad” and Mac Lane popularised it.

Example.

1. Any adjunction $F \dashv G$ induces a monad $(GF, \eta, G\varepsilon_F)$ on \mathbf{C} and a *comonad* $(FG, \varepsilon, F\varepsilon_G)$ on \mathbf{D} .
2. Let M be a monoid. The functor $(M \times -) : \mathbf{Set} \rightarrow \mathbf{Set}$ has a monad structure with unit given by $\eta_A(a) = (a_M, a)$ and multiplication $\mu_A(m, m', a) = (mm', a)$. The monad identities follow from the monoid ones.
3. Let \mathbf{C} be any category with finite products and $A \in \text{ob } \mathbf{C}$. The functor $(A \times -) : \mathbf{C} \rightarrow \mathbf{C}$ has a comonad structure with counit $\varepsilon_B : A \times B \rightarrow B$ given by π_2 and comultiplication $\delta_B : A \times B \rightarrow A \times A \times B$ given by (π_1, π_1, π_2) .

Does every monad comes from an adjunction? The answer is yes and is given independently in 1965 by Eilenberg, Moore and Kleisli. We will cover both.

In example 2 above we have the category $[M, \mathbf{Set}]$. Its forgetful functor to \mathbf{Set} has a left adjoint, sending A to $M \times A$ with M acting by multiplication on the left factor. This adjunction gives rise to the monad.

Definition (Eilenberg-Moore algebra). Let \mathbb{T} be a monad on \mathbf{C} . A \mathbb{T} -algebra is a pair (A, α) with $A \in \text{ob } \mathbf{C}$ and $\alpha : TA \rightarrow A$ satisfying commutative diagrams 4, 5

$$\begin{array}{ccc} A & \xrightarrow{\eta_A} & TA \\ & \searrow 1_A & \downarrow \alpha \\ & & A \end{array} \quad \begin{array}{ccc} TTA & \xrightarrow{T\alpha} & TA \\ & \downarrow \mu_A & \downarrow \alpha \\ TA & \xrightarrow{\alpha} & A \end{array}$$

A homomorphism $f : (A, \alpha) \rightarrow (B, \beta)$ is a morphism $f : A \rightarrow B$ such that diagram 6

$$\begin{array}{ccc} TA & \xrightarrow{Tf} & TB \\ \downarrow \alpha & & \downarrow \beta \\ A & \xrightarrow{f} & B \end{array}$$

commutes.

The category of \mathbb{T} -algebras is denoted $\mathbf{C}^{\mathbb{T}}$.

Lemma 5.1. *The forgetful functor $G^{\mathbb{T}} : \mathbf{C}^{\mathbb{T}} \rightarrow \mathbf{C}$ has a left-adjoint $F^{\mathbb{T}}$ and the adjunction induces \mathbb{T} .*

Proof. We define a “free” \mathbb{T} -algebra functor, mimicking that in monoid case. Define $F^{\mathbb{T}}A = (TA, \mu_A)$ (an algebra by 2, 3) and $F^{\mathbb{T}}(A \xrightarrow{f} B) = Tf$ (a homomorphism by naturality of μ). Clearly $G^{\mathbb{T}}F^{\mathbb{T}} = T$, the unit of the adjunction is η . We define the counit $\varepsilon_{(A, \alpha)} = \alpha : (TA, \mu_A) \rightarrow (A, \alpha)$ (a homomorphism by 5) and is natural by 6. The triangle identities

$$\begin{aligned} \varepsilon_{FA}(F\eta_A) &= 1_{FA} \\ G\varepsilon_{(A, \alpha)}\eta_A &= 1_A \end{aligned}$$

are given by 1 and 4. Finally, the monad induced by $F^{\mathbb{T}} \dashv G^{\mathbb{T}}$ has functor T and unit η , and

$$G^{\mathbb{T}}\varepsilon_{F^{\mathbb{T}}A} = \mu_A$$

by definition of $F^{\mathbb{T}}A$. □

Kleisli took a “minimalist” approach: if $F : \mathbf{C} \rightarrow \mathbf{D}, G : \mathbf{D} \rightarrow \mathbf{C}$ induces \mathbb{T} then so does $F : \mathbf{C} \rightarrow \mathbf{D}', G1_{\mathbf{D}'} : \mathbf{D}' \rightarrow \mathbf{C}$ where \mathbf{D}' is the full subcategory of \mathbf{D} on objects FA . So in trying to construct \mathbf{D} , we may assume F is surjective (or indeed bijective) on objects. But then morphisms $FA \rightarrow FB$ correspond bijectively to morphisms $A \rightarrow GFB = TB$ in \mathbf{C} . This leads us half way through as we still have to specify how to compose morphisms (in general domains and codomains don’t match up).

Definition (Kleisli category). Given a monad \mathbb{T} on \mathbf{C} , the *Kleisli category* $\mathbf{C}_{\mathbb{T}}$ has $\text{ob } \mathbf{C}_{\mathbb{T}} = \text{ob } \mathbf{C}$ and morphisms $A \xrightarrow{\text{blue}} B$ (we use blue arrow to signify morphism in $\mathbf{C}_{\mathbb{T}}$) are $A \rightarrow TB$ in \mathbf{C} . The composite $A \xrightarrow{f} B \xrightarrow{g} C$ is

$$A \xrightarrow{f} TB \xrightarrow{Tg} TTC \xrightarrow{\mu_C} C$$

and the identity $A \xrightarrow{\text{blue}} A$ is $A \xrightarrow{\eta_A} TA$.

To verify associativity, suppose given $A \xrightarrow{f} B \xrightarrow{g} C \xrightarrow{h} D$ then

$$\begin{array}{ccccccc} A & \xrightarrow{f} & TB & \xrightarrow{Tg} & TTC & \xrightarrow{TT h} & TTTD & \xrightarrow{T\mu D} & TTD \\ & & & & \downarrow \mu_C & & \downarrow \mu_{TD} & & \downarrow \mu_D \\ & & & & TC & \xrightarrow{Th} & TTD & \xrightarrow{\mu_D} & TD \end{array}$$

commutes: the upper way monad is $(hg)f$ and the lower is $h(gf)$. (used diagram 3 in rightmost square)

The unit laws similarly follow from

$$\begin{array}{ccc} A \xrightarrow{f} TB & \xrightarrow{T\eta_B} & TTB \\ & \searrow 1_{TB} & \downarrow \mu_B \\ & & TB \end{array} \quad \begin{array}{ccc} A \xrightarrow{f} TB & & \\ \downarrow \eta_A & & \downarrow 1_{TB} \\ TA \xrightarrow{Tf} TTB & \xrightarrow{\mu_B} & TB \end{array}$$

(used diagram 1 and 2 respectively in the triangles)

Lemma 5.2. *There exists an adjunction $F_{\mathbb{T}} : \mathbf{C} \rightarrow \mathbf{C}_{\mathbb{T}}, G_{\mathbb{T}} : \mathbf{C}_{\mathbb{T}} \rightarrow \mathbf{C}$ inducing the monad \mathbb{T} .*

Proof. We define $F_{\mathbb{T}}A = A, F_{\mathbb{T}}(A \xrightarrow{f} B) = A \xrightarrow{f} B \xrightarrow{\eta_B} TB$. $F_{\mathbb{T}}$ preserves identities by definition. For composites, consider $A \xrightarrow{f} B \xrightarrow{g} C$, we get

$$\begin{array}{ccc} A \xrightarrow{f} B & \xrightarrow{\eta_B} & TB \\ \downarrow g & & \downarrow Tg \\ C \xrightarrow{\eta_C} & TC & \xrightarrow{T\eta_C} & TTC \\ & \searrow 1_{TC} & & \downarrow \mu_C \\ & & & TC \end{array}$$

(used diagram 1 in the triangle)

We define

$$\begin{aligned} G_{\mathbb{T}}A &= TA \\ G_{\mathbb{T}}(A \xrightarrow{f} B) &= TA \xrightarrow{Tf} TTB \xrightarrow{\mu_B} TB \end{aligned}$$

$G_{\mathbb{T}}$ preserves identities by diagram 1. For composite, consider $A \xrightarrow{f} B \xrightarrow{g} C$, we get

$$\begin{array}{ccccc} TA & \xrightarrow{Tf} & TTB & \xrightarrow{TTg} & TTTC & \xrightarrow{T\mu_C} & TTC \\ & & \downarrow \mu_B & & \downarrow \mu_{TC} & & \downarrow \mu_C \\ & & TB & \xrightarrow{Tg} & TTC & \xrightarrow{\mu_C} & TC \end{array}$$

(used diagram 3 is last square)

Have

$$\begin{aligned} G_{\mathbb{T}}F_{\mathbb{T}}A &= TA \\ G_{\mathbb{T}}T_{\mathbb{T}}f &= \mu_B(T\eta_B)Tf = TF \end{aligned}$$

so we take $\eta : 1_{\mathbf{C}} \rightarrow T$ as the unit of $F_{\mathbb{T}} \dashv G_{\mathbb{T}}$. The counit $TA \xrightarrow{\varepsilon_A} A$ is 1_{TA} . To verify naturality consider the square

$$\begin{array}{ccc} TA & \xrightarrow{F_{\mathbb{T}}G_{\mathbb{T}}f} & TB \\ \downarrow \varepsilon_A & & \downarrow \varepsilon_B \\ A & \xrightarrow{f} & B \end{array}$$

This expands to

$$\begin{array}{ccccccc} TA & \xrightarrow{Tf} & TTB & \xrightarrow{\mu_B} & TB & \xrightarrow{\eta_{TB}} & TTB \\ & & & & \searrow 1_{TB} & & \downarrow \mu_B \\ & & & & & & TB \end{array}$$

(used diagram 2 in triangle) so ε is natural.

Finally, $G_{\mathbb{T}}(TA \xrightarrow{\varepsilon_A} A) = \mu_A$ so

$$G_{\mathbb{T}}(\varepsilon_A)\eta_{G_{\mathbb{T}}A} = \mu_A\eta_{TA} = 1_{TA}$$

and $(\varepsilon_{F_{\mathbb{T}}A})(F_{\mathbb{T}}\eta_A)$ is

$$\begin{array}{ccccc} A & \xrightarrow{\eta_A} & TA & \xrightarrow{\eta_{TA}} & TTA \\ & & \searrow 1_{TA} & & \downarrow \mu_A \\ & & & & TA \end{array}$$

(used diagram 1 in triangle) which is $(1_{F_{\mathbb{T}}A})$. Also $G_{\mathbb{T}}(\varepsilon_{F_{\mathbb{T}}A}) = \mu_A$ so $F_{\mathbb{T}} \dashv G_{\mathbb{T}}$ induces \mathbb{T} . \square

Theorem 5.3. *Given a monad \mathbb{T} on \mathbf{C} , let $\mathbf{Adj}(\mathbb{T})$ be the category whose objects are the adjunctions $F : \mathbf{C} \rightarrow \mathbf{D}, G : \mathbf{D} \rightarrow \mathbf{C}$ inducing \mathbb{T} , and whose morphisms*

$$(\mathbf{C} \xrightleftharpoons[G]{F} \mathbf{D}) \longrightarrow (\mathbf{C} \xrightleftharpoons[G']{F'} \mathbf{D}')$$

are functors $H : \mathbf{D} \rightarrow \mathbf{D}'$ satisfying $HF = F'$ and $G'H = G$. Then the Kleisli adjunction is an initial object of $\mathbf{Adj}(\mathbb{T})$ and Eilenberg-Moore adjunction is terminal.

