University of
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Mathematics Tripos

Part II

Algebraic Topology

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0 Introduction

Question. Is the Hopf link really linked? More formally, is there a homeomorphism $\mathbb{R}^3 \to \mathbb{R}^3$ taking H to U?

H can be realised as $S^1 \amalg S^1 \to \mathbb{R}^3$. For U, we can consider $S^1 \amalg S^1$ as boundary of $D^1 \amalg D^1$ and the map extends to a map to discs.

So it makes sense to phrase the question as

Question. Does the Hopf link $\eta : S^1 \amalg S^1 \to \mathbb{R}^3$extend to a map of discs?

This is an example of an *extension problem.*

Here is another example. Define the n-sphere $S^{n-1} := \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i^2 = 1\}$, which sits inside $D^n = \{x \in \mathbb{R}^n : \sum_{i=1}^n x_i^2 = 1\}$. We can ask:

Question. Does the identity map $\mathrm{id}_{S^{n-1}} : S^{n-1} \to S^{n-1}$ factor through D^n?

To gain some intuition, let's consider small n. For $n = 1$, $S^0 = \{-1, 1\}$. The answer is no by Intermediate Value Theorem, or connectedness from topology. For $n = 2$, this answer is again no by winding number argument. What about $n \geq 3$?

These problems are hard because we have to consider continuous maps between two spaces, which are in general very big and hard to compute. On the other hand, a comparable algebraic problem is

Question. Does the map $\mathrm{id} : \mathbb{Z} \to \mathbb{Z}$ factor through 0?

Well that's much much easier!

1 The fundamental group

Throughout this course, "maps" mean continuous maps.

1.1 Deforming maps and spaces

Definition (homotopy)**.** Let $f_0, f_1 : X \to Y$ be maps. A *homotopy* between f_0 and f_1 is a map $F : X \times [0,1] \to X$ such that $F(x,0) = f_0(x)$ and $F(x,1) = f_1(x)$ for all $x \in X$.

If F exists, we say that f_0 is *homotopic* to f_1 and write $f_0 \simeq f_1$, or to emphasise the homotopy, $f_0 \simeq_F f_1$.

Notation. $I = [0,1]$. We often write $f_t(x) = F(x,t)$.

Example. If Y is a convex region in $\mathbb{R}^n$ then for any $f_0, f_1 : X \to Y$, the *straightline homotopy* $F(x,y) = tf_1(x) + (1-t)f_0(x)$ is a homotopy $f_0 \simeq f_1$.

Definition (relative homotopy)**.** If $Z \subseteq X$ and $F(z,t) = f_0(z) = f_1(z)$ for all $z \in Z, t \in I$, then F is a *homotopy relative to* Z, write $f_0 \simeq_F f_1 \operatorname{rel} Z$.

Lemma 1.1. *The relation* $\simeq$ $(\operatorname{rel} Z)$ *is an equivalence relation on maps* $X \to Y$.

Proof. Reflexivity and symmetry are easy. For transitivity, suppose $f_0 \simeq_{F_0} f_1 \simeq_{F_1} f_2$. Let

$$F(x,t) = \begin{cases} F_0(x,2t) & t \leq \frac{1}{2} \\ F_1(x,2t-1) & t \geq \frac{1}{2} \end{cases}$$

which is the homotopy we need. □

Definition (homotopy equivalence)**.** $f : X \to Y$ and $g : Y \to X$ is a *homotopy equivalence* if $g \circ f \simeq \mathrm{id}_X$ and $f \circ g \simeq \mathrm{id}_Y$. In this case we say X is homotopy equivalent to Y and write $X \simeq Y$.

Example. Let $X = *$, the space with one point and $Y = \mathbb{R}^n$. Let $f : * \mapsto 0$, g be the unique map $Y \to X$. Then $g \circ f = \mathrm{id}_X$, and $f \circ g = 0 \simeq \mathrm{id}_Y$ via the straightline homotopy. Therefore $\mathbb{R}^n$ is homotopy equivalent to $*$.

Definition (contractible)**.** A space X is *contractible* if $X \simeq *$.

Example. Let $X = S^1, Y = \mathbb{R}^2 - \{0\}$. Let $f : X \to Y$ be the natural inclusion nad $g : Y \to X, x \mapsto \frac{x}{\|x\|}$. Then

$$g \circ f = \mathrm{id}_X$$
$$f \circ g(x) = \frac{x}{\|x\|} \in \mathbb{R}^2$$

Although Y is not convex, for all x, t, straightline homotopy $F(x,t)$ between $f \circ g$ and id_Y satisfies $F(x,t) \neq 0$ so $f \circ g \simeq_F \mathrm{id}_Y$. Thus $X \simeq Y$.

Definition (retract, deformation retract)**.** Let $f : X \to Y$ and $g : Y \to X$. If $g \circ f = \mathrm{id}_X$ then X is a *retract* of Y.

If in addition $f \circ g \simeq \mathrm{id}_Y \operatorname{rel} f(X)$ then we say X is a *deformation retract* of Y.

Note that whenever we have $g \circ f = \mathrm{id}_X$, f is injective so we can think X as being embedded in Y. Informally, Y is "as complicated" as X.

Lemma 1.2. *Homotopy equivalence is an equivalence on topological spaces.*

Proof. Symmetry and reflexivity are obvious. For transitivity, consider

$$X \underset{g}{\overset{f}{\rightleftarrows}} Y \underset{g}{\overset{f}{\rightleftarrows}} Z$$

Need to show that $g \circ (g' \circ f') \circ f \simeq \mathrm{id}_X$ (and the other direction will follow similarly). By hypothesis $g' \circ f' \simeq_{F'} \mathrm{id}_Y$. Now

$$g(F'(f(x), t))$$

is a homotopy

$$g \circ g' \circ f' \circ f \simeq g \circ \mathrm{id}_Y \circ f = g \circ f \simeq \mathrm{id}_X .$$

□

1.2 The fundamental group

Definition (path, loop)**.** A *path* (from x_0 to x_1) is a continuous map $\gamma : I \to X$ (with $\gamma(0) = x_0, \gamma(1) = x_1$).

A *loop* (based at x_0) is a path from x_0 to x_0.

Definition (homotopy of path)**.** Let γ_0, γ_1 be paths from x_0 to x_1. A *homotopy (of path)* from γ_0 to γ_1 is a homotopy

$$\gamma_0 \simeq_F \gamma_1 \operatorname{rel}\{0, 1\}.$$

Definition (concatenation of path, constant path, inverse path)**.** Let γ be a path from x to y and δ a path from y to z.

1. The *concatenation* of γ and δ is

$$(\gamma \cdot \delta)(t) = \begin{cases} \gamma(2t) & t \leq \frac{1}{2} \\ \delta(2t-1) & t \geq \frac{1}{2} \end{cases}$$

2. The *constant* path (at x) is $c_x(t) = x$.

3. The *inverse path* to γ is $\overline{\gamma}(t) = \gamma(1-t)$.

Theorem 1.3 (fundamental group). *Let* $x_0 \in X$. *Let*

$$\pi_1(X, x_0) = \{\textit{loops based at } x_0\}/\simeq .$$

This has a group structure with

- $[\gamma][\delta] = [\gamma \cdot \delta]$,
- *identity* $[c_{x_0}]$,
- $[\gamma]^{-1} = [\overline{\gamma}]$.

We call $\pi_1(X, x_0)$ *the* fundamental group *of* X *(based at* x_0*).*

Proof. To prove the theorem, we need to check that multiplication and inverses are well-defined and the group axioms are satisfied.

Lemma 1.4. *If* γ_0, γ_1 *are paths to* y *and* δ_0, δ_1 *are paths from* y *and* $\gamma_0 \simeq \gamma_1, \delta_0 \simeq \delta_1$, *then*

$$\gamma_0 \cdot \delta_0 \simeq \gamma_1 \cdot \delta_1.$$

Also $\overline{\gamma}_0 \simeq \overline{\gamma}_1$.

Proof. We only show for concatenation. Inverses are similar. Let $\gamma_0 \simeq_F \gamma_1, \delta_0 \simeq_G \delta_1$. (proof by picture) Algebraically, the homotopy is given by

$$H(s,t) = \begin{cases} F(s, 2t) & t \leq \frac{1}{2} \\ G(s, 2t-1) & t \geq \frac{1}{2} \end{cases}$$

□

Now we check that the group axioms are satisfied.

Lemma 1.5.

1. $(\alpha \cdot \beta) \cdot \gamma \simeq \alpha \cdot (\beta \cdot \gamma)$.
2. $\alpha \cdot c_x \simeq \alpha \simeq c_w \cdot \alpha$.
3. $\alpha \cdot \overline{\alpha} \simeq c_w$.