Proof. Let FG be an object of $\mathbf{Adj}(\mathbb{T})$. We define the *Eilenberg-Moore comparison functor* $K : \mathbf{D} \rightarrow \mathbf{C}^{\mathbb{T}}$ by $KB = (GB, G\varepsilon_B)$ where ε is the counit of $F \dashv G$. Note this is an algebra by one of the triangular identities for $F \dashv G$ and naturality of ε , and $K(B \xrightarrow{g} B') = Gg$, a homomorphism by naturality of ε .

Clearly $G^{\mathbb{T}}K = G$ and

$$\begin{aligned} KFA &= (GFA, G\varepsilon_{FA}) = (TA, \mu_A) = F^{\mathbb{T}}A \\ KF(A \xrightarrow{f} A') &= Tf = F^{\mathbb{T}}f \end{aligned}$$

so K is a morphism of $\mathbf{Adj}(\mathbb{T})$.

Suppose $K' : \mathbf{D} \rightarrow \mathbf{C}^{\mathbb{T}}$ is another such, then since $G^{\mathbb{T}}K' = G$ we know $K'B = (GB, \beta_B)$ where β is a natural transformation $GFG \rightarrow G$. Also since $K'F = F^{\mathbb{T}}$, we have

$$\beta_{FA} = \mu_A = G\varepsilon_{FA}.$$

Now given any $B \in \text{ob } \mathbf{D}$, consider the diagram.

Also since $K'F = F^{\mathbb{T}}$ we have $\beta_{FA} = \mu_A = G\varepsilon_{FA}$.

Now given any $B \in \text{ob } \mathbf{D}$, consider the diagram

$$\begin{array}{ccc} GFGFGB & \xrightarrow{GFG\varepsilon_B} & GFGB \\ \begin{array}{c} G\varepsilon_{FGB} \downarrow \\ \beta_{FGB} \downarrow \\ G\varepsilon_{FGB} \downarrow \end{array} & & \begin{array}{c} G\varepsilon_B \downarrow \\ \beta_B \downarrow \\ G\varepsilon_B \downarrow \end{array} \\ GFGB & \xrightarrow{G\varepsilon_B} & GB \end{array}$$

Both squares commute so $G\varepsilon_B$ and β_B have the same composite with $GFG\varepsilon_B$. But this is split epic with splitting $GFG\varepsilon_B$ so $\beta = G\varepsilon$. Hence $K' = K$.

We now define the Kleisli comparison functor $L : \mathbf{C}_{\mathbb{T}} \rightarrow \mathbf{D}$ by $LA = FA$,

$$L(A \xrightarrow{f} B) = FA \xrightarrow{Ff} FGF B \xrightarrow{\varepsilon_{FB}} FB.$$

L preserves identities by one of the triangular identities for $F \dashv G$. Given $A \xrightarrow{f} B \xrightarrow{g} C$, we have

$$\begin{array}{ccccc} FA & \xrightarrow{Ff} & FGF B & \xrightarrow{FGFg} & FGF GFC & \xrightarrow{FG\varepsilon_{FC}} & FGFC \\ & & \downarrow \varepsilon_{FB} & & \downarrow \varepsilon_{FGFC} & & \downarrow \varepsilon_{FC} \\ & & FB & \xrightarrow{Fg} & FGFC & \xrightarrow{\varepsilon_{FC}} & FC \end{array}$$

Also

$$\begin{aligned} GLA &= TA = G_{\mathbb{T}}A \\ GL(A \xrightarrow{f} B) &= (G\varepsilon_{FB})(FGf) = \mu_B(Tf) = G_{\mathbb{T}}f \\ GF_{\mathbb{T}}A &= FA \\ LF_{\mathbb{T}}(A \xrightarrow{f} B) &= (\varepsilon_{FB})(F\eta_B)(Ff) = Ff \end{aligned}$$

For future reference, note that L is full and faithful: its effect on morphisms (with blue domains and codomains) is that of transposition across $F \dashv G$.

Finally for uniqueness, suppose $L' : \mathbf{C}_{\mathbb{T}} \rightarrow \mathbf{D}$ is a morphism of $\mathbf{Adj}(\mathbb{T})$. We must have $L'A = FA$ and L' maps the counit $TA \rightarrow A$ to the counit $FGFA \xrightarrow{\varepsilon_{FA}} FA$. For any $A \xrightarrow{f} B$, we have

$$f = 1_{TA}(F_{\mathbb{T}}f)$$

so $L'(f) = \varepsilon_{FA}(Ff) = Lf$. \square

If \mathbf{C} has coproducts then so does $\mathbf{C}_{\mathbb{T}}$ since $F_{\mathbb{T}}$ preserves them. But in general it has few other limits or colimits. In contrast, we have

Theorem 5.4.

1. The forgetful functor $G : \mathbf{C}^{\mathbb{T}} \rightarrow \mathbf{C}$ creates all limits which exists in \mathbf{C} .
2. If \mathbf{C} has colimits of shape \mathbf{J} then $G : \mathbf{C}^{\mathbb{T}} \rightarrow \mathbf{C}$ creates them if and only if T preserves them.

Proof.

1. Suppose given $D : \mathbf{J} \rightarrow \mathbf{C}^{\mathbb{T}}$. Write $D(j) = (GD(j), \delta_j)$ and suppose $(L, (\mu_j : L \rightarrow GD(j) : j \in \text{ob } \mathbf{J}))$ is a limit cone for GD . Then the composites

$$TL \xrightarrow{T\mu_j} TGD(j) \xrightarrow{\delta_j} GD(j)$$

form a cone over GD since the edges of GD are homomorphisms, so they induce a unique $\lambda : TL \rightarrow L$ such that $\mu_j \lambda = \delta_j(T\mu_j)$ for all j . The fact that λ is a \mathbb{T} -algebra structure on L follows from the fact that the δ_j are algebra structure and uniqueness of factorisations through limits. So $((L, \lambda), (\mu_j : j \in \text{ob } \mathbf{J}))$ is the unique lifting of the limit cone over GD to a cone over D , and it's a limit, since given a cone over D with apex (A, α) , we get a unique factorisation $A \xrightarrow{f} L$ in \mathbf{C} , and F is an algebra homomorphism by uniqueness of factorisation through L .

For \Leftarrow direction, suppose given $D : \mathbf{J} \rightarrow \mathbf{C}^{\mathbb{T}}$ as in 1, and a colimit cone $(GD(j) \xrightarrow{\mu_j} L : j \in \text{ob } \mathbf{J})$ in \mathbf{C} , then $(TGD(j) \xrightarrow{T\mu_j} TL : j \in \text{ob } \mathbf{C})$ is also a colimit cone, so the composite

$$TGD(j) \xrightarrow{f_j} GD(j) \xrightarrow{\mu_j} L$$

induces a unique $\lambda : TL \rightarrow L$. The rest of the argument is like 1. □

Definition (monadicity). Given an adjunction $F \dashv G$, we say the adjunction (or the functor G) is *monadic* if the comparison functor $K : \mathbf{D} \rightarrow \mathbf{C}^{\mathbb{T}}$ is part of an equivalence of categories.

Note that since the Kleisli comparison $\mathbf{C}_{\mathbb{T}} \rightarrow \mathbf{D}$ is full and faithful, it's part of an equivalence if and only if it (equivalently, F) is essentially surjective on objects.

Remark. Given any adjunction $F \dashv G$, for each object B of \mathbf{D} we have a diagram

$$FGFGB \begin{array}{c} \xrightarrow{FG\varepsilon_B} \\ \xleftarrow{\varepsilon_{FGB}} \end{array} FGB \xrightarrow{\varepsilon_B} B$$

with equal composites. The ‘‘primeval’’ monadicity theorem asserts that $\mathbf{C}^{\mathbb{T}}$ is characterised in $\mathbf{Adj}(\mathbb{T})$ by the fact that these diagrams are all coequalisers.

Definition (reflexivity, split coequaliser).

1. We say a parallel pair $f, g : A \rightarrow B$ is *reflexive* if there exists $B \xrightarrow{r} A$ such that $fr = gr = 1_B$.

We say \mathbf{C} has reflexive coequalisers if it has coequalisers of all reflexive pairs. Equivalently, colimits of shape

$$\begin{array}{c} \curvearrowright \\ \cdot \\ \curvearrowleft \end{array} \rightleftarrows \cdot$$

2. By a *split coequaliser diagram* we mean a diagram

$$\begin{array}{ccc} A & \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} & B & \xrightarrow{h} & C \\ & \curvearrowleft & & \curvearrowleft & \\ & t & & s & \end{array}$$

satisfying $hf = hg, hs = 1_C, gt = 1_B$ and $ft = sh$.

These equalisers imply that h is a coequaliser of (f, g) , if $B \xrightarrow{x} D$ satisfies $xf = xg$ then

$$x = xgt = xfg = xsh$$

so x factors through h and the factorisation is unique since its split monic.

Note that split coequalisers are preserved by *all* functors.

3. Given a functor $G : \mathbf{D} \rightarrow \mathbf{C}$, a parallel pair $A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B$ is called *G-split* if there exists a split coequaliser diagram

$$\begin{array}{ccc} GA & \begin{array}{c} \xrightarrow{Gf} \\ \xrightarrow{Gg} \end{array} & GB & \xrightarrow{h} & C \\ & \curvearrowleft & & \curvearrowleft & \\ & t & & s & \end{array}$$

in \mathbf{C} .

Note that $FGFGB \begin{array}{c} \xrightarrow{FG\varepsilon_B} \\ \xrightarrow{\varepsilon_{FGB}} \end{array} FGB$ is G -split, since

$$\begin{array}{ccc} GFGFGB & \begin{array}{c} \xrightarrow{GFG\varepsilon_B} \\ \xrightarrow{G\varepsilon_{FGB}} \end{array} & GFGB & \xrightarrow{G\varepsilon_B} & C \\ & \curvearrowleft & & \curvearrowleft & \\ & \eta_{GFGB} & & \eta_{GB} & \end{array}$$

Note that the aforementioned pair

$$FGFGB \begin{array}{c} \xrightarrow{FG\varepsilon_B} \\ \xrightarrow{\varepsilon_{FGB}} \end{array} FGB$$

is reflexive with $r = F\eta_{GB}$.

Lemma 5.5. *Suppose given an adjunction $\mathbf{C} \xrightleftharpoons[G]{F} \mathbf{D}$ where $F \dashv G$, inducing a monad \mathbb{T} on \mathbf{C} . Then $K : \mathbf{D} \rightarrow \mathbf{C}^{\mathbb{T}}$ has a left adjoint provided, for every \mathbb{T} -algebra (A, α) , the pair $FGFA \xrightleftharpoons[\varepsilon_{FA}]{F\alpha} FA$ has a coequaliser in \mathbf{D} .*

Proof. We define $L : \mathbf{C}^{\mathbb{T}} \rightarrow \mathbf{D}$ by taking $FA \rightarrow L(A, \alpha)$ to be a coequaliser for $(F\alpha, \varepsilon_{FA})$. Note that this is a functor $\mathbf{C}^{\mathbb{T}} \rightarrow \mathbf{D}$. Recall that K is defined by $KB = (GB, G\varepsilon_B)$. For any B , morphisms $LA \rightarrow B$ correspond bijectively to morphisms $FA \xrightarrow{f} B$ satisfying $f(F\alpha) = f(\varepsilon_{FA})$. These correspond to morphisms $A \xrightarrow{\check{f}} GB$ satisfying

$$\check{f}\alpha = Gf = G(\varepsilon_B(F\check{f})) = (G\varepsilon_B)(T\check{f})$$

i.e. to algebra homomorphisms $(A, \alpha) \rightarrow KB$. And these bijections are natural in (A, α) and in B . \square

Theorem 5.6 (precise monadicity theorem). *$G : \mathbf{D} \rightarrow \mathbf{C}$ is monadic if and only if G has a left adjoint and creates coequalisers of G -split pairs.*

Theorem 5.7 (refined/reflexive monadicity theorem). *Suppose \mathbf{D} has and $G : \mathbf{D} \rightarrow \mathbf{C}$ preserves reflexive coequalisers, and that G reflects isomorphisms and has a left adjoint. Then G is monadic.*

Proof. Theorem 5.6 \implies : sufficient to show that $G^{\mathbb{T}} : \mathbf{C}^{\mathbb{T}} \rightarrow \mathbf{C}$ creates coequalisers of $G^{\mathbb{T}}$ -split pairs. But this follows from the argument of 5.4 (2), since if $f, g : (A, \alpha) \rightarrow (B, \beta)$ is a $G^{\mathbb{T}}$ -split pairs, the coequalisers of $A \xrightleftharpoons[g]{f} B$ is preserved by T and TT .

Theorem 5.6 \longleftarrow and Theorem 5.7: Let \mathbb{T} denote the monad induced by $F \dashv G$. For any \mathbb{T} -algebra (A, α) , the pair $FGFA \xrightleftharpoons[\varepsilon_{FA}]{F\alpha} A$ is both reflexive and G -split, so has a coequaliser in \mathbf{D} and hence by Lemma 5.5, $K : \mathbf{D} \rightarrow \mathbf{C}^{\mathbb{T}}$ has a left adjoint L . Then unit of $L \dashv K$ at an algebra (A, α) , the coequaliser defining $L(A, \alpha)$ is mapped by K to the diagram

$$\begin{array}{ccc} F^{\mathbb{T}}TA \xrightleftharpoons[\mu_A]{F^{\mathbb{T}}\alpha} F^{\mathbb{T}}A & \longrightarrow & KL(A, \alpha) \\ & \searrow \alpha & \uparrow \iota_{(A, \alpha)} \\ & & (A, \alpha) \end{array}$$

and $\iota_{(A, \alpha)}$ is the factorisation of this through the $G^{\mathbb{T}}$ -split coequaliser α . But either set of hypothesis implies that G preserves the coequaliser defining $L(A, \alpha)$, so $\iota_{(A, \alpha)}$ is an isomorphism. For the counit $\xi : LKB \rightarrow B$, we have a coequaliser

$$\begin{array}{ccc} FGFGB \xrightleftharpoons[\varepsilon_{FGB}]{FG\varepsilon_B} FGB & \longrightarrow & LKB \\ & \searrow \varepsilon_B & \downarrow \xi_B \\ & & B \end{array}$$

Again, either set of hypothesis implies that ε_B is a coequaliser of $(FG\varepsilon_B, \varepsilon_{FGB})$ so ξ_B is an isomorphism. \square

Example.

1. The forgetful functors $\mathbf{Gp} \rightarrow \mathbf{Set}, \mathbf{Rng} \rightarrow \mathbf{Set}, \mathbf{Mod}_R \rightarrow \mathbf{Set} \dots$ all satisfy the hypothesis of refined monadicity theorem, for the reflexive coequalisers, use example sheet 4 Q3 which shows that if

$$A \begin{array}{c} \xrightarrow{f} \\ \rightrightarrows \\ \xrightarrow{g} \end{array} B \xrightarrow{h} C$$

is a reflexive coequaliser diagram in \mathbf{Set} then so is

$$A^n \begin{array}{c} \xrightarrow{f^n} \\ \rightrightarrows \\ \xrightarrow{g^n} \end{array} B^n \xrightarrow{h^n} C^n.$$

2. Any reflection is monadic: this follows from example sheet 3 Q3, but can also be proved using precise monadicity theorem. Let \mathbf{D} be a reflective (full) subcategory of \mathbf{C} , and suppose a pair $f, g : A \rightarrow B$ in \mathbf{D} fits into a split coequaliser diagram

$$A \begin{array}{c} \xrightarrow{f} \\ \rightrightarrows \\ \xrightarrow{g} \end{array} B \begin{array}{c} \xrightarrow{h} \\ \leftarrow s \\ \leftarrow t \end{array} C$$

in \mathbf{C} . Then t and $ft = sh$ belongs to \mathbf{D} since \mathbf{D} is full and hence s is in \mathbf{D} since it's an equaliser of $(1_B, sh)$ and \mathbf{D} is closed under limits in \mathbf{C} . Hence also $h \in \text{mor } \mathbf{D}$.

3. Consider the composite adjunction

$$\mathbf{Set} \begin{array}{c} \xrightarrow{F} \\ \leftarrow U \end{array} \mathbf{AbGp} \begin{array}{c} \xrightarrow{L} \\ \leftarrow I \end{array} \mathbf{tfAbGp}$$

These two factors are monadic by the above two examples respectively, but the composite isn't, since the monad it induces on \mathbf{Set} is isomorphic to that induced by $F \dashv U$.

4. Consider the forgetful functor $U : \mathbf{Top} \rightarrow \mathbf{Set}$. This is faithful and has both left and right adjoint (so preserves all coequalisers), but the monad induced on \mathbf{Set} is $(1, 1, 1)$ and the category of algebras is \mathbf{Set} .
5. One may get the impression that monadicity is something only shared by algebraic construction but not topological constructions. Consider the composite adjunction

$$\mathbf{Set} \begin{array}{c} \xrightarrow{D} \\ \leftarrow U \end{array} \mathbf{Top} \begin{array}{c} \xrightarrow{\beta} \\ \leftarrow I \end{array} \mathbf{KHaus}$$

We shall show that this satisfies the hypothesis of precise monadicity theorem. Let

$$X \begin{array}{c} \xrightarrow{f} \\ \rightrightarrows \\ \xrightarrow{g} \end{array} Y \begin{array}{c} \xrightarrow{h} \\ \leftarrow s \\ \leftarrow t \end{array} Z$$

be a split coequaliser in **Set** where X and Y have compact Hausdorff topologies and f, g are continuous. Note that the quotient topology on $Z \cong Y/R$ is compact, so it's the only possible candidate for a compact Hausdorff topology making h continuous.

We use the lemma from general topology: if Y is compact Hausdorff, then a quotient Y/R is Hausdorff if and only if $R \subseteq Y \times Y$ is closed. We note

$$\begin{aligned} R &= \{(y, y') : h(y) = h(y')\} \\ &= \{(y, y') : sh(y) = sh(y')\} \\ &= \{(y, y') : ft(y) = ft(y')\} \end{aligned}$$

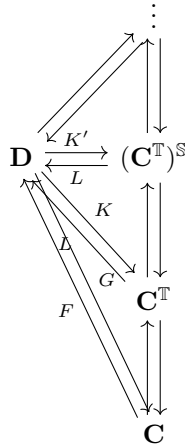
so if we define $S = \{(x, x') : f(x) = f(x')\} \subseteq X \times X$ then $R \subseteq (g \times g)(S)$, but this reverse inclusion also holds. But

$$S \longrightarrow X \times X \begin{array}{c} \xrightarrow{f\pi_1} \\ \xrightarrow{f\pi_2} \end{array} Y$$

is an equaliser, Y is Hausdorff, so S is closed on $X \times X$ and hence compact. So $R = (g \times g)(S)$ is compact and hence closed in $Y \times Y$.

Morally, a category that is monadic over **Set** can be thought as an algebraic object, in the sense that it is defined by algebraic equations. This applies to **KHaus** by including “infinitary equations”.

Definition (monadic tower). Let $\mathbf{C} \begin{array}{c} \xrightarrow{F} \\ \xleftarrow{G} \end{array} \mathbf{D}$ be an adjunction and suppose \mathbf{D} has reflexive coequalisers. The *monadic tower* of $F \dashv G$ is the diagram



where \mathbb{T} is the monad induced by $F \dashv G$, K is as in 5.7, L as in 5.11 (comparison?) \mathbb{S} is the monad induced by $L \dashv K$ and so on.

We say $F \dashv G$ has *monadic length* n if we reach an equivalence after n steps.

Monadic length 0: already equivalence. Monadic length 1: monad

For example, the adjunction of 5.14c has monadic length 1. The adjunction of 5.14d has monadic length ∞ .

We will need one more result for topos theory, which we state without proof. (lifting of left adjoint in categorical algebra)

Theorem 5.8. *Suppose given an adjunction $\mathbf{C} \xrightleftharpoons[R]{L} \mathbf{D}$ and monads \mathbb{T}, \mathbb{S} on \mathbf{C}, \mathbf{D} respectively, and a functor $\bar{R} : \mathbf{D}^{\mathbb{S}} \rightarrow \mathbf{C}^{\mathbb{T}}$ such that*

$$\begin{array}{ccc} \mathbf{D}^{\mathbb{S}} & \xrightarrow{\bar{R}} & \mathbf{C}^{\mathbb{T}} \\ \downarrow G^{\mathbb{S}} & & \downarrow G^{\mathbb{T}} \\ \mathbf{D} & \xrightarrow{R} & \mathbf{C} \end{array}$$

commutes up to isomorphism. Suppose also $\mathbf{D}^{\mathbb{S}}$ has reflexive coequalisers. Then \bar{R} has a left adjoint \bar{L} .

Proof. Note that if \bar{L} exists we must have $\bar{L}F^{\mathbb{T}} \cong F^{\mathbb{S}}L$ by 3.6. So we'd expect $\bar{L}(A, \alpha)$ to be a coequaliser of two morphisms $F^{\mathbb{S}}LTA \xrightarrow[\text{?}]{F^{\mathbb{S}}L\alpha} F^{\mathbb{S}}LA$. To construct the second morphism, note first that we assume $\text{wlog } G^{\mathbb{T}}\bar{R} = RG^{\mathbb{G}}$, by transporting \mathbb{T} -algebra structurrs along the isomorphism $G^{\mathbb{T}}R(B, \beta) \rightarrow RB$.