Proof. We show 1. The other two are similar. Let

$$\delta = \begin{cases} \alpha(3t) & t \leq \frac{1}{3} \\ \beta(3t-1) & \frac{1}{3} \leq t \leq \frac{2}{3} \\ \gamma(3t-2) & \frac{2}{3} \leq t \leq 1 \end{cases}$$

Let

$$f_0(t) = \begin{cases} \frac{4}{3}t & t \leq \frac{1}{2} \\ \frac{1}{3} + \frac{2}{3}t & t \geq \frac{1}{2} \end{cases}$$

and

$$f_1(t) = \begin{cases} \frac{2}{3}t & t \leq \frac{1}{2} \\ -\frac{1}{3} + \frac{4}{3}t & t \geq \frac{1}{2} \end{cases}$$

Note that $f_0 \simeq f$ as *paths* via the straightline homotopy in I. But

$$\begin{aligned}(\alpha \cdot \beta) \cdot \gamma &= \delta \circ f_0 \\ \alpha \cdot (\beta \cdot \gamma) &= \delta \circ f_1\end{aligned}$$

so they are homotopic as path. □

□

Example. Let $X = \mathbb{R}^n, x_0 = 0$. Consider a loop γ in $\mathbb{R}^n$ based at 0. The straightline homotopy shows that $\gamma \simeq c_0$ as path. Therefore $\pi_1(\mathbb{R}^n, 0) \cong 1$.

Lemma 1.6. *Let $f : X \to Y$ be such that $f(x_0) = y_0$. There is a well-defined homomorphism*

$$\begin{aligned}f_* : \pi_1(X, x_0) &\to \pi_1(Y, y_0) \\ [\gamma] &\mapsto [f \circ \gamma]\end{aligned}$$

Furthermore,

1. *if $f \simeq f' \operatorname{rel}\{x_0\}$ then $f_* = f'_*$.*
2. *if $g : Y \to Z$ is another map then $f_* \circ g_* = (f \circ g)_*$.*
3. $(\mathrm{id}_X)_* = \mathrm{id}_{\pi_1(X,x_0)}$.

Proof. Easy. □

We'd like to eliminate the dependence of $\pi_1(X, x_0)$ on x_0, at least when X is path-connected. Suppose $x_0, x_1 \in X$. What do $\pi_1(X, x_0)$ and $\pi_1(X, x_1)$ have to do with each other, where X is path-connected?

Fix α a path from x_0 to x_1.

Lemma 1.7. *There is a well-defined group homomorphism*

$$\begin{aligned}\alpha_\# : \pi_1(X, x_0) &\to \pi_1(X, x_1) \\ [\gamma] &\mapsto [\overline{\alpha} \cdot \gamma \cdot \alpha]\end{aligned}$$

Furthermore

1. *if $\alpha \simeq \alpha'$ then $\alpha_\# = \alpha'_\#$,*
2. $(c_{x_0})_\# = \mathrm{id}_{\pi_1(X,x_0)}$,
3. *if β is a path from x_1 to x_2, $\beta_\# \circ \alpha_\# = (\alpha \cdot \beta)_\#$.*
4. *if $f : X \to Y$ then $(f \circ \alpha)_\# \circ f_* = f_* \circ \alpha_\#$.*

Now it makes sense to talk about isomorphism type of the fundamental group of a path-connected space.

Definition (simply connected)**.** If X is path-connected and $\pi_1(X, x_0) \cong 1$ for some (i.e. any) $x_0 \in X$ then we say X is *simply connected*.

Our last task is to understand what homotopies that don't fix basepoints do to the fundamental group.

Lemma 1.8. *Suppose $f, g : X \to Y$ is such that $f \simeq_F g$. Define $\alpha(t) = F(x_0, t)$, a path from $f(x_0)$ to $g(x_0)$. Then the following diagram commutes:*

$$\begin{tikzcd}
 & \pi_1(Y, f(x_0)) \arrow[dd, "\alpha_\#"] \\
\pi_1(X, x_0) \arrow[ur, "f_*"] \arrow[dr, "g_*"'] & \\
 & \pi_1(Y, g(x_0))
\end{tikzcd}$$

i.e. $g_* = \alpha_\# \circ f_*$.

Proof. Let $[\gamma] \in \pi_1(X, x_0)$. We need to show that

$$[g \circ \gamma] = g_*[\gamma] = \alpha_\# \circ f_*[\gamma] = [\overline{\alpha} \cdot (f \circ \gamma) \cdot \alpha]$$

which is saying

$$g \circ \gamma \simeq \overline{\alpha} \cdot (f \circ \gamma) \cdot \alpha$$

as paths. Consider

$$\begin{aligned} I \times I &\to Y \\ (s, t) &\mapsto F(\gamma(s), t) \end{aligned}$$

Let H be the straightline homotopy in $I \times I$ between the yellow path and the brown path. Then $G \circ H$ is the homotopy we need. $\square$

Theorem 1.9. *If $f : X \to Y, g : Y \to X$ is a pair of homotopy equivalences and $x_0 \in X$ then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is an isomorphism.*

Proof. Suffices to prove that f_* is bijective. Let $g \circ f \simeq_F \mathrm{id}_X$ and α be the path defined from F as above. Then

$$g_* \circ f_* = (g \circ f)_* = \alpha_\# \circ \mathrm{id}_{\pi_1(X, x_0)} = \alpha_\#$$

so f_* is injective. Similarly it is surjective. $\square$

Corollary 1.10. *Contractible spaces are simply connected.*

2 Covering spaces

2.1 Definition and first examples

Definition (covering space)**.** Let $p : \hat{X} \to X$ be a map. An open set $U \subseteq X$ is *evenly covered* if there is a discrete space Δ_U and an identification $p^{-1}(U) = \Delta_U \times U$ such that on $p^{-1}(U)$, p coincides with projection to the second factor.

If every $x \in X$ has an everly covered neighbourhood, we say that p is a *covering map* and $\hat{X}$ is a *covering space*

Alternatively, write $U_\delta = \{\delta\} \times U$. Then $p^{-1}(U) = \coprod_{\delta \in \Delta_U} U_\delta$. Write $p|_\delta = p|_{U_\delta}$ which is a homeomorphism.

Example.

1. Let $\hat{X} = \mathbb{R}, X = S^1$ and define
$$\begin{aligned} p : \mathbb{R} &\to S^1 \\ t &\mapsto e^{2\pi i t} \end{aligned}$$
Let $1 \in U \subsetneq S^1$. Choose a branch of log well-defined on U such that $\log 1 = 0$. Every point $\hat{z} \in p^{-1}(U)$ can be written uniquely as
$$\hat{z} = k + \frac{\log(z)}{2\pi i}$$
where $z = p(\hat{z}) \in U$ and $k \in \mathbb{Z}$, i.e. $p^{-1}(U) = \mathbb{Z} \times U$. Thus U is evenly covered. The same proof shows that p is a covering map.

2. Let $\hat{X} = X = S^1$. Define
$$\begin{aligned} p_n : S^1 &\to S^1 \\ z &\mapsto z^n \end{aligned}$$
This is also a covering map by essentially the same proof by choosing a nth root of unity. In this case Δ_n is the nth roots of unity.

3. Let $\hat{X} = S^2$ and $G = \mathbb{Z}/2\mathbb{Z}$ acts on S^2 via the antipodal map. Let
$$X = \hat{X}/G = \{\{x, -x\} : x \in S^2\}$$
and $p : \hat{X} \to X$ be the quotient map. The orbit space X can be identified with straightlines in $\mathbb{R}^3$ passing through the origin. Given a line ℓ through the origin, let
$$C_\ell = \{y \in S^2 : y \text{ perpendicular to } \ell\}.$$
Then $S^2 - C_\ell = U_+ \amalg U_-$. Let $U = p(U_+ \amalg U_-)$, an open neighbourhood of ℓ in X. Note that $p|_{U_+}$ and $p|_{U_-}$ are both homeomorphisms onto U. Thus U is evenly covered and p is a covering map. $X = \mathbb{R}P^2$ is the *real projective plane.*

Note that in all three examples, for all points $x \in X$, the number of copies of U in $p^{-1}(U)$ is the same. We give a name to such covering spaces:

> **Definition** (n-sheeted)**.** A covering map $p : \hat{X} \to X$ is *n-sheeted* where $n \in \mathbb{N} \cup \{\infty\}$ if for all $x \in X$, $\#p^{-1}(x) = n$.

2.2 Lifting properties

Let $p : \hat{X} \to X$ be a covering map throughout the section.

> **Definition** (lift)**.** A *lift* of $f : Y \to X$ to $\hat{X}$ is a map $\hat{f} : Y \to \hat{X}$ such that $f = p \circ \hat{f}$, i.e. the following diagram commutes:
>
> $$\begin{tikzcd} & \hat{X} \arrow[d, "p"] \\ X \arrow[ur, dashed, "\hat{f}"] \arrow[r, "f"] & X \end{tikzcd}$$

> **Lemma 2.1** (uniqueness of lift)**.** *Suppose $f : Y \to X$ where Y is connected and locally path-connected. Let $\hat{f}_1, \hat{f}_2 : Y \to \hat{X}$ are both lifts of f. If there exists $y \in Y$ such that $\hat{f}_1(y) = \hat{f}_2(y)$ then $\hat{f}_1 = \hat{f}_2$.*

Proof. Consider
$$S = \{y \in Y : \hat{f}_1(y) = \hat{f}_2(y)\}.$$
Claim that S is both open and closed, from which the lemma follows immediately. Given $y_0 \in Y$, let U be an evenly covered neighbourhood of $f(y_0)$ and $V \subseteq \hat{f}^{-1}(U)$ a path-connected neighbourhood of y_0. Let $y \in V$ be arbitrary. Need to show that $y_0 \in S$ if and only if $y \in S$. If $y_0 \in S$ then $\hat{f}_1(y_0) = \hat{f}_2(y_0) \in U_\delta$ for some $\delta \in \Delta_U$. Let α be a path in V from y_0 to y. Then $f \circ \alpha$ is a path from $f(y_0)$ to $f(y)$. Then $\hat{f}_i \circ \alpha$ is a path in $p^{-1}(U)$ from $\hat{f}_i(y_0)$ to $\hat{f}_i(y)$. It follows that $\hat{f}_i(y) \in U_\delta$ so $\hat{f}_1(y) = (\delta, f(y)) = \hat{f}_2(y)$ so $y \in S$. The converse is identical. ☐

> **Definition** (lift at a point)**.** Let $\gamma : I \to X$ be a path with $\gamma(0) = x_0$. A (unique) lift of γ to $\hat{X}$ such that $\hat{\gamma}(0) = \hat{x}_0 \in p^{-1}(x_0)$ is called the *lift of γ at $\hat{x}_0$*.