We obtain $\phi : TR \rightarrow RS$ by starting from

$$R \xrightarrow{R\iota} RS = RG^{\mathbb{S}}F^{\mathbb{S}} = G^{\mathbb{T}}\bar{R}F^{\mathbb{S}},$$

conjugate by F to get

$$F^{\mathbb{T}}R \rightarrow \bar{R}F^{\mathbb{S}},$$

and finally (?)

$$TR = G^{\mathbb{T}}F^{\mathbb{T}}R \xrightarrow{\phi} G^{\mathbb{T}}\bar{R}F^{\mathbb{S}} = RG^{\mathbb{S}}F^{\mathbb{S}} = RS.$$

Convert it into $\varphi : LT \rightarrow SL$ by

$$LT \xrightarrow{LT\gamma} LTRL \xrightarrow{L\theta_L} LRS L \xrightarrow{\delta_{SL}} SL$$

where γ and δ are the unit and counit of $L \dashv R$. Transposing across $F^{\mathbb{S}} \dashv G^{\mathbb{S}}$, we get $\bar{\varphi} : F^{\mathbb{S}}LT \xrightarrow{F^{\mathbb{S}}} L$. The pair $(F^{\mathbb{S}}L\alpha, \bar{\varphi}_A)$ is relexive, with common splitting $F^{\mathbb{S}}L\eta$. ($\bar{\varphi}$ is the question mark in the diagram)

It can be verified (albeit extremely tedious) that the coequaliser of this pair has the unviersal property we require for $\bar{L}(A, \alpha)$. \square

6 Cartesian closed categories

Definition (exponentiable object, cartesian closed category). Let \mathbf{C} be a category with finite products. We say $A \in \text{ob } \mathbf{C}$ is *exponentiable* if the functor $(-) \times A : \mathbf{C} \rightarrow \mathbf{C}$ has a right adjoint $(-)^A$.

If every object of \mathbf{C} is exponentiable, we say \mathbf{C} is *cartesian closed*.

Intuitively, exponential object “lifts” morphisms to an object, instead of a set. “internalisation”

Example.

1. **Set** is cartesian closed, with $B^A = \mathbf{Set}(A, B)$. A function $f : C \times A \rightarrow B$ corresponds to $\bar{f} : C \rightarrow B^A$.
2. **Cat** is cartesian closed with $\mathbf{D}^{\mathbf{C}} = [\mathbf{C}, \mathbf{D}]$. In fact we have implicitly used this idea when discussing limits in functor categories.
3. In **Top**, if an exponential Y^X exists, its points must be the continuous maps $X \rightarrow Y$. The *compact-open* topology on $\mathbf{Top}(X, Y)$ has the universal property of an exponential if and only if X is locally compact.

Note that finite products of exponential objects are exponentiable: since

$$(-) \times (A \times B) \cong (- \times A) \times B,$$

we have $(-)^{A \times B} \cong ((-)^B)^A$. However, even if X and Y are locally compact, Y^X needn't be. So the exponentiable objects don't form a cartesian closed full subcategory.

4. A cartesian closed poset is called a *Heyting semilattice*: it's a poset with finite meet \wedge and a binary operation \implies satisfying

$$a \leq (b \implies c) \text{ if and only if } a \wedge b \leq c.$$

For example, a complete poset is a Heyting semilattice if and only if it satisfies the infinite distributive law

$$a \wedge \bigvee_{i \in I} \{b_i\} = \bigvee_{i \in I} \{a \wedge b_i\}.$$

For any topological space, the lattice $\mathcal{O}(X)$ of open sets satisfies this condition, since \wedge and \vee coincide with \cap and \cup . (However it does not satisfy the dual condition: arbitrary intersection needs to take interior).

Recall from example sheet 2 that, if $B \in \text{ob } \mathbf{C}$, we define the *slice category* \mathbf{C}/B to have objects which are morphisms $A \rightarrow B$ in \mathbf{C} and morphisms are commutative triangles

$$\begin{array}{ccc} A & \longrightarrow & A' \\ & \searrow & \downarrow \\ & & B \end{array}$$

The forgetful functor $\mathbf{C}/B \rightarrow \mathbf{C}$ will be denoted Σ_B . If \mathbf{C} has finite products, Σ_B has a right adjoint B^* which sends A to

$$A \times B \xrightarrow{\pi_2} B$$

since morphisms

$$\begin{array}{ccc} C & \xrightarrow{(f,g)} & A \times B \\ & \searrow g & \downarrow \pi_2 \\ & & B \end{array}$$

correspond to morphisms $f : \Sigma_B g = C \rightarrow A$.

Lemma 6.1. *If \mathbf{C} has all finite limits then an object B is exponentiable if and only if $B^* : \mathbf{C} \rightarrow \mathbf{C}/B$ has a right adjoint Π_B .*

Proof.

1. \Leftarrow : The composite $\Sigma_B B^*$ is equal to $(-) \times B$ so we take $(-)^B$ to be $\Sigma_B B^*$.
2. \Rightarrow : If B is exponentiable, for any $f : A \rightarrow B$ we define $\Pi_B(f)$ to be the pullback

$$\begin{array}{ccc} \Pi_B(f) & \longrightarrow & A^B \\ \downarrow & & \downarrow f^B \\ 1 & \xrightarrow{\bar{\pi}_2} & B^B \end{array}$$

where $\bar{\pi}_2$ is the transpose of the projection (?). Then morphisms $C \rightarrow \Pi_B(f)$ corresponding to morphisms $C \rightarrow A^B$ making

$$\begin{array}{ccc} C & \longrightarrow & A^B \\ \downarrow & & \downarrow f^B \\ 1 & \xrightarrow{\bar{\pi}_2} & B^B \end{array}$$

commute, i.e. to morphisms $C \times B \rightarrow A$ making

$$\begin{array}{ccc} C \times B & \longrightarrow & A \\ & \searrow \pi_2 & \downarrow f \\ & & B \end{array}$$

commute.

□

Lemma 6.2. *Suppose \mathbf{C} has finite limits. If A is exponentiable in \mathbf{C} then B^*A is exponentiable in \mathbf{C}/B for any B . Moreover B^* preserves exponentials.*

Proof. Given an object $\begin{array}{c} C \\ \downarrow f \\ B \end{array}$, form the pullback

$$\begin{array}{ccc} P & \longrightarrow & C^A \\ \downarrow f^{B^*A} & & \downarrow f^A \\ B & \xrightarrow{\bar{\pi}_1} & B^A \end{array}$$

Then for any $\begin{array}{c} D \\ \downarrow g \\ B \end{array}$, morphisms $g \rightarrow f^{B^*A}$ in \mathbf{C}/B correspond to morphisms

$D \xrightarrow{\bar{h}} C^A$ making

$$\begin{array}{ccc} D & \xrightarrow{\bar{h}} & C^A \\ \downarrow g & & \downarrow f^A \\ B & \xrightarrow{\bar{\pi}_1} & B^A \end{array}$$

commute, and hence to morphisms $D \times A \xrightarrow{h} C$ making

$$\begin{array}{ccc} D \times A & \xrightarrow{h} & C \\ & \searrow g\pi_1 & \downarrow f \\ & & B \end{array}$$

commute. But

$$\begin{array}{ccc} D \times A & \xrightarrow{g \times 1_A} & B \times A \\ \downarrow \pi_1 & & \downarrow \pi_1 \\ D & \xrightarrow{g} & B \end{array}$$

is a pullback in \mathbf{C} , i.e. a product in \mathbf{C}/B .

For the second assertion, note that if $C \xrightarrow{f} B$ is of the form $B \times E \xrightarrow{\pi_1} B$ then the pullback defining f^{B^*A} becomes

$$\begin{array}{ccc} B \times E^A & \xrightarrow{\bar{\pi}_1 \times 1_{E^A}} & B^A \times E^A \\ \downarrow \pi_1 & & \downarrow \pi_1 \\ B & \xrightarrow{\bar{\pi}_1} & B^A \end{array}$$

so $f^{B^*A} \cong B^*(E^A)$. □

Remark. \mathbf{C}/B is isomorphic to the category of coalgebra for the monad structure on $(-) \times B$ (5.2c). So the first part of the lemma could be proved using lift of adjoint (last theorem of chapter 5).

Definition (locally cartesian closed). We say \mathbf{C} is *locally cartesian closed* if it has all finite limits and each \mathbf{C}/B is cartesian closed.

Note that this includes the fact that $\mathbf{C} \cong \mathbf{C}/1$ is cartesian closed. So the usage is to the contrary of normal usage of “locally” as it imposes a stricter condition.

Example.

1. **Set** is locally cartesian closed since $\mathbf{Set}/B \simeq \mathbf{Set}^B$ for any B .
2. For any small category \mathbf{C} , $[\mathbf{C}, \mathbf{Set}]$ is cartesian closed: by Yoneda

$$G^F(A) \cong [\mathbf{C}, \mathbf{Set}](\mathbf{C}(A, -), G^F) \cong [\mathbf{C}, \mathbf{Set}](\mathbf{C}(A, -) \times F, G)$$

so we take RHS as a definition of $G^F(A)$ and define G^F on morphisms $A \xrightarrow{f} B$ by composition with $\mathbf{C}(f, -) \times 1_F$. Note that the class of functors H for which we have

$$[\mathbf{C}, \mathbf{Set}](H, G^F) \cong [\mathbf{C}, \mathbf{Set}](H \times F, G)$$

is closed under colimits. But every functor $\mathbf{C} \rightarrow \mathbf{Set}$ is a colimit of representables.

In fact $[\mathbf{C}, \mathbf{Set}]$ is locally cartesian closed since all its slice categories $[\mathbf{C}, \mathbf{Set}]/F$ are of the same form. See example sheet 4 Q6.

3. Any Heyting semilattice H is locally cartesian closed since $H/b \cong \downarrow(b)$, the poset of elements $\leq b$, and $b^* = (-) \wedge b$ is surjective.
4. **Cat** is *not* locally cartesian closed, since not all strong epis are regular. c.f. example sheet 3 Q6.

Note that given $\begin{array}{c} A \\ \downarrow f \\ B \end{array}$ in \mathbf{C}/B , the iterated slice $(\mathbf{C}/B)/f$ is isomorphic to

\mathbf{C}/A , and this identifies $f^* : \mathbf{C}/B \rightarrow (\mathbf{C}/B)/f$ with the operation of pulling back morphisms along f . So by 6.3 \mathbf{C} is locally cartesian closed if and only if it has finite limits and $f^* : \mathbf{C}/B \rightarrow \mathbf{C}/A$ has a right adjoint Π_f for every $A \xrightarrow{f} B$ in \mathbf{C} . This can be taken as the definition of locally cartesian closed category.

Theorem 6.3. *Suppose \mathbf{C} is locally cartesian closed and has reflexive co-equalisers. Then every morphism $A \xrightarrow{f} B$ factors as*

$$A \xrightarrow{q} I \xrightarrow{m} B$$

where q is regular epic and m is monic.

Proof. First form the pullback

$$\begin{array}{ccc} R & \xrightarrow{a} & A \\ \downarrow b & & \downarrow f \\ A & \xrightarrow{f} & B \end{array}$$

and then form the coequaliser

$$R \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{b} \end{array} A \xrightarrow{q} I.$$

Since $fa = fb$, f factors as

$$A \xrightarrow{q} I \xrightarrow{m} B.$$

Suppose given $D \begin{array}{c} \xrightarrow{g} \\ \xrightarrow{h} \end{array} I$ with $mg = mh$, form the pullback

$$\begin{array}{ccc} E & \xrightarrow{n} & D \\ \downarrow (k,\ell) & & \downarrow (g,h) \\ A \times A & \xrightarrow{q \times q} & I \times I \end{array}$$

Since $q \times q$ factors as $(q \times 1_I)(1_A \times q)$ and both factors are pullbacks of q , it is an epimorphism and so is n . Now

$$fk = m q k = m g n = m h n = m g \ell = f \ell$$

so there exists $E \xrightarrow{p} R$ with $ap = k, bp = \ell$.

Now

$$qk = qap = qbp = q\ell,$$

i.e. $gn = hn$. But n is epic so $g = h$. Hence m is monic. \square

Note that this implies any strong epi $A \xrightarrow{f} B$ is regular since the monic part of its image factorisation is an isomorphism. In particular, regular epimorphisms are stable under composition.

Definition. If \mathbf{C} and \mathbf{D} are cartesian closed categories and $F : \mathbf{C} \rightarrow \mathbf{D}$ preserves products then for each pair of objects (A, B) of \mathbf{C} , we get a natural morphism $\theta : F(B^A) \rightarrow FB^{FA}$, namely the transpose of

$$F(B^A) \times FA \cong F(B^A \times A) \xrightarrow{F(\text{ev})} FB$$

where ev is the counit of $((-) \times A \dashv (-)^A)$.

We say F is a *cartesian closed functor* if θ is an isomorphism for every pair (A, B) . Note that the second part of 6.4 says that if \mathbf{C} is locally cartesian closed then $f^* : \mathbf{C}/B \rightarrow \mathbf{C}/A$ is a continuous cartesian functor for any $A \xrightarrow{f} B$.

Theorem 6.4. *Let \mathbf{C} and \mathbf{D} be cartesian closed categories and $F : \mathbf{C} \rightarrow \mathbf{D}$ a functor having a left adjoint L . Then F is cartesian closed if and only if the canonical morphism $\varphi_{A,B}$*

$$L(B \times FA) \xrightarrow{(L\pi_1, L\pi_2)} LB \times LFA \xrightarrow{1_{LA} \times \varepsilon_B} LB \times A$$

is an isomorphism for all $A \in \text{ob } \mathbf{C}, B \in \text{ob } \mathbf{D}$. This condition is called

▮ Frobenius reciprocity.

Note that if \mathbf{C} is locally cartesian closed then $f^* : \mathbf{C}/B \rightarrow \mathbf{C}/A$ has a left adjoint Σ_f given by composition with f , and it's easy to verify that

$$\Sigma_f(g \times f^*h) \cong \Sigma_f g \times h.$$

Proof.

- \implies : Given an inverse for $\theta : F(C^A) \rightarrow FA^{FA}$, we define $\varphi_{A,B}^{-1}$ to be the composite

$$\begin{array}{ccc} LB \times B & \xrightarrow{L\lambda \times 1} & L((B \times FA)^{FA}) \times A \xrightarrow{L(\eta^{FA}) \times 1} L(FL(B \times FA)^{FA}) \times B \\ & & \downarrow L\theta^{-1} \times 1 \\ L(B \times FA) & \xleftarrow{\text{ev}} & L(B \times FA)^A \times A \xleftarrow{\varepsilon \times 1} LF(L(B \times FA)^A) \times A \end{array}$$

The verification is a tedious exercise.

- \impliedby : Given an inverse for φ , we define θ^{-1} to be

$$\begin{array}{ccc} F((L(FC^{FA} \times A))^A) & \xleftarrow{F\lambda} & FL(FC^{FA}) \xleftarrow{\eta} FC^{FA} \\ \downarrow F(\varphi^{-1A}) & & \\ F((L(FC^{FA} \times FA))^A) & \xrightarrow{F((L(\text{ev}))^A)} & F((LFC)^A) \xrightarrow{F(\varepsilon^A)} F(C^A). \end{array}$$

□

▮ **Corollary 6.5.** *Suppose \mathbf{C} and \mathbf{D} are cartesian closed, and $F : \mathbf{C} \rightarrow \mathbf{D}$ has a left adjoint L which preserves finite products. Then F is cartesian closed if and only if F is full and faithful.*

Proof.

- \implies : L preserves 1 so if we substitute B in the definition of φ above, we get $LFA \xrightarrow{\varepsilon^A} A$. But ε is an isomorphism if and only if F is full and faithful.
- \impliedby : If L preserves binary products and ε is an isomorphism. Then both factors in the definition of φ are isomorphisms.

□

▮ **Definition** (exponential ideal). Let \mathbf{C} be a cartesian closed category. By an *exponential ideal* of \mathbf{C} we mean a class of objects (or a full subcategory) \mathcal{E} such that $B \in \mathcal{E}$ implies $B^A \in \mathcal{E}$ for all $A \in \text{ob } \mathbf{C}$.

Example.

1. We say A is *subterminal* if $A \rightarrow 1$ is monic. In any cartesian closed category \mathbf{C} , the class $\text{Sub}_{\mathbf{C}}(1)$ is an exponential ideal since A is subterminal if and only if there exists at most 1 morphisms $B \rightarrow A$ for any B , if and only if at most 1 morphism $C \times B \rightarrow A$ for any B and C , so by adjunction if and only if at most 1 morphism $C \rightarrow A^B$, if and only if A^B is subterminal. More generally, if \mathbf{C} is locally cartesian closed then $\text{Sub}_{\mathbf{C}}(A)$ is an exponential ideal in \mathbf{C}/A for any A . if \mathbf{C} also satisfies the hypotheses of 6.7 then $\text{Sub}_{\mathbf{C}}(A)$ is reflexive in \mathbf{C}/A .

2. Let X be a topological space. By a *presheaf* of X we mean a functor $F : \mathcal{O}(X)^{\text{op}} \rightarrow \mathbf{Set}$ where $\mathcal{O}(X)$ is the partial order of open subsets of X . So F has sets $F(U)$ for each open U and restriction maps $F(U) \rightarrow F(V) : x \mapsto x|_V$ whenever $V \subseteq U$.

We say F is a *sheaf* if whenever $U = \bigcup_{i \in I} U_i$ and we're given $x_i \in F(U_i)$ for each i , such that

$$x_i|_{U_i \cap U_j} = x_j|_{U_i \cap U_j}$$

for all (i, j) , then exists a unique $x \in F(U)$ such that $x|_{U_i} = x_i$ for all i .

We write $\mathbf{Sh}(X) \subseteq [\mathcal{O}(X)^{\text{op}}, \mathbf{Set}]$ for the full subcategory of sheaves. We'll show $\mathbf{Sh}(X)$ is an exponential ideal: given presheaves F, G , $G^F(U)$ is the set of natural transformations $F \times \mathcal{O}(X)(-, U) \rightarrow G$ or equivalently, the set of natural transformations $F|_U \rightarrow G|_U$ where $F|_U : \mathcal{O}(X)^{\text{op}} \rightarrow \mathbf{Set}$ is the presheaf obtained by restricting F to open sets in U . Now suppose G is a sheaf. Suppose $U = \bigcup_{i \in I} U_i$ and suppose given $\alpha_i : F|_{U_i} \rightarrow G|_{U_i}$ for each i such that

$$\alpha_i|_{U_i \cap U_j} = \alpha_j|_{U_i \cap U_j}$$

for all i, j . Given $x \in F(V)$ where $V \subseteq U$, write $V_i = V \cap U_i$, then $V = \bigcup_{i \in I} V_i$. The elements $x|_{V_i}$, $i \in I$, satisfying the compatibility condition and hence so do the elements $(\alpha_i)_{V_i}(x|_{V_i}) \in G(V_i)$. So there's a unique $y \in G(V)$ such that

$$y|_{V_i} = (\alpha_i)_{V_i}(x|_{V_i})$$

for all i and we define this to be $\alpha_V(x)$. This defines natural transformations $\alpha : F|_U \rightarrow G|_U$ and it's the unique transformation whose restriction to U_i is α_i for each i .

Note that since $\mathbf{Sh}(X)$ is closed under finite products (and in fact all limits) in $[\mathcal{O}(X)^{\text{op}}, \mathbf{Set}]$, it is itself cartesian closed.

Lemma 6.6. *Suppose \mathbf{C} is cartesian closed and $\mathbf{D} \subseteq \mathbf{C}$ is a (full) reflective subcategory, with reflector $L : \mathbf{C} \rightarrow \mathbf{D}$. Then \mathbf{D} is an exponential ideal if and only if L preserves binary products.*

Proof.

- \implies : Suppose $A, B \in \text{ob } \mathbf{C}, C \in \text{ob } \mathbf{D}$. Then we have bijections

$$\begin{aligned}
 A \times B &\rightarrow C \\
 A &\rightarrow C^B \\
 LA &\rightarrow C^B \\
 LA \times B &\rightarrow C \\
 B &\rightarrow C^{LA} \\
 LB &\rightarrow C^{LA} \\
 LA \times LA &\rightarrow C
 \end{aligned}$$

so $LA \times LB$ has the universal property of $L(A \times B)$.

- \impliedby : Suppose $B \in \text{ob } \mathbf{D}, A, C \in \text{ob } \mathbf{C}$. We have bijections

$$\begin{aligned}
 C &\rightarrow B^A \\
 C \times A &\rightarrow B \\
 LC \times LA &\cong L(C \times A) \rightarrow B \\
 L(LC \times A) &\rightarrow B \\
 LC \times A &\rightarrow B \\
 LC &\rightarrow B^A
 \end{aligned}$$

so every $C \rightarrow B^A$ factors throu $C \rightarrow LC$, hence $B^A \in \text{ob } \mathbf{D}$.

□

7 Toposes

Topos has its origin in the French school of algebraic geometry. In 1963, when studying cohomologies in geometry, Grothendieck studied toposes as categories of “generalised sheaves”. As sheaves can be seen as representation of spaces and properties of the space can be detected from sheaves, toposes become generalised spaces. Thus the name topos: something more fundamental than topology.

J. Giraud gave a characterisation of such categories by (set-theoretic) categorical properties. F. W. Lawvere and M. Tierney (1969 - 1970) investigated the elementary categorical properties of these categories and come up with the elementary definition. In fact a Grothendieck topos is exactly a Lawvere-Tierney topos which is (co)complete and locally small, and has a separating set of objects (which is what Grothendieck should, but didn't, come up with, by the way).