> **Lemma 2.2** (path-lifting lemma)**.** *Let $\gamma : I \to X$ be a path with $\gamma(0) = x_0$. For any $\hat{x}_0 \in p^{-1}(x_0)$ there is a uniqueness $\hat{\gamma}$ of γ at $\hat{x}_0$.*

Proof. Uniqueness follows from the more general uniquenss of lift so suffices to show existence. Consider
$$S = \{t \in I : \text{ lift of } \gamma|_{[0,t]} \text{ at } \hat{x}_0 \text{ exists}\},$$
as $0 \in S$, the lemma follows if we can show S is both open and closed. Let $t_0 \in I$. Then $\gamma(t_0) \in U$ for some evenly covered neighbourhood U. There exists a path-connected neighbourhood V of t_0 such that $\gamma(V) \subseteq U$. Let $t \in V$. We'll prove that $t_0 \in S$ if and only if $t \in S$. By symmetry suffices to show one direction.

Suppose $t_0 \in S, t \notin S$. Since $t_0 \in S$, $\hat{\gamma}(t_0)$ is well-defined so let $\hat{\gamma}(t_0) \in U_\delta$. Since $[t_0, t] \subseteq V$ (as $t \notin S$), $\gamma([t_0, t]) \subseteq U$ so the path

$$s \mapsto \begin{cases} \hat{\gamma}(s) & s \leq t_0 \\ p_\delta^{-1} \circ \gamma & t_0 \leq s \leq t \end{cases}$$

is a lift of $\gamma|_{[0,t]}$ so $t \in S$. Contradiction. ☐

Lemma 2.3. *If X is path-connected the p is n-sheeted for some $n \in \mathbb{N} \cup \{\infty\}$.*

Proof. Let $x, y \in X$ and α a path between them. Let $\hat{x} \in p^{-1}(x)$ and let $\hat{\alpha}_{\hat{x}}$ be the unique lift of α at $\hat{x}$. Define a map

$$\begin{aligned} p^{-1}(x) &\to p^{-1}(y) \\ \hat{x} &\mapsto \hat{\alpha}_{\hat{x}}(1) \end{aligned}$$

Now replacing α with $\overline{\alpha}$ defines an inverse to this map. ☐

Definition (degree of covering map)**.** n is called the *degree* of p.

Lemma 2.4 (homotopy lifting lemma)**.** *Let $f_0 : Y \to X$ be a map where Y is path-connected. Let $F : Y \times I \to X$ be a homotopy with $F(\cdot, 0) = f_0$. Let $\hat{f}_0 : Y \to \hat{X}$ be a lift of f_0 to $\hat{X}$. Then there is a unique lift $\hat{F}$ of F to $\hat{X}$ such that $\hat{F}(\cdot, 0) = \hat{f}_0$.*

Proof. Let $y_0 \in Y$. Let $\gamma_{y_0}(t) = F(y_0, t)$ be a path. By path lifting lemma, there is a unique lift $\hat{\gamma}_{y_0}$ such that $\hat{\gamma}_{y_0}(0) = \hat{f}_0(y_0)$ such that $\hat{F}(y_0, t) = \hat{\gamma}_{y_0}(t)$. By uniqueness of path lifting, this is the only choice for $\hat{F}$, but it is not clear that $\hat{F}$ is continuous.

We will construct a map that is obviously continuous and argue that it is also a lift. Fix y_0. For all t there exists U_t an evenly covered neighbourhood of $F(y_0, t)$. By definition of product topology,

$$(y_0, t) \in V_t \times J_t \subseteq F^{-1}(U_t).$$

Compactnss of I implies that $\{y_0\} \times I$ is covered by $V_1 \times J_1, \dots, V_n \times J_n$ where $t_i \in J_i$. Setting $V = \bigcap_{i=1}^n V_i$ (and passing to a path-connected subset), we have $\{y_0\} \times I$ covered by $V \times J_1, \dots, V \times J_n$. Now define $\tilde{F}$ on $V \times I$ by

$$\tilde{F}(y, t) = p_{\delta_i}^{-1} \circ F(y, t)$$

for $y \in V, t \in J_i$. Need to check that $\tilde{F}$ is well-defined. Suppose $t \in J_i \cap J_j$. Let $y \in V$. Choose α in V from y_0 to y and let $\alpha_t(s) = F(\alpha(s), t)$. Now $p_{\delta_i}^{-1} \circ \alpha_t$ is the lift of α_t at $\hat{F}(y_0, t)$. Same for $p_{\delta_j}^{-1} \circ \alpha_t$ so they are equal. Therefore their endpoints coincide: $p_{\delta_i}^{-1} \circ F(y, t) = p_{\delta_j}^{-1} \circ F(y, t)$. Thus $\tilde{F}$ is well-defined.

$\tilde{F}$ is clearly continuous and a lift of F, so it remains to check that $\tilde{F} = \hat{F}$ on $V \times I$. By construction $\tilde{F}(y_0, 0) = \hat{F}(y_0, 0)$. Now $\tilde{F}(\alpha(\cdot), 0)$ is a lift of $f_0 \circ \alpha$, so will agree with $\hat{f}_0 \circ \alpha$. So $\tilde{F}(y,0) = \hat{f}_0(y)$ for all $y \in V$. Finally $\tilde{F}(y,\cdot)$ is a lift of γ_y starting at $\hat{f}_0(y)$, so by uniqueness again, $\tilde{F}(y,t) = \tilde{\gamma}_y(t) = \hat{F}(y,t)$ for all $y \in V, t \in I$. □

We have discussed lifts of maps, paths and homotopies. Recall that homotopy of paths is a slightly stronger form of homotopy and the next lemma shows that indeed the lift of a homotopy of paths is a homotopy of paths:

Lemma 2.5. *Let $F : I \times I \to X$ be a homotopy of paths and $\hat{F}$ be a lift of F to $\hat{X}$. Then $\hat{F}$ is also a homotopy of paths.*

Proof. As F is a homotopy of path, $F(0,t) = x_0$ for all t. Consider $\hat{F}(0,\cdot) : I \to \hat{X}$. For any $t \in I$ we have

$$\hat{F}(0,t) \in p^{-1}(F(0,t)) = p^{-1}(x_0)$$

which is discrete. As I is connected $\hat{F}(0,\dots)$ is constant. Same for $\hat{F}(1,\dots)$ so $\hat{F}$ is a homotopy of paths. □

2.3 Applications to calculations of fundamental groups

Lemma 2.6. *If $p : \hat{X} \to X$ is a map, $x \in X$ and $\hat{x} \in p^{-1}(x)$ then*

$$p_* : \pi_1(\hat{X}, \hat{x}) \to \pi_1(X, x)$$

is an injection.

Proof. Suppose $[\hat{\gamma}] \in \ker p_*$, i.e. $p_*([\hat{\gamma}]) = [p \circ \hat{\gamma}] = [\gamma] = 1 \in \pi_1(X,x)$. Then γ is homotopic to the constant path. But by homotopy lifting lemma this lifts to homotopy between $\hat{\gamma}$ and constant path. □

As last time, path lifting defines an *action* of $\pi_1(X,x)$ on $p^{-1}(x)$ by

$$\begin{aligned} \pi_1(X,x) \times p^{-1}(x) &\to p^{-1}(x) \\ ([\gamma], \hat{x}) &\mapsto \hat{x}.\gamma \end{aligned}$$

where $\hat{x}.\gamma$ is the endpoint of the lift of γ at $\hat{x}$. Note that by Lemma 2.5 this is indeed in the fibre of x. Furthermore it shows that this is well-defined. Finally note that this is a *right action* (ultimately because we defined concatenation of paths from left to right).

Given G action on X, orbit-stabiliser says that there is a bijection between the left cosets of stabiliser G_x of an element x and the orbit G^x. Furthermore, G has a natural action on the left cosets G/G_x such that the bijection is G-equivariant. Spelling this out (and use right action instead of left), we have

Lemma 2.7. *Suppose $\hat{X}$ is path-connected and $x \in X$. Let $\hat{x} \in p^{-1}(x)$. Then*

$$\begin{aligned} p_*\pi_1(\hat{X}, \hat{x}) \backslash \pi_1(X,x) &\to p^{-1}(x) \\ (p_*\pi_1(\hat{X}, \hat{x}))[\gamma] &\mapsto \hat{x}.\gamma \end{aligned}$$

Furthermore, the map is equivariant.