Definition (subobject classifier, topos, logical functor).

1. Let \mathcal{E} be a category with finite limits. A *subobject classifier* for \mathcal{E} is a monomorphism $\top : \Omega' \rightarrow \Omega$ such that for every mono $m : A' \rightarrow A$ in \mathcal{E} , there is a unique $\chi_m : A \rightarrow \Omega$ for which there is a pullback square

$$\begin{array}{ccc} A' & \longrightarrow & \Omega' \\ \downarrow m & & \downarrow \top \\ A & \xrightarrow{\chi_m} & \Omega \end{array}$$

Note that, for any A , there is a unique $A \rightarrow \Omega$ which factors through $\top : \Omega' \rightarrow \Omega$, so the domain of \top is actually a terminal object.

If \mathcal{E} is well-powered, we have a functor $\text{Sub}_{\mathcal{E}} : \mathcal{E} \rightarrow \mathbf{Set}$ sending A to the set of (isomorphism classes) of subobjects of A and acting on morphisms by pullback, and a subobject classifier is a representation of this functor.

2. A *topos* is a category which has finite limits, is cartesian closed and has a subobject classifier.
3. If \mathcal{E} and \mathcal{F} are toposes, a *logical functor* $F : \mathcal{E} \rightarrow \mathcal{F}$ is one which preserves finite limits, exponentials and the subobject classifier.

Example.

1. \mathbf{Set} is a topos, with $\Omega = \{0, 1\}$ and $\top = 1 : 1 \mapsto \{0, 1\}$. Have

$$\chi(a) = \begin{cases} 1 & a \in A \\ 0 & a \notin A \end{cases}$$

So also is the category \mathbf{Set}_f of finite set, or the category $\mathbf{Set}_{<\kappa}$ of sets of cardinality $< \kappa$, where κ is an infinite cardinal such that if $\lambda < \kappa$ then $2^\lambda < \kappa$.

2. For any small category \mathbf{C} , $[\mathbf{C}^{\text{op}}, \mathbf{Set}]$ is a topos: we've seen that it's cartesian, and Ω is determined by Yoneda:

$$\Omega(A) \cong [\mathbf{C}^{\text{op}}, \mathbf{Set}](\mathbf{C}(-, A), \Omega) \cong \{\text{subfunctors of } \mathbf{C}(-, A)\}.$$

So we define $\Omega(A)$ to be the set of *sieves* on A , i.e. sets R of morphisms with codomain A such that if $f \in R$ then $fg \in R$ for any g .

Given $f : B \rightarrow A$ and a sieve R on A , we define f^*R to be the set of g with codomain B such that $fg \in R$. This makes Ω into a functor $\mathbf{C}^{\text{op}} \rightarrow \mathbf{Set}$: $\top : 1 \rightarrow \Omega$ is defined by

$$\top_A(*) = \{\text{all morphisms with codomain } A\}.$$

Given a subfunctor $m : F' \rightarrow F$, we define $\chi_m : F \rightarrow \Omega$ by

$$(\chi_m)_A(x) = \{f : B \rightarrow A : Ff(x) \in F'(B)\}.$$

This is the unique natural transformation making

$$\begin{array}{ccc} F' & \longrightarrow & 1 \\ \downarrow m & & \downarrow \top \\ F & \xrightarrow{\chi_m} & \Omega \end{array}$$

a pullback.

- For any space X , $\mathbf{Sh}(X)$ is a topos. It's cartesian closed. For the subobject classifier we take

$$\Omega(U) = \{V \in \mathcal{O}(X) : V \subseteq U\}$$

and $\Omega(U' \rightarrow U)$ is the map $V \mapsto V \subseteq U'$. Ω is a sheaf since if we have $U = \bigcup_{i \in I} U_i$ and $U_i \subseteq U_j$ such that $V_i \cap U_j = V_j \cap U_i$ for each i, j then $V = \bigcup_{i \in I} V_i$ is the unique open subset of U with $V \cap U_i = V_i$ for each i .

If $m : F' \rightarrow F$ is a subsheaf then for any $x \in F(U)$ the sieve

$$\{V \subseteq U : x|_V \in F'(V)\}$$

has a greatest element since F' is a sheaf. So we define $\chi_m : F \rightarrow \Omega$ to send x to this object.

- Let \mathbf{C} be a group G . The topos structure on $[G, \mathbf{Set}]$ is particularly simple: B^A is the set of all G -equivariant maps $f : A \times G \rightarrow B$ but such an f is determined by its values at elements of the form $(a, 1)$ since $f(a, g) = g.f(g^{-1}.a, 1)$, and this restriction can be any mapping $A \times \{1\} \rightarrow B$. So we can take B^A to be the set of functions $A \rightarrow B$ with G acting by

$$(g.f)(a) = g(f(g^{-1}.a))$$

and $\Omega = \{0, 1\}$ with trivial G -action. So the forgetful functor $[G, \mathbf{Set}] \rightarrow \mathbf{Set}$ is logical, as is the functor which equips a set A with trivial G -action.

Moreover, even if G is infinite, $[G, \mathbf{Set}]$ is a topos and this inclusion $[G, \mathbf{Set}_f] \rightarrow [G, \mathbf{Set}]$ is logical. Similarly if \mathcal{G} is a large group (i.e. the underlying space may not be a set), then $[\mathcal{G}, \mathbf{Set}]$ is a topos.

- Let \mathbf{C} be a category such that every \mathbf{C}/A is equivalent to a finite category. Then $[\mathbf{C}^{\text{op}}, \mathbf{Set}_f]$ is a topos. Similarly if \mathbf{C} is large but all \mathbf{C}/A are small, then $[\mathbf{C}^{\text{op}}, \mathbf{Set}]$ is a topos. In particular $[\mathbf{On}, \mathbf{Set}]$ is a topos, but it's not locally small.

Lemma 7.1. *Suppose \mathcal{E} has finite limits and a subobject classifier. Then every monomorphism in \mathcal{E} is regular. In particular \mathcal{E} is balanced.*

Proof. The universal monomorphism $\top : 1 \rightarrow \Omega$ is split and hence regular. But any pullback of a regular mono is regular: if f is an equaliser of (g, h) then $K^*(f)$ is an equaliser of (gk, hk) . The second assertion follows since a regular mono that is also epic is an isomorphism. \square

Given an object A in a topos \mathcal{E} , we write PA for the exponential Ω^A . and $\exists_A \rightarrow PA \times A$ for the subobject corresponding to $\text{ev} : PA \times A \rightarrow \Omega$. This has the property that, for any B and any $m : R \rightarrow B \times A$, there is a unique $\ulcorner m \urcorner : B \rightarrow PA$ such that

$$\begin{array}{ccc} R & \longrightarrow & \exists_A \\ \downarrow m & & \downarrow \\ B \times A & \xrightarrow{\ulcorner m \urcorner \times 1_A} & PA \times A \end{array}$$

is a pullback.

Definition (power-object). By a *power-object* for A in a category \mathcal{E} with finite limits, we mean an object PA equipped with $\exists_A \rightarrow PA \times A$ satisfying the above.

We say \mathcal{E} is a *weak topos* if every $A \in \text{ob } \mathcal{E}$ has a power-object.

Similarly we say $F : \mathcal{E} \rightarrow \mathcal{F}$ is *weakly logical* if $F(\exists_A) \rightarrow F(PA) \times FA$ is a power-object for FA for every $A \in \text{ob } \mathcal{E}$.

A power-object for the terminal object is the same as a subobject classifier.

This is an ad hoc definition and we will soon show that \mathcal{E} is cartesian closed and therefore we can safely drop the adjective “weak”. Consequently this may be taken as the definition of topos.

Lemma 7.2. *P is a functor $\mathcal{E}^{op} \rightarrow \mathcal{E}$. Moreover it is self-adjoint on the right.*

Compare this with the contravariant power set functor on **Set**, which is a special case.

Proof. Given $f : A \rightarrow B$, we define $Pf : PB \rightarrow PA$ to correspond to the pullback

$$\begin{array}{ccc} E_f & \longrightarrow & \exists_B \\ \downarrow & & \downarrow \\ PB \times A & \xrightarrow{1 \times f} & PB \times B \end{array}$$

For any $\ulcorner m \urcorner : c \rightarrow PB$, it’s easy to see that $(Pf)\ulcorner m \urcorner$ corresponds to $(1_C \times f)^*(m)$, hence $f \mapsto Pf$ is functorial. For any A and B , we have a bijection between subobjects of $A \times B$ and of $B \times A$ given by composition with $(\pi_2, \pi_1) : A \times B \rightarrow B \times A$. This yields a (natural) bijection between morphisms $A \rightarrow PB$ and $B \rightarrow PA$. \square

We write $\{ \}_A : A \rightarrow PA$ (pronounced “singleton”) for the morphism corresponding to $(1_a, 1_a) : A \rightarrow A \times A$.

Lemma 7.3. *Given $f : A \rightarrow B$, $\{\}_B f$ corresponds to $(1_A, f) : A \rightarrow A \times B$ and $(Pf)\{\}_B$ corresponds to $(f, 1_A) : A \rightarrow B \times A$.*

Proof. The square

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ \downarrow (1, f) & & \downarrow (1, 1) \\ A \times B & \xrightarrow{f \times 1} & B \times B \end{array}$$

is a pullback. Similarly for the second assertion. \square

Corollary 7.4.

1. $\{\}_A : A \rightarrow A$ is monic.
2. P is faithful.

Proof.

1. If $\{\}f = \{\}g$ then $(1_A, f)$ and $(1_A, g)$ are isomorphic as subobjects of $A \times B$, which forces $f = g$.
2. Similarly if $Pf = Pg$ then $(Pf)\{\} = (Pg)\{\}$ so we again deduce $f = g$.

\square

Given a mono $f : A \rightarrow B$ in \mathcal{E} , we define $\exists f : PA \rightarrow PA$ to correspond to the composite

$$\exists_A \rightarrow PA \times A \xrightarrow{1 \times f} PA \times B.$$

Then for any $\ulcorner m \urcorner : C \rightarrow PA$, $(\exists f)\ulcorner m \urcorner$ corresponds to

$$R \xrightarrow{m} C \times A \xrightarrow{1 \times f} C \times B$$

so $f \mapsto \exists f$ is a functor $\mathbf{Mono}(\mathcal{E}) \rightarrow \mathcal{E}$.

Lemma 7.5 (Beck-Chevalley condition). *Suppose*

$$\begin{array}{ccc} D & \xrightarrow{h} & A \\ \downarrow k & & \downarrow f \\ B & \xrightarrow{g} & C \end{array}$$

is a pullback with f monic. Then the diagram

$$\begin{array}{ccc} PA & \xrightarrow{\exists f} & PC \\ \downarrow Ph & & \downarrow Pg \\ PD & \xrightarrow{\exists k} & PB \end{array}$$

commutes.

Proof. Consider the diagram

$$\begin{array}{ccc}
 E_n & \longrightarrow & \exists_A \\
 \downarrow & & \downarrow \\
 PA \times D & \xrightarrow{1 \times h} & PA \times A \\
 \downarrow 1 \times k & & \downarrow 1 \times f \\
 PA \times B & \xrightarrow{1 \times g} & PA \times C
 \end{array}$$

The lower square is a pullback so the upper square is a pull back. This is equivalent to the composite is a pullback. \square

Theorem 7.6 (Paré). *The functor $P : \mathcal{E}^{op} \rightarrow \mathcal{E}$ is monadic.*

Proof. It has a left adjoint $P : \mathcal{E} \rightarrow \mathcal{E}^{op}$ by 7.5 (the “self-adjoint on the right” lemma). It’s faithful by 7.7 (ii) and hence reflects isomorphisms by 7.3. \mathcal{E}^{op} has coequalisers since \mathcal{E} has equalisers. Suppose

$$A \begin{array}{c} \xrightarrow{f} \\ \xrightarrow{g} \end{array} B$$

is a coreflexive pair in \mathcal{E} , then f and g are (split) monic and the equaliser $e : E \rightarrow A$ makes

$$\begin{array}{ccc}
 E & \xrightarrow{e} & A \\
 \downarrow e & & \downarrow g \\
 A & \xrightarrow{f} & B
 \end{array}$$

a pullback square. Since any cone over

$$\begin{array}{ccc}
 & & A \\
 & & \downarrow g \\
 A & \xrightarrow{f} & B
 \end{array}$$

has both legs equal so by Beck-Chevalley condition we have $(Pf)(\exists g) = (\exists e)(Pe)$. But we also have $(Pg)(\exists g) = 1_{FA}$ since

$$\begin{array}{ccc}
 A & \xrightarrow{1} & A \\
 \downarrow 1 & & \downarrow g \\
 A & \xrightarrow{g} & B
 \end{array}$$

is a pullback, and similarly $(PE)(\exists e) = 1_{PE}$. So

$$\begin{array}{ccc}
 PB & \begin{array}{c} \xrightarrow{Pf} \\ \xrightarrow{Pg} \end{array} & PA & \xrightarrow{Pe} & PE \\
 \swarrow \exists g & & \swarrow \exists e & & \\
 & & & &
 \end{array}$$

is a split coequaliser and in particular a coequaliser. Hence by 5.13 P is monadic. \square

Corollary 7.7.

1. A weak topos has finite colimits. Moreover if it has any infinite limits then it has the corresponding colimits. In particular if it is complete then it is cocomplete.
2. If a weakly logical functor has a left adjoint then it has a right adjoint.

Proof.

1. P creates all limits which exist, by 5.e.
2. By definition if F is weakly logical then

$$\begin{array}{ccc} \mathcal{E}^{\text{op}} & \xrightarrow{F} & \exists^{\text{op}} \\ \downarrow P & & \downarrow P \\ \mathcal{E} & \xrightarrow{F} & \exists \end{array}$$

commutes up to isomorphism. So this follows from 5.16. □

Lemma 7.8. *Let \mathcal{E} be a category with finite limits and suppose $A \in \text{ob } \mathcal{E}$ has a power-object PA . Then, for any B , $B^*(PA)$ is a power-object for B^*A in \mathcal{E}/B .*

Proof. Given $\begin{array}{c} C \\ \downarrow g \\ B \end{array}$ we have a pullback square

$$\begin{array}{ccc} C \times A & \xrightarrow{g \times 1} & B \times A \\ \downarrow \pi_1 & & \downarrow \pi_1 \\ C & \xrightarrow{g} & B \end{array}$$

so $\Sigma_B(g \times B^*A) \cong C \times A$. Hence

$$\mathbf{Sub}_{\mathcal{E}/B}(g \times B^*A) \cong \mathbf{Sub}_{\mathcal{E}}(C \times A),$$

but if $h : C \rightarrow PA$ corresponds to $\begin{array}{c} R \\ \downarrow \\ C \times A \end{array}$ then the upper square of the diagram

$$\begin{array}{ccc} R & \longrightarrow & B \times \exists_A \\ \downarrow & & \downarrow \\ C \times A & \xrightarrow{(g,h) \times 1_A} & B \times PA \times A \\ & \searrow g\pi_1 & \swarrow \pi_1 \\ & B & \end{array}$$

is a pullback. So $B \times PA$ equipped with $B^*(\exists_A) \mapsto B^*(PA \times A)$ is a power object for B^*A . \square

Theorem 7.9. *Suppose \mathcal{E} is a weak topos. Then for any $B \in \text{ob } \mathcal{E}$, \mathcal{E}/B is a weak topos and $B^* : \mathcal{E} \rightarrow \mathcal{E}/B$ is weakly logical.*

Proof. The second assertion follows from the previous lemma. For the first,

we need to construct a power object for an arbitrary \downarrow_f in \mathcal{E}/B . Then the pullback

$$\begin{array}{ccc} \Sigma_B(g \times f) & \longrightarrow & A \\ \downarrow & & \downarrow f \\ C & \xrightarrow{g} & B \end{array}$$

is a subobject of $C \times A$, namely the equaliser of

$$C \times A \xrightarrow[\text{g}\pi_2]{\text{f}\pi_1} B$$

Define $\wedge : PA \times PA \rightarrow PA$ to correspond to the intersection of $\pi_{13}^*(\exists_A \mapsto PA \times A)$ and $\pi_{23}^*(\exists_A \mapsto PA \times A)$ and define $P_1A \mapsto PA \times PA$ to be the equaliser of $PA \times PA \xrightarrow[\pi_1]{\wedge} PA$. Then, for any C , $C \xrightarrow[\pi_1]{(\text{r}_m^{\neg}, \text{r}_n^{\neg})} PA \times PA$ factors through P_1A if and only if $m \leq n$ in $\mathbf{Sub}_{\mathcal{E}}(C \times A)$. Now form the pullback

$$\begin{array}{ccc} Q & \longrightarrow & P_1A \\ \downarrow (h,k) & & \downarrow \\ PA \times B & \xrightarrow{1 \times \{\}} & PA \times PB \xrightarrow{1 \times Pf} PA \times PA \end{array}$$

Given any \downarrow_g morphisms $g \xrightarrow{\ell} k$ in \mathcal{E}/B correspond to morphisms $C \xrightarrow{h\ell} PA$

such that the subobject named by $h\ell$ is contained in that named by $(Pf)(\{\})g$. But the latter is indeed $\Sigma_B(g \times f) \mapsto C \times A$ so k is a power object for f in \mathcal{E}/B . \square

Corollary 7.10. *A weak topos is locally cartesian closed. In particular, it is a topos.*

Proof. For any $f : A \rightarrow B$ in \mathcal{E} , we can define $(\mathcal{E}/B)/f$ with \mathcal{E}/A and $f^* : \mathcal{E}/B \rightarrow \mathcal{E}/A$ with pullback along f . Hence all such functors are weakly logical.

But f^* has a left adjoint Σ_f so by 7.7 (2) (in lecture 7.10 (ii)) it has a right adjoint Π_f . Hence by 6.3 \mathcal{E}/B is cartesian closed for any B . \square

Remark. It can be shown that a weakly logical functor is cartesian closed, and hence logical.

Corollary 7.11.

1. Any epimorphism in a topos is regular.
2. Any $f : A \rightarrow B$ in a topos factors uniquely up to isomorphism as

$$A \xrightarrow{q} I \xrightarrow{m} B$$

Proof. \mathcal{E} is locally cartesian closed by the above corollary and has coequalisers by 7.10(i) so by 6.7 every f factors uniquely as regular epi plus mono. If f itself is epic then the monic part of the factorisation is isomorphism by 7.3, so f is regular epic. \square

7.1 Sheaves and local operators*

Recall that $\mathbf{Sh}(X) \subseteq [\mathcal{O}(X)^{\text{op}}, \mathbf{Set}]$ is a full subcategory closed under limits. In fact it's reflective and the reflector $L : [\mathcal{O}(X)^{\text{op}}, \mathbf{Set}] \rightarrow \mathbf{Sh}(X)$ preserves finite limits. This suggests considering reflective subcategories $\mathcal{D} \subseteq \mathcal{E}$ for which the reflector preserves finite limits (equivalently, pullbacks).

Lemma 7.12. *Given such a reflective subcategory and a monomorphism $A' \hookrightarrow A$ in \mathcal{E} , define $c(A') \hookrightarrow A$ by the pullback diagram*

$$\begin{array}{ccc} c(A') & \longrightarrow & LA' \\ \downarrow & & \downarrow \\ A & \xrightarrow{\eta_A} & LA \end{array}$$

Then $A' \mapsto c(A')$ is a closure operation on $\mathbf{Sub}_{\mathcal{E}}(A)$ and commutes with pullback along a fixed morphism of \mathcal{E} .

By “closure” we mean an order-preserving inflationary idempotent operator.

Proof. Since

$$\begin{array}{ccc} A & \xrightarrow{\eta_{A'}} & LA' \\ \downarrow & & \downarrow \\ A & \xrightarrow{\eta_A} & LA \end{array}$$

and $A' \leq A''$ in $\mathbf{Sub}(A)$ implies $LA' \leq LA''$ in $\mathbf{Sub}(LA)$ and hence $c(A') \leq c(A'')$.

Since $L\eta$ is an isomorphism,

$$\begin{array}{ccc} LA' & \xrightarrow{L\eta_{A'}} & LLA' \\ \downarrow & & \downarrow \\ LA & \xrightarrow{L\eta_A} & LLA \end{array}$$

is a pullback, and since L preserves pullbacks we deduce $Lc(A') \cong LA'$ in $\mathbf{Sub}(LA)$. Hence $c(c(A')) \cong c(A')$.

For stability under pullback, suppose

$$\begin{array}{ccc} A' & \longrightarrow & B' \\ \downarrow & & \downarrow \\ A & \xrightarrow{f} & B \end{array}$$

is a pullback. Then in the cube

$$\begin{array}{ccccc} c(A') & \longrightarrow & LA' & & \\ \downarrow & \searrow & \downarrow & \searrow & \\ & & c(B') & \longrightarrow & LB' \\ \downarrow & & \downarrow & & \downarrow \\ A & \xrightarrow{\eta_A} & LA & \xrightarrow{Lf} & LB \\ \downarrow & \searrow & \downarrow & \searrow & \\ & & B & \xrightarrow{\eta_B} & LB \end{array}$$

the front, back and right faces are pullbacks, whence the left face is too. \square

Definition (local operator). Let \mathcal{E} be a topos. By a *local operator* in \mathcal{E} we mean a morphism $j : \Omega \rightarrow \Omega$ satisfying the commutative diagrams

$$\begin{array}{ccc} 1 & \xrightarrow{\top} & \Omega & \xleftarrow{j} & \Omega \\ & \searrow & \downarrow j & \swarrow j & \\ & & \Omega & & \end{array} \quad \begin{array}{ccc} \Omega_1 & \longrightarrow & \Omega_1 \\ \downarrow & & \downarrow \\ \Omega \times \Omega & \xrightarrow{j \times j} & \Omega \times \Omega \end{array}$$

where Ω_1 is the order relation on Ω define in 7.12.