Proof. Suffices to show that the action is transitive and the stabiliser of $\hat{x}$ is $p_*\pi_1(\hat{X}, \hat{x})$. As $\hat{X}$ is path-connected there exists a path $\hat{\gamma}$ between any two points in $p^{-1}(x)$, whose image γ under p is a loop bases at x, and is the only loop whose lift is $\hat{\gamma}$ by uniquenss. The stabiliser of $\hat{x}$ are precisely the homotopy classes of loops based at x whose lifts are loops baesd at $\hat{x}$, which is precisely $p_*\pi_1(\hat{X}, \hat{x})$. □

Definition (universal cover)**.** If $p : \tilde{X} \to X$ is a covering map with X path-connected and $\tilde{X}$ simply connected then $\tilde{X}$ is called a *universal cover* of X.

Corollary 2.8. *If $p : \tilde{X} \to X$ is a universal cover and $p(\tilde{x}) = x$ then*

$$\begin{aligned} \pi_1(X, x) &\to p^{-1}(x) \\ [\gamma] &\mapsto \tilde{x}.\gamma \end{aligned}$$

is an equivariant bijection.

The map is not only bijective, but also equivariantly so. Thus by looking into the universal cover we can recover information about the fundamental group of the base space.

Example (fundamental group of S^1)**.** Consider $p : \mathbb{R} \to S^1, t \mapsto e^{2\pi it}$ is a covering map. Since $\mathbb{R}$ is contractible, this is the universal cover so

$$\begin{aligned} \pi_1(S^1, 1) &\to p^{-1}(1) = \mathbb{Z} \\ [\gamma] &\mapsto 0.\gamma \end{aligned}$$

is a bijection. Therefore we can write down representative loops for each element of $\pi_1(S^1, 1)$. For $n \in \mathbb{Z}$, let $\tilde{\gamma}_n(t) = nt$ so $\gamma_n = p \circ \tilde{\gamma}_n$ is a loop in S^1 based at 1. As $[\gamma_n] \mapsto n$, these represent every element of $\pi_1(S^1, 1)$ uniquely.

To recover the group structure, note that for any $m, n \in \mathbb{Z}$, $m + \tilde{\gamma}_n$ is the lift of γ_n at m. On the other hand, the endpoint of the lift of $\gamma_m \cdot \gamma_n$ at 0 is $m + n$, which is the endpoint of $m + \tilde{\gamma}_n$. So

$$m + n : [\gamma_m \cdot \gamma_n] \mapsto m + n$$

is a homomorphism. Thus

$$\pi_1(S^1, 1) \cong \mathbb{Z}.$$

2.4 The fundamental group of S^1

Theorem 2.9. id_{S^1} *does not extend over D^2, i.e. S^1 is not a retract of D^2.*

Proof. Suppose otherwise and $r : D^2 \to S^1$ is a retraction. Then $\mathrm{id}_{S^1} = r \circ i$:

$$\begin{tikzcd} S^1 \arrow[r, "\mathrm{id}"] \arrow[dr, "i"'] & S^1 \\ & D^2 \arrow[u, "r"] \end{tikzcd}$$

Look at the induced fundamental groups, we have

$$\mathrm{id}_{\mathbb{Z}} = r_* \circ i_*$$

so

$$\begin{tikzcd} \mathbb{Z} \arrow[r, "\mathrm{id}"] \arrow[dr, "i_*"'] & \mathbb{Z} \\ & 0 \arrow[u, "r_*"'] \end{tikzcd}$$

Absurd. □

> **Corollary 2.10** (Brouwer fixed point theorem). *Every continuous map $f : D^2 \to D^2$ has a fixed point.*

Proof. If there exists f such that $f(x) \neq x$ for all $x \neq D^2$ then we can construct a continuous retraction $r : D^2 \to S^1$: for all $x \in D^2$, let $r(x)$ be the intersection of the ray from $f(x)$ to x with S^1 (well-defined since $f(x) \neq x$). It is continuous. As r fixes S^1 this is a retract. □

> **Theorem 2.11** (fundamental theorem of algebra). *Every nonconstant polynomial $p : \mathbb{C} \to \mathbb{C}$ has a root.*

Sketch of proof. Suppose $p(z) = z^d + a_{d-1}z^{d-1} + \cdots + a_1 z + a_0$ has no root. Then $p : \mathbb{C} \setminus \{0\} \to \mathbb{C} \setminus \{0\}$. Let

$$\begin{aligned} r : \mathbb{C} \setminus \{0\} &\to S^1 \\ z &\mapsto \frac{z}{|z|} \end{aligned}$$

be the usual retraction. Let $\lambda_R(z) = Rz$ for $R > 0$ and consider f_R which is the composition

$$S^1 \xrightarrow{\lambda_R} \mathbb{C} \setminus \{0\} \xrightarrow{p} \mathbb{C} \setminus \{0\} \xrightarrow{r} S^1$$

as all these maps are homotopic, they induce the same map $f_* : \mathbb{Z} \to \mathbb{Z}$ which is multiplication by some number m, independent of R. When R is small, we can argue that f_R is homotopic to a constant map so $m = 0$. When R is large, p is approximately $z \mapsto z^d$ so $m = d$, contradiction. □

2.5 Existence of universal covers

> **Theorem 2.12.** *If X is path-connected and locally simply connected then X has a universal cover.*

Sketch of proof. [non-examinable] Let

$$\mathfrak{X} = \{\gamma : I \to X : \gamma(0) = x_0\}$$

and define $\tilde{X} = \mathfrak{X}/\simeq$, the homotopy classes of paths. Define

$$\begin{aligned} p : \tilde{X} &\to X \\ [\gamma] &\mapsto \gamma(1) \end{aligned}$$

The verification is omitted. □

2.6 The Galois correspondence

Definition (covering space isomorphism)**.** Let X be a path-connected topological space and $p_1 : \hat{X}_1 \to X, p_2 : \hat{X}_2 \to X$ are covering spaces of X. An *isomorphism of covering spaces* is a map $\varphi : \hat{X}_1 \to \hat{X}_2$ such that $p_2 \circ \varphi = p_1$.

If $\hat{x}_1, \hat{x}_2$ are bases points and $\varphi(\hat{x}_1) = \hat{x}_2$, we say φ is *based*.

Remark. φ is a lift of p_1 to $\hat{X}_2$.

Theorem 2.13 (Galois correspondence with base points)**.** *Let X be path-connected, locally simply connected space and $x_0 \in X$. Then there is a bijection between based isomorphism class of path-connected covering space $p : (\hat{X}, \hat{x}_0) \to (X, x_0)$ and subgroups of $\pi_1(X, x_0)$, given by*

$$\hat{X} \mapsto p_* \pi_1(\hat{X}, \hat{x}_0).$$

Proof. Non-examinable and omitted. □

Example. Let $X = S^1$, we have path-connteced covering space $p : \mathbb{R} \to S^1, t \mapsto e^{2\pi i t}$ and $p_n : S^1 \to S^1, z \mapsto z^n$. The subgroups of $\mathbb{Z}$ are precisely $n\mathbb{Z}$. It is easy to see that p corresponds to 0 and p_n correponds to $n\mathbb{Z}$. Galois correspondence then tells us that these are all the path-connected covering space of S^1 up to isomorphism.

Corollary 2.14. *Let X be "reasonable". Then any two universal covers $p_1 : \tilde{X}_1 \to X, p_2 : \tilde{X}_2 \to X$ are isomorphic.*

Proof. Exercise. □

Corollary 2.15. *Let X be path-connected, locally simply connected and $x_0 \in X$. Then there is a bijection between isomorphism class of path-connected covering space $p : (\hat{X}, \hat{x}_0) \to (X, x_0)$ and subgroups of $\pi_1(X, x_0)$ modulo conjugation, given by*

$$\hat{X} \mapsto p_* \pi_1(\hat{X}, \hat{x}_0).$$

Proof. Surjectivity of the map follows from immediately from the previous theorem. We need to prove that if $p_{1*}\pi_1(\hat{X}_1, \hat{x}_1)$ and $p_{2*}\pi_1(\hat{X}_2, \hat{x}_2)$ are conjugate then $\hat{X}_1$ and $\hat{X}_2$ are isomorphic covering spaces. So let

$$p_{1*}\pi_1(\hat{X}_1, \hat{x}_1) = [\gamma] p_{2*}\pi_1(\hat{X}_2, \hat{x}_2)[\overline{\gamma}]. \tag{$*$}$$

Let $\hat{\overline{\gamma}}$ be the lift of $\overline{\gamma}$ and $\hat{x}_2' = \hat{\overline{\gamma}}(1)$. ($*$) then tells us that

$$\partial_{1*}\pi_1(\hat{X}_1, \hat{x}_1) = p_{2*}\hat{\gamma}_\# \pi_1(\hat{X}_2, \hat{x}_2) = p_{2*}\pi_1(\hat{X}_2, \hat{x}_2').$$

Then by the original Galois correspondence, there is a based isomorphism between $\hat{X}_1$ and $\hat{X}_2$. Of course they are isomorphic. □

Definition (covering transformation)**.** Let $p : \hat{X} \to X$ be a covering space. A *covering transformation* or *deck transformation* $\hat{X} \to \hat{X}$ is a homeomorphism that is also a cover isomorphism.