Given a closure operator on subobjects in the above lemma, define $J \rightrightarrows \Omega$ to be the closure of $\top : 1 \rightrightarrows \Omega$ and $j : \Omega \rightarrow \Omega$ to be the classifying map of $J \rightrightarrows \Omega$. Then, for any $A' \rightrightarrows A$ with classifying map $\chi_m : A \rightarrow \Omega$, the composite $j\chi_m$ classifies $c(A') \rightrightarrows A$.

Given a pullback-stable closure operation c on subobjects, we say $A' \rightrightarrows A$ is *dense* if $c(A') \rightrightarrows A$ is isomorphism, and *closed* if $A' \rightrightarrows c(A')$ is isomorphism.

Lemma 7.13. *Suppose given a commutative square*

$$\begin{array}{ccc} B' & \xrightarrow{f'} & A' \\ \downarrow n & & \downarrow m \\ B & \xrightarrow{f} & A \end{array}$$

with n dense and m closed. Then there is a unique $g : B \rightarrow A'$ with $mg = f$

| (and $gn = f'$).

Proof. We have $n \leq f^*(m)$ in $\mathbf{Sub}(B)$ so

$$1_B \cong c(n) \leq f^*(c(m)) \cong f^*(m)$$

so we define g as

$$B \xrightarrow{\cong} f^*(A') \longrightarrow A'$$

□

Note that $c(A')$ may be characterised as the unique (up to isomorphism) subobject A'' such that $A' \twoheadrightarrow A''$ is dense and $A'' \twoheadrightarrow A$ is closed.

Lemma 7.14. *Suppose c is induced as in Lemma 7.12 by a reflector $L : \mathcal{E} \rightarrow \mathcal{D}$ preserving finite limits. Then an object A of \mathcal{E} belongs to \mathcal{D} (up to isomorphism) if and only if, given any diagram*

$$\begin{array}{ccc} B' & \xrightarrow{f'} & A \\ \downarrow m & & \\ B & & \end{array}$$

with m dense, there exists a unique $f : B \rightarrow A$ with $fm = f'$.

Proof. Note first that m is dense if and only if Lm is an isomorphism: \Leftarrow follows from the definition. \Rightarrow follows since, by the proof of Lemma 7.12, we know $L(B')$ and $L(c(B'))$ are isomorphic in $\mathbf{Sub}(B)$.

Given this, if A is in \mathcal{D} then the given diagram extends uniquely to

$$\begin{array}{ccccc} B' & \xrightarrow{\eta_{B'}} & LB' & \longrightarrow & A \\ \downarrow & & \downarrow \cong & \nearrow & \\ B & \xrightarrow{\eta_B} & LB & & \end{array}$$

Conversely, suppose A satisfies the condition. Let $R \xrightarrow[a]{a} A$ be the kernel-pair of $\eta_A : A \rightarrow LA$ and $d : A \twoheadrightarrow R$ the factorisation of $(1_A, 1_A)$ through (a, b) . Since $L\eta_A$ is isomorphism and L preserve pullbacks, Ld is isomorphism so d is dense. This forces $a = b$ so η_A is monic. And η_A is dense so we get a unique $r : LA \rightarrow A$ with $r\eta_A = 1_A$. Now $\eta_A r \eta_A = \eta_A$ and since LA satisfies the condition we have $\eta_A r = 1_{LA}$. □

We say A is a *sheaf* (for c , or for j) if it satisfies the condition in the previous lemma. Given a local operator j on \mathcal{E} , we write $\mathbf{sh}_j(\mathcal{E})$ for the full subcategory of j -sheaves in \mathcal{E} .

Our aim is to show $\mathbf{sh}_j(\mathcal{E})$ is a topos.

| **Lemma 7.15.** *$\mathbf{sh}_j(\mathcal{E})$ is closed under limits in \mathcal{E} and an exponential ideal.*

Proof. The first assertion follows since the definition involves only morphisms with codomain A . For the second, note that if $m : B' \twoheadrightarrow B$ is dense then so is $\times 1_C : B' \times C \twoheadrightarrow B \times C$ for any C (since it's $\pi_1^*(m)$) and so if A is a sheaf then any morphism $B' \rightarrow A^C$ extends uniquely to a morphism $B \rightarrow A^C$. □

Lemma 7.16. *If A is a sheaf then a subobject $m : A' \rightarrow A$ in \mathcal{E} is a sheaf if and only if it is closed.*

Proof.

- \Leftarrow : Immediate from Lemma 7.13.
- \Rightarrow : Consider

$$A' \xrightarrow{p} c(A') \xrightarrow{q} A$$

p is dense so if A is a sheaf we get a unique $r : c(A') \rightarrow A'$ with $rp = 1_{A'}$. But $c(A')$ is a sheaf so $prp = p$ and thus $pr = 1_{c(A')}$. □

We define $\Omega_j \rightarrow \Omega$ to be the equaliser of $\Omega \begin{array}{c} \xrightarrow{j} \\ \xrightarrow{1_\Omega} \end{array} \Omega$. Then for any A , morphisms $A \rightarrow \Omega_j$ corresponding to closed subobjects of A .

Lemma 7.17. *Ω_j is a j -sheaf.*

Proof. We want to show that if $m : B \rightarrow A$ is a dense mono the pullback along m yields a bijection from closed subobjects of A to closed subobjects of B . If $n : A' \rightarrow A$ is closed then the pullback

$$\begin{array}{ccc} B' & \xrightarrow{m'} & A' \\ \downarrow n' & & \downarrow n \\ B & \xrightarrow{m} & A \end{array}$$

m' is dense so $A' \rightarrow A$ is the closure of $B' \rightarrow B \rightarrow A$. It remains to show that if $B' \rightarrow B$ is closed, it is iso up to the pullback of its closure in A . But (writing $A' \rightarrow A$ for the closure) we have a factorisation $B' \rightarrow f^*A'$ which is dense since $B' \rightarrow A'$ is dense, and closed since $B' \rightarrow B$ is closed. □

Theorem 7.18. *For any local operator j on \mathcal{E} , $\mathbf{sh}_j(\mathcal{E})$ is a topos. Moreover it's reflective in \mathcal{E} and the reflector preserves finite limits.*

Proof. $\mathbf{sh}_j(\mathcal{E})$ is cartesian closed and has a subobject classifier Ω_j by the lemma. To construct the reflect, consider the composite

$$A \xrightarrow{\{\}} \Omega^A \xrightarrow{j^A} \Omega_j^A$$

this corresponds to the closure $(a, b) : \bar{A} \rightarrow A \times A$ of the diagonal subobject $(1_A, 1_A) : A \rightarrow A \times A$. I claim (with proof omitted) that $\bar{A} \begin{array}{c} \xrightarrow{a} \\ \xrightarrow{b} \end{array} A$ is the kernel-pair of f . Hence any morphism $g : A \rightarrow B$ where B is a sheaf satisfies $ga = gb$. So if we form the image

$$A \xrightarrow{q} I \xrightarrow{m} \Omega_j^A$$

of f any such g factors uniquely through q .

Now Ω_j^A is a sheaf by the lemmas so if we form the closure $LA \rightarrow \Omega_j^A$ of m , we get a morphism $A \rightarrow LA$ through which any morphism from A to a sheaf factors uniquely. Hence L becomes a functor $\mathcal{E} \rightarrow \mathbf{sh}_j(\mathcal{E})$, left adjoint to the inclusion.

By 6.13, we know L preserves finite products. In fact it preserves equalisers as well (can be checked but omitted here). \square

Finally, we state the relation between topos and the topos Grothendieck was interested in (i.e. sheaves):

Theorem 7.19. *For a category \mathcal{E} , TFAE:*

1. \mathcal{E} is a topos, complete and locally small and has a separating set of objects.
2. there exists a small category \mathbf{C} and a local operators on $[\mathbf{C}^{op}, \mathbf{Set}]$ such that

$$\mathcal{E} \simeq \mathbf{sh}_j([\mathbf{C}^{op}, \mathbf{Set}]).$$

Sketch of proof.

1. $1 \implies 2$: since $\mathbf{sh}_j([\mathbf{C}^{op}, \mathbf{Set}])$ has given properties.
2. $1 \Leftarrow 2$: take \mathbf{C} to be the full subcategory of \mathcal{E} on the separating set and consider

$$\mathcal{E} \xrightarrow{Y} [\mathcal{E}^{op}, \mathbf{Set}] \longrightarrow [\mathbf{C}^{op}, \mathbf{Set}].$$

\square

Index

- adjoint functor theorem
 - general, 30
 - special, 32
- adjoints on the right, 17
- adjunction, 16
- algebra, 36
- arrow category, 17

- Beck-Chevalley condition, 57

- category, 2
 - balanced, 9
 - cartesian closed, 46
 - cocomplete, 27
 - complete, 27
 - discrete, 3
 - locally cartesian closed, 48
 - locally small, 10
 - opposite, 3
 - skeletal, 8
 - small, 2
 - well-powered, 32
- closed, 62
- closure, 61
- coequaliser, 13, 25
 - split, 41
- comonad, 35
- cone, 23
- congruence, 2
- coproduct, 13, 24
- counit, 19
- create limit, 27

- dense, 62
- detecting family, 14
- detector, 14
- diagram, 23
- direct limit, 25

- Eilenberg-Moore comparison
 - functor, 39
- epimorphism, 9
- equaliser, 13, 25
- equivalence, 6
- exponentiable object, 46
- exponential ideal, 51

- Frobenius reciprocity, 51

- functor, 3
 - contravariant, 4
 - essentially surjective, 7
 - faithful, 7
 - forgetful, 4
 - full, 7
 - representable, 12

- Heyting semilattice, 46

- initial object, 16
- injective, 15
- inverse limit, 25

- Kleisli category, 37

- local operator, 62
- logical functor, 54, 56

- monad, 35
- monadic tower, 44
- monadicity, 40
- monadicity theorem, 42
- monoid, 3
- monomorphism, 9
 - regular, 13
 - split, 9
- multiplication, 35

- natural transformation, 5

- Paré theorem, 58
- power-object, 56
- preserve limit, 27
- presheaf, 52
- product, 13, 24
- projective, 15
- pullback, 25
- pushout, 25

- quotient object, 31

- reflection, 21
- reflective subcategory, 21
- reflexivity, 41
- right adjoint preserves limits, 29

- separating family, 14
- separator, 14
- sheaf, 52
- sieve, 55

- singleton, 56
- skeleton, 8
- slice category, 46
- Stone-Čech compactification, 22, 33
- subobject, 31
- subobject classifier, 54
- subterminal, 52
- terminal object, 17
- topos, 54, 56
- triangular identities, 19
- unit, 19, 35
- universal element, 12
- Yoneda embedding, 11
- Yoneda lemma, 10