Corollary 2.16. *Let X be "reasonable", path-connected and locally simply connected and $p : \tilde{X} \to X$ a universal cover. Let $x_0 \in X$ and $\tilde{x}_0 \in p^{-1}(x_0)$. Let $\tilde{x} \in p^{-1}(x_0)$. Then there is a unique covering transformation $\varphi_{\tilde{x}} : \tilde{X} \to \tilde{X}$ such that $\varphi_{\tilde{x}}(\tilde{x}_0) = \tilde{x}$.*

Proof. Both $(\tilde{X}, \tilde{x}_0)$ and $(\tilde{X}, \tilde{x})$ correspond ot the trivial subgroup of $\pi_1(X, x_0)$ so the result follows from 2.27. ☐

Now we have two different correspondences:

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In fact these are isomorphic. automorphism of universal covers is isomorphic to fundamental group of base group.

We can thus make $\pi_1(X, x_0)$ act on $\tilde{X}$ on the *left* by covering transformation.

Remark. Left vs. right action. Abelian group in case of S^1.

3 Seifert-van Kampen theorem

So far we have only seen one space with nontrivial fundamental group. In general, the fundamental groups are notoriously difficult to compute. In this chapter, we will develop the machinery needed to divide and conquer the problem of finding the fundamental group of a complex space. Specifically, given $X = Y_1 \cup Y_2$, we will ultimate describe $\pi_1 X$ in terms of $\pi_1 Y_1, \pi_1 Y_2$ and $\pi_1(Y_1 \cap Y_2)$. But before that, we have to develop more group theory.

3.1 Free groups and presentations

We have seen groups described in the following form in IA Groups:

$$D_{2n} = \langle r, s | s^2 = r^n = e, srs = r^{-1} \rangle$$

where we impose *relations* on the right on the group generated by the *generators* on the left. This is an example of a *presentation*. What should be the group generated by the generators be? Should it, for example, have an elemnet of order 2? Morally, the answer should be "no" as we should move all relations to the right. This leaves us with a free group, which is a group with no relation. Given a set A of generators, called an *alphabet*, FA is the free group generated by A. Thus a free group has presentation

$$FA = \langle a \in A \rangle.$$

Formally

Definition (free group)**.** A group $F(A)$ equipped with a map of set $A \to F(A)$ is the *free group* on A if it satisfies the following universal property: whenever G is a group and $A \to G$ is a set map there is a unique canonical homomorphism $f : F(A) \to G$ such that

$$\begin{tikzcd} F(A) \arrow[dr, "f"] & \\ A \arrow[u] \arrow[r] & G \end{tikzcd}$$

commutes.

Example.

1. $F(\emptyset) \cong 1$.
2. Let $A = \{a\}$. If $A \to G, a \mapsto g$, define $f : \mathbb{Z} \to G, n \mapsto g^n$. Then the diagram

   $$\begin{tikzcd} \mathbb{Z} \arrow[dr, "f"] & \\ A \arrow[u] \arrow[r] & G \end{tikzcd}$$

   commutes. Thus $\mathbb{Z}$ is the free group on A.

Remark.

1. Free group is defined uniquely up to a unique isomorphism: suppose $A \to F'(A)$ also satisfies the universal property. Take $G = F'(A)$ in the universal property for $F(A)$, then there is a canonical homomorphism $f : F(A) \to F'(A)$ such that

$$\begin{tikzcd} F(A) \arrow[dr] & \\ A \arrow[u] \arrow[r] & F'(A) \end{tikzcd}$$

commutes. Conversely, take $G = F(A)$ in the universal property for $F'(A)$, then there is a canonical homomorphism $f' : F'(A) \to F(A)$ such that the corresponding diagram commutes. Now both $\mathrm{id}_{F(A)}$ and $f' \circ f$ both make the diagram commute so by uniqueness $f' \circ f = \mathrm{id}_{F(A)}$. Likewise $f \circ f' = \mathrm{id}_{F'(A)}$ so f and f' are isomorphisms.

2. The definition does not guarantee the existence of free groups. We'll cover this later.

Notation. We identify $a \in A$ with its image in $F(A)$.

Definition (presentation)**.** Let A be an *alphabet*. A subset $R \subseteq F(A)$ defines a *(group) presentation*

$$\langle A | R \rangle = F(A)/\langle\langle R \rangle\rangle$$

where $\langle\langle R \rangle\rangle$ is the normal closure of R in $F(A)$.

Example.

1. $\langle a | a^n \rangle \cong \mathbb{Z}/n\mathbb{Z}$.
2. $\langle r, s | r^n, s^2, srsr \rangle \cong D_{2n}$.

Lemma 3.1 (universal property of group presentation)**.** *Given a presentation $\langle A|R \rangle$ and the quotient map $q : F(A) \to \langle A|R \rangle$, for any homomorphism $g : F(A) \to G$ such that $g(r) = 1$ for all $r \in R$, there exists a unique homomorphism $f : \langle A|R \rangle \to G$ such that $f \circ q = g$. In other words, the following diagram commutes:*

$$\begin{tikzcd} \langle A|R \rangle \arrow[dr, "f"] & \\ F(A) \arrow[u, "q"] \arrow[r, "g"] & G \end{tikzcd}$$

Proof. Follows easily from universal property of quotient map. $\square$

Definition (pushout)**.** Let $i : C \to A, j : C \to B$ be group homomorphisms. Homomorphism $k : A \to \Gamma, \ell : B \to \Gamma$ is a *pushout* if it satisfies the following property: for any group G and homomorphisms $f : A \to G, g : B \to G$ such that $f \circ i = g \circ j$, then there is a unique homomorphism $\varphi : \Gamma \to G$ such that

$f = \varphi \circ k, g = \varphi \circ \ell$. In other words the following diagram commutes.

$$\begin{tikzcd}
C \arrow[r, "i"] \arrow[d, "j"] & A \arrow[d, "k"] \arrow[ddr, bend left, "f"] & \\
B \arrow[r, "\ell"] \arrow[drr, bend right, "g"] & \Gamma \arrow[dr, dashed, "\varphi"] & \\
& & G
\end{tikzcd}$$

Again Γ is uniquely defined by the universal property.

We mainly care about special cases of the definition.

Definition (free product, amalgamated free product)**.** If $C \cong 1$, then Γ is called the *free product* of A and B, denoted $A * B$.

More generally, if i and j are injective then Γ is called the *amalgamated free product*, denoted $A *_C B$.

Example. $\mathbb{Z} * \mathbb{Z} \cong F_2$ since they satisfy the same universal property. More generally, we can check that

$$\underbrace{\mathbb{Z} * \mathbb{Z} * \cdots * \mathbb{Z}}_{r} \cong F_r.$$

Notation. Write F_n for the free group with n generators.

Lemma 3.2.

$$\begin{tikzcd}
C \arrow[r, "i"] \arrow[d] & A \arrow[d] \\
1 \arrow[r] & A/\langle\langle i(C) \rangle\rangle
\end{tikzcd}$$

is a pushout.

Proof. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Etiam lobortis facilisis sem. Nullam nec mi et neque pharetra sollicitudin. Praesent imperdiet mi nec ante. Donec ullamcorper, felis non sodales commodo, lectus velit ultrices augue, a dignissim nibh lectus placerat pede. Vivamus nunc nunc, molestie ut, ultricies vel, semper in, velit. Ut porttitor. Praesent in sapien. Lorem ipsum dolor sit amet, consectetuer adipiscing elit. Duis fringilla tristique neque. Sed interdum libero ut metus. Pellentesque placerat. Nam rutrum augue a leo. Morbi sed elit sit amet ante lobortis sollicitudin. Praesent blandit blandit mauris. Praesent lectus tellus, aliquet aliquam, luctus a, egestas a, turpis. Mauris lacinia lorem sit amet ipsum. Nunc quis urna dictum turpis accumsan semper. □

presentation for free group with amalgamation

3.2 Seifert-van Kampen theorem for wedges

Definition (wedge)**.** Given two pointed spaces $(X, x_0), (Y, y_0)$, the *wedge* is

$$X \vee Y = X \amalg Y/(x_0 \sim y_0).$$

Usually X and Y are path-connected so we can define wedges $X \vee Y$ without specifying basepoints.

Theorem 3.3 (Seifert-van Kampen for wedges)**.** *If Y_1, Y_2 are path-connected and x_0 is the wedge point of $X = Y_1 \vee Y_2$. Then*

$$\pi_1(X, x_0) = \pi_1(Y_1, x_0) * \pi_1(Y_2, x_0).$$

Sketch of proof. non-examinable

Suppose $f_1 : \pi_1(Y_i, x_0) \to G$ are group homomorphisms for $i = 1, 2$. We need to find a unique $\phi : \pi_1(X, x_0) \to G$ such that ϕ restricts to f_i on $\pi_1(Y_i, x_0)$.

First replace X by X' (drawing) with $X \simeq X'$. Let $\gamma : I \to X'$ be a based loop. We can "straighten" γ so that it is of the form

$$\gamma = \alpha_1 \cdot \beta_1 \cdot \alpha_2 \cdot \beta_2 \cdots \alpha_n \cdot \beta_n$$

where α_i's are in $\pi_1(Y_1, x_0)$ and β_i's are in $\pi_1(Y_1, x_0)$. Define

$$\phi(\gamma) = f_1(\alpha_1) f_2(\beta_2) f_1(\alpha_2) \cdots f_2(\beta_{n-1}) f_1(\alpha_n) f_2(\beta_n)$$

uniquely. This is easily seen to be a homomorphism but we need to prove that ϕ is well-defined. Let $\gamma' \simeq_F \gamma$ with

$$\gamma' = \alpha_1' \cdot \beta_1' \cdots \alpha_m' \beta_m'$$

so

$$\phi(\gamma') = f_1(\alpha_1') f_2(\beta_1') \cdots f_1(\alpha_m') f_2(\beta_m'),$$

we need to prove that $\phi(\gamma) = \phi(\gamma')$. The key idea is to "straighten" F so that it is "transverse" to x_0: this means that $F^{-1}(x_0) \subseteq I \times I$ consists of a finite union of circles and intervales embedded in $I \times I$. If there is a cirlce $S! \subseteq F^{-1}(0)$ then we can "cut it out" and remove it. An arc with both endpoints on γ exhibit a subarc $\delta \subseteq \Gamma$ such that $\delta \simeq c_{x_0}$ in Y_1 or Y_2, reducing n without changing $\phi(\gamma)$. After finitely many of these moves, we are left with a picture of the following form (drawing). Therefore $m = n$ and $\alpha_i \simeq \alpha_i', \beta_i \cong \beta_i'$ as paths so $\phi(\gamma) = \phi(\gamma')$ as required. □

Example. Let $X = S^1 \vee S^1$, then

$$\pi_1 X \cong \pi_1 S^1 * \pi_1 S^1 \cong \mathbb{Z} * \mathbb{Z} \cong F_2.$$

More generally, let $X_r = \bigvee_{i=1}^r S^1$, sometimes called a bouquet, then

$$\pi_1 X_r \cong \underbrace{\mathbb{Z} * \cdot * \mathbb{Z}}_{r} \cong F_r.$$

3.3 Seifert-van Kampen theorem

Theorem 3.4 (Seifert-van Kampen)**.** *If $X = Y_1 \cup_Z Y_2$ with Y_1, Y_2, Z open and path-connected and $x_0 \in Z$ then the diagram*

$$\begin{array}{ccc} \pi_1(Z, x_0) & \xrightarrow{i_{1*}} & \pi_1(Y_1, x_0) \\ \downarrow{\scriptstyle i_{2*}} & & \downarrow{\scriptstyle j_{1*}} \\ \pi_1(Y_2, x_0) & \xrightarrow{j_{2*}} & \pi_1(X, x_0) \end{array}$$

is a pushout.

Proof. Omitted. □

Example. Let $X = S^n$ where $n \geq 2$. Let $x_\pm = (\pm 1, 0, \dots, 0)$ be the north/south poles and define

$$\begin{aligned} U_\pm &= S^n - \{x_\mp\} \\ V &= U_+ \cap U_- = S^n - \{x_\pm\} \end{aligned}$$

Then $X = U_+ \cup_V U_-$. Stereographic projection tells us that $U_\pm \cong \mathbb{R}^n$. Project V radially onto the cylinder $(-1, 1) \times S^{n-1}$, which is a homeomorphism so $V \cong (-1, 1) \times S^{n-1} \simeq S^{n-1}$. S^{n-1} is path-connected for $n \geq 2$ so by Seifert-van Kampen the following diagram is a pushout:

$$\begin{array}{ccc} \pi_1(S^{n-1}, x_0) & \longrightarrow & 1 \\ \downarrow & & \downarrow \\ 1 & \longrightarrow & \pi_1(S^n, x_0) \end{array}$$

so $\pi_1(S^n, x_0)$ is a quotient of 1 so is trivial.

Definition (neighbourhood deformation retract)**.** A subset $Y \subseteq X$ is called a *neighbourhood deformation retract* if there exists $Y \subseteq V \subseteq X$ where V is open in X such that Y is a deformation rectraction of V.

Corollary 3.5. *If $X = Y_1 \cup_Z Y_2$ with Y_1, Y_2, Z path-connected and closed and Z a neighbourhood deformation retract of Y_1 and Y_2 and $x_0 \in Z$ then*

$$\begin{array}{ccc} \pi_1(Z, x_0) & \longrightarrow & \pi_1(Y_1, x_0) \\ \downarrow & & \downarrow \\ \pi_1(Y_2, x_0) & \longrightarrow & \pi_1(X, x_0) \end{array}$$

is a pushout.

Proof. See online notes. □

3.4 Attaching cells

Definition (cell)**.** An *n-cell* is a copy of D^n, the closed ball in $\mathbb{R}^n$.

Definition. Let $\alpha : S^{n-1} = \partial D^n \to X$ be a continuous map. The space

$$X \cup_\alpha D^n := X \amalg D^n / \sim$$

where $\sim$ is the finest equivalence relation such that $\alpha(\theta) \sim \theta$ for all $\theta \in S^{n-1}$, is called an *attaching cell*.

What effect does attaching an n-cell have on π_1?
Let's start with $n \geq 3$:

Lemma 3.6. *If $n \geq 3$ and $\alpha : S^{n-1} \to X$ is a continuous map. Let $x_0 = \alpha(\theta_0)$ for $\theta_0 \in S^{n-1}$. Then the (not necessarily injective) inclusion map $i : X \to X \cup_\alpha D^n$ induces an isomorphism $i_* : \pi_1(X, x_0) \to \pi_1(X \cup_\alpha D^n, x_0)$.*

Proof. The main obstacle is that α might not be injective. However, we can divide D^n into two parts and attach D^n in two stages: the mapping cylinder of α is

$$M_\alpha := X \amalg (S^{n-1} \times I) / \sim$$

where $\alpha(\theta) \sim (\theta, 0)$ for all $\theta \in S^{n-1}$. Note that

1. X is a deformation retract of M_α.
2. $S^{n-1} \times \{1\} \subseteq M_\alpha$ is a neighbourhood deformation retract.
3. $S^{n-1} \subseteq D^n$ is a neighbourhood deformation retract.

If we choose $\theta_1 \in S^{n-1}$, the previous corollary tells us that

$$\begin{tikzcd}
\pi_1(S^{n-1}, \theta_1) \arrow[r] \arrow[d] & \pi_1(M_\alpha, \theta_1) \arrow[d, "j_*"] \\
\pi_1(D^n, \theta_1) \arrow[r] & \pi_1(M_\alpha \cup_{S^{n-1}} D^n, \theta_1)
\end{tikzcd}$$

is a pushout. Therefore the inclusion $j : M_\alpha \to M_\alpha \cup_{S^{n-1}} D^n$ induces an isomorphism on π_1. Since $X \cup_\alpha D^n = M_\alpha \cup_{S^{n-1}} D^n$ and M'_α deformation retracts to X, the result follows. □

What about $n = 2$?

Lemma 3.7. *If $\alpha : S^1 \to X$ is a continuous map and $x_0 = \alpha(\theta_0)$ where $\theta_0 \in S^1$. Then*

$$\pi_1(X \cup_\alpha D^2, \theta_0) \cong \pi_1(X, x_0) / \langle\langle [\alpha] \rangle\rangle$$

and the inclusion map $X \hookrightarrow X \bigcap_\alpha D^2$ induces the quotient map

$$\pi_1(X, x_0) \to \pi_1(X \cup_\alpha D^2, x_0).$$

Proof. As in the proof of the previous lemma, the diagram

$$\begin{tikzcd} \pi_1(S^1, \theta_0) \arrow[r, "\alpha_*"] \arrow[d] & \pi_1(X, x_0) \arrow[d, "i_*"] \\ \pi_1(D^2, \theta_0) \arrow[r] & \pi_1(X \cup_\alpha D^2, x_0) \end{tikzcd}$$

is a pushout. By lemma 3.2 the result follows. □

Theorem 3.8. *If $G = \langle A|R \rangle$ with A, R both finite then it is the fundamental group of some space. Moreover the spaces can be taken to be compact.*

In fact, we don't have to restrict our attention to finitely generated or finitely presented groups. So every group is the fundamental group of some space (although not compact in general).

Proof. If $R = \{r_1, \dots r_n\}$ then

$$\begin{aligned} G &= F(A)/\langle\langle r_1, \dots, r_n \rangle\rangle \\ &\cong (F(A)/\langle\langle r_1, \dots, r_{n-1} \rangle\rangle)/\langle\langle r_n \rangle\rangle \\ &\cong \dots \\ &\cong (\dots (F(A)/\langle\langle r_1 \rangle\rangle) \dots)/\langle\langle r_n \rangle\rangle, \end{aligned}$$

one way to check this is to show they satisfy the same universal property. Now induciton on n, with the base case $n = 1$ being the wedge of $|A|$ circles. □

3.5 Classification of surfaces

Definition (topological manifold)**.** An *n-dimensional (topological) manifold* is a Hausdorff space M such that every $x \in M$ has an open neighbourhood U homeomorphic to an open subset of $\mathbb{R}^n$.

Definition (surface)**.** A 2-dimensional manifold is called a *surface*.

Example. Let $\alpha : S^1 \to *$. Consider $X = * \cup_\alpha D^2$. Note that $\int D^2 \cong \mathbb{R}^2$ and $S^2 - \{x_+\} \cong \mathbb{R}^2$ via stereographic projection. Moreover the homeomorphism $i : \int D^2 \to S^1 - \{x_+\}$ extends to a unique continuous bijection $X \to S^2$, so a homeomorphism. In particular S^2 is a surface.

Example. Let $\Gamma_{2g} = \bigvee_{i=1}^{2g} S_i^1$, with each $S_i' \cong S^1$. Choose unit speed loops $\alpha_1, \dots, \alpha_g$ and $\beta_1, \dots, \beta_g$ in the circles. Let

$$\rho_g = (\alpha_1 \cdot \beta_1 \cdot \overline{\alpha}_1 \cdot \overline{\beta}_1) \cdot (\alpha_2 \cdot \beta_2 \cdot \overline{\alpha}_2 \cdot \overline{\beta}_2) \dots (\alpha_g \cdot \beta_g \cdot \overline{\alpha}_g \cdot \overline{\beta}_g)$$

and let

$$\Sigma_g = \Gamma_{2g} \cup_{\rho_g} D^2.$$

Claim Σ_g is a surface. There are three cases to consider. If a point in the interior of D^2 then it has a neighbourhood homeomorphic to an open disk. If a point is in the interior of image of a path the "two parts" glue together to form an open disk. Similary all the edges are identified together.

Σ_0 is just S^2. Σ_1 is the square with two sides identified to a torus. In general Σ_g is called the (orientable surface) with genus g.

Example. Let $\Gamma_{g+1} = \bigvee_{i=0}^{g} S_i^1$ and let

$$\sigma_j = \alpha_0 \cdot \alpha_0 \cdot \alpha_1 \cdot \alpha_1 \dots \alpha_g \cdot \alpha_g$$

let

$$S_g = \Gamma_{g+1} \cup_{\sigma_g} D^2.$$

Similarly we can check these are surfaces. This is the *non-orientable surface* of genus g. $S_0 = \mathbb{R}P^2$ and S_1 is the Klein bottle.

We have

$$\begin{aligned} \pi_1 \Sigma_g &= \langle a_1, \dots, a_g, b_1, \dots, b_g | a_1 b_1 a_1^{-1} b_1^{-1} \cdots a_g b_g a_g^{-1} b_g^{-1} \rangle \\ \pi_1 S_g &= \langle a_0, \dots, a_g | a_0^2 a_1^2 \cdots a_g^2 \rangle \end{aligned}$$

We state without proof

Theorem 3.9 (classification of compact surfaces)**.** *If M is a compact surface then either $M \cong \Sigma_g$ or $M \cong S_g$.*

We won't prove this but a good point to start is to consider given an identification of S^1 of D^2, how can we convert it into one of the two forms?

We also ask the following question: are $\{\Sigma_g\}$ and $\{S_g\}$ pairwise non-homeomorphic? What about homotopy equivalence? The only tool available to us is π_1. The strategy is to that the fundamental groups map onto different abelian groups.

Lemma 3.10. *Let $g \in \mathbb{N}$.*

1. *The group $\pi_1 \Sigma_g$ surjects $\mathbb{Z}^{2g}$ but not $\mathbb{Z}^{2g} \oplus (\mathbb{Z}/(2))$.*
2. *The group $\pi_1 S_g$ surjects $\mathbb{Z}^g \oplus (\mathbb{Z}/(2))$ but not $\mathbb{Z}^{g+1}$.*

Proof. Easy. See notes. □

And as a result we get want we want

Corollary 3.11. *The strategy works.*

4 Simplicial complexes

We have seen that the fundamental groups are useful, and for example, when it works, it tells us S^n is contractible for $n > 1$. There are higher dimensional analogues of π_1, called the *homotopy groups* π_n. However, they are notoriously difficult to compute. Instead, we will use (more or less) the only thing in mathematics we understand fully (again, more or less) — linear algebra. This is called *homology*.

There are many types of homologies and we'll only define *simplicial homologies* in this course.

4.1 Simplices and stuff

Definition. A finite set $V \subseteq \mathbb{R}^n$ is in *general position* if the smallest affine subspace containing V is of dimension $|V| - 1$.

This is quite an abstract definition, but there are a few equivalent notions. For example, if $V = \{v_0, \dots, v_n\}$ then for any $t_0, \dots t_n$ such that $\sum_{i=0}^n t_i = 0$, if $\sum_{i=0}^n t_i v_i = 0$ then $t_i = 0$ for all i.

Definition (simplex)**.** For $n \geq 0, V = \{v_0, \dots, v_n\} \subseteq \mathbb{R}^m$. The *span* of V is

$$\langle V \rangle = \{\sum_{i=0}^n t_i v_i : t_i \geq 0, \sum_{i=0}^n t_i = 1\}.$$

If V is in general position, $\sigma = \langle V \rangle$ is an *n-simplex*.

Definition (face)**.** Let $V = \{v_0, \dots, v_n\}$ in general position. If $U \subseteq V$ then $\langle U \rangle$ is called a *face* of $\langle V \rangle$, write $\langle U \rangle \leq \langle V \rangle$.

If $U \neq V$ then $\langle U \rangle$ is called a *proper* face.

Definition (simplicial complex, dimension, skeleton)**.** A *simplicial complex* is a finite set of simplices K in some $\mathbb{R}^m$ satisfying the following condition:

1. if $\sigma \in K$ and $\tau \leq \sigma$ then $\tau \in K$,
2. if $\sigma, \tau \in K$ then $\sigma \cap \tau \leq \sigma$ and $\sigma \cap \tau \leq \tau$.

The *dimension* of K, denoted $\dim K$, is the largest n such that K contains an n-simplex.

The *d-skeleton* of K is

$$K_{(d)} = \{\sigma \in K : \dim \sigma \leq d\}.$$

Example.

1. If σ is a simplex then $K = \{\tau : \tau \leq \sigma\}$ is a simplicial complex.
2. If σ is a simplex then the set of proper faces of σ, denoted $\partial\sigma$, is a simplicial complex. It is called the *boundary* of σ. The set of points in σ not in a simplex of $\partial\sigma$ is called the *interior*, denoted by $\mathring{\sigma}$.

Note that if σ is a 0-simplex then $\mathring{\sigma} = \sigma$.

Definition (realisation/polyhedron)**.** The *realisation* or *polyhedron* of a simplicial complex K is the union of the simplices in K, denoted by $|K|$.

Example.

1. In $\mathbb{R}^{n+1}$, the standard basis $\{e_0, \dots e_n\}$ is in general position. The simplex it spans $\sigma_n = \langle e_0, \dots, e_n \rangle$ is called the *standard n-simplex.*
2. The *standard (simplicial) $(n-1)$-sphere* is $\partial \sigma_n$.

Definition (triangulation)**.** A *triangulation* of a space X is a homeomorphism $h : |K| \to X$.

It's not hard to see that there is a triangulation $h : |\partial \sigma_n| \to S^{n-1}$.

Example. Here is another way of triangulating S^n. For now set $n = 2$. The convex hull of $\{\pm e_0, \pm e_1, \pm e_2\}$ is a surface of an octahedron, which is triangulation of S^2. In general, let $\{e_0, \dots, e_n\}$ be the standard basis for $\mathbb{R}^{n+1}$ and $E = \{\pm e_0, \dots, \pm e_n\}$. Let

$$E_0 = \{S \subseteq E : \text{ for all } i \text{ exactly one of } \pm e_i \text{ is in } S\}.$$

Let $K = \{\langle S \rangle : S \in E_0\}$. This is the *octahedral n-sphere* and there exists a triangulation $|K| \to S^n$.

Definition (simplicial map)**.** Let K, L be simplicial complexes. A *simplicial map* $f : K \to L$ is a map such that for all $\langle v_0, \dots, v_n \rangle \in K$,

$$f(\langle v_0, \dots, v_n \rangle) = \langle f(v_0), \dots, f(v_n) \rangle$$

where $f(\{v_i\}) = \{f(v_i)\}$.

The *realisation* of $f : K \to L$ is the continuous map $|f| : |K| \to |L|$ defined on $\sigma = \langle v_0, \dots, v_n \rangle$ to be

$$f_\sigma \left(\sum_{i=0}^n t_i v_i \right) = \sum_{i=0}^n t_i f(v_i).$$

Note that if $\tau \leq \sigma$ then $f_\tau = f_\sigma|_\tau$, so $|f|$ is well-defined and continuous.

Example. (drawing)

4.2 Barycentric subdivision

Realisaition of simplicial maps are piecewise linear and thus very rigid. On the other hand, the realisations of simplicial complexes, as topological spaces, are "deformable". Is every continuous map $|K| \to |L|$ homotopic to a realisation of a simplicial map? For example for $K = L = \partial \sigma_2$, there are infinitely many homotopy classes of continuous maps, which are in bijection with $\pi_1(S^1)$. On the other hand there are only finitely many simplicial map $K \to L$, and thus at most that many realisations. To establish the correspondence, we need subdivision.

Definition (barycentre)**.** If $\sigma = \langle v_0, \dots, v_n \rangle$, the *barycentre* of σ is

$$\hat{\sigma}_n = \frac{1}{n+1} \sum_{i=0}^{n} v_i.$$

Definition (barycentric subdivision)**.** Suppose K is a simplicial complex. The *barycentric subdivision* of K is K' with vertices $\{\hat{\sigma} : \sigma \in K\}$. A collection of barycentres $\{\hat{\sigma}_0, \dots, \hat{\sigma}_n\}$ spans a simplex in K' whenever $\sigma_0 \leq \sigma_1 \leq \dots \leq \sigma_n$.

Lemma 4.1. *K' is a simplicial complex and $|K'| = |K|$.*

Proof. See online notes. $\square$

Definition. We define the rth barycentric subdivision to be

$$\begin{aligned} K^{(0)} &= K \\ K^{(r)} &= (K^{(r-1)})' \end{aligned}$$

Definition (mesh)**.** Let K be a simplicial complex. define the *mesh* of K to be

$$\operatorname{mesh}(K) = \max_{\langle v_0, v_1 \rangle \in K} \|v_0 - v_1\|_2.$$

Here the 2-norm is just taken for the sake of convenience and concreteness.

Lemma 4.2. *If $\dim K = n$ then*

$$\operatorname{mesh}(K^{(r)}) \leq \left(\frac{n}{n+1} \right)^r \operatorname{mesh}(K).$$

In particular

$$\lim_{r \to \infty} \operatorname{mesh}(K^{(r)}) = 0.$$

Proof. $\dim K' = \dim K = n$ so by induction it suffices to show that

$$\operatorname{mesh}(K') \leq \frac{n}{n+1} \operatorname{mesh}(K).$$

A 1-simplex in K' is of the form $\langle \hat{\tau}, \hat{\sigma} \rangle$ where $\tau \leq \sigma$. Note that K' is a finite set and mesh is realised by some pairs of vertices. By a bit geometric reasoning this is achieved by some vertex. We may thus assume that $\hat{\tau} = v_0$, a vertex of

$\sigma = \langle v_0, \dots, v_m \rangle$. Thus

$$\begin{aligned}
\|\hat{\tau} - \hat{\sigma}\| &= \left\| v_0 - \frac{1}{m+1} \sum_{i=0}^{m} v_i \right\| \\
&= \left\| \frac{m}{m+1} v_0 - \frac{1}{m+1} \sum_{i=1}^{m} v_i \right\| \\
&= \frac{1}{m+1} \left\| \sum_{i=1}^{m} (v_0 - v_i) \right\| \\
&\le \frac{1}{m+1} \sum_{i=1}^{m} v_0 - v_1 \\
&\le \frac{m}{m+1} \operatorname{mesh}(K) \\
&\le \frac{n}{n+1} \operatorname{mesh}(K)
\end{aligned}$$

□

4.3 Simplicial approximation theorem

Definition (star)**.** Let v be a vertex of K. The *star* of v is

$$\mathrm{St}_K(v) = \bigcup_{v \in \sigma \in K} \mathring{\sigma}$$

Definition (simplicial approximation)**.** Let $\phi : |K| \to |L|$ be a continuous map. A simplicial map $f : K \to L$ is a *simplicial approximation* of ϕ if for every vertex v of K,

$$\phi(\mathrm{St}_K(v)) \subseteq \mathrm{St}_L(f(v)).$$

Lemma 4.3. *If $f : K \to L$ is a simplicial approximation to $\phi : |K| \to |L|$ then $|f| \simeq \phi$.*

Proof. Suppose $|L| \subseteq \mathbb{R}^m$ as usual. Consider the straightline homotopy H between $|f|$ and φ. We will prove that H stays inside $|L|$. Let $x \in \mathring{\sigma}$ and let $\phi(x) \in \mathring{\tau}$. We'll show that $f(\sigma) \le \tau$. The result then follows because τ is a convex subset of R^m.

Let $\sigma = \langle v_0, \dots, v_n \rangle$. For each i, $x \in \mathrm{St}_K(v_i)$ so

$$\phi(x) \in \phi(\mathrm{St}_K(v_i)) \subseteq \mathrm{St}_L(f(v_i))$$

so $f(v_i)$ is a vertex of τ. So $f(\sigma)\tau$ as desired. □

Theorem 4.4 (simplicial approximation theorem)**.** *Let K, L be simplicial complexes and $\phi : |K| \to |L|$ a continuous map. For some $r \in \mathbb{N}$ there is a implicial approximation to ϕ, $f : K^{(r)} \to L$.*

Proof. Let
$$U = \{\phi^{-1}(\mathrm{St}_L(u)) : u \text{ a vertex of } L\}$$
which is an open cover of $|K|$. By Lebesgue number lemma there is $\delta > 0$ such that for all $x \in |K|$, there exists a vertex of L such that
$$B(x, \delta) \subseteq \phi^{-1}(\mathrm{St}_L(u)).$$
Choose r large enough such that $\mathrm{mesh}(K^{(r)}) < \delta$. Then for any vertex v of $K^{(r)}$,
$$\mathrm{St}_{K^{(r)}}(v) \subseteq B(v, \delta) \subseteq \phi^{-1}(\mathrm{St}_L(u))$$
for some u. Set $f(v) = u$ for some such u. Left to check this is a simplicial map, i.e. for all $\sigma \in K^{(r)}, f(\sigma) \in L$. But as in the proof of the previous lemma, if $x \in \mathring{\sigma}$ and $\phi(x) \in \mathring{\tau}$ then $f(\sigma)$ must be a face of τ. $\square$

5 Homology

5.1 Simplicial homology

The analogue in simplices of a path is a *chain*, which is a formal sum of simplices. If we interpret positive coefficient as copies of a simplex, what does it mean to have a negative simplex? To make sense of this we need the notion of *oriented simplex*.

Definition (orientation)**.** Let $V = (v_0, \dots, v_n)$ be an ordered set of points in general position in $\mathbb{R}^M$. Consider the natural action of S_{n+1} on V. The subgroup $A_{n+1} \leq S_{n+1}$ has 2 orbits on V, as long as $n \geq 1$. An *orientation* on $\sigma = \langle V \rangle$ is a choice of A_{n+1}-orbit under the action on V.

We will abuse notation and write $\langle v_0, \dots, v_n \rangle$ for the simplex $\langle v_0, \dots, v_n \rangle$ equipped with the orientation which is the A_{n+1}-orbit of $(v_0, \dots, v_n)$.

Example. Let $V = \{v_0, v_1\}$. The two possible orientations are $\langle v_0, v_1 \rangle$ and $\langle v_1, v_0 \rangle$, which corresponds to "arrows going in opposite directions".

Example. Let $V = \{v_0, v_1, v_2\}$. There are two orientations, for exmaple $\langle v_0, v_1, v_2 \rangle$ and $\langle v_2, v_1, v_0 \rangle$ are two representatives.

Definition (chain)**.** Let K be a simplicial complex. The group of *n-chains* on K is

$$C_n(K) = \bigoplus_{\sigma \in K, \dim \sigma = n} \langle \sigma \rangle.$$

In particular if there are no n-simplex (e.g. $n > \dim K$ or $n < 0$) then $C_n(K) \cong 0$. Arbitrarily choose orientations on the simplices of K and then identify $-\sigma$ with the opposite oriented simplex. Note that this arbitrary choice isn't important — it could be realised by an automorphism of $C_n(K)$.

Remark. Note that these groups are abelian, which is a huge advantage compared to homotopy groups if you actually want to do anything with them. On the other hand, it also means that there are things that a homotopy group can see but homology groups cannot.

Definition (boundary homomorphism)**.** The *(nth) boundary homomorphism ∂_n*, usually just written as ∂, is defined by

$$\begin{aligned} C_n(K) &\to C_{n-1}(K) \\ \langle v_0, \dots, v_n \rangle &\mapsto \sum_{i=0}^{n} (-1)^i \langle v_0, \dots, \hat{v}_i, \dots, v_n \rangle \end{aligned}$$

where $\hat{v}_i$ means that the vertex v_i is omitted.

Note this is well-defined.

Example. Let $\sigma = \langle v_0, v_1 \rangle$. Then $\partial(\sigma) = \langle v_1 \rangle - \langle v_0 \rangle$.

Example. Let $\sigma = \langle v_0, v_1, v_2 \rangle$. Then

$$\begin{aligned}\partial(\sigma) &= \langle v_1, v_2 \rangle - \langle v_0, v_2 \rangle + \langle v_0, v_1 \rangle \\ &= \langle v_1, v_2 \rangle + \langle v_2, v_0 \rangle + \langle v_0, v_1 \rangle\end{aligned}$$

Definition (cycle, boundary)**.** Let $n \in \mathbb{Z}$. The group

$$Z_n(K) = \ker \partial_n \leq C_n(K)$$

is the group of *n-cycles.*

The group

$$B_n(K) = \operatorname{im} \partial_{n+1} \leq C_n(K)$$

is the group of *n-boundaries.*

These are analogous to loops and homotopies respectively.

Lemma 5.1. *Every n-boundary is an n-cycle, i.e.*

$$B_n(K) \leq Z_n(K),$$

i.e.

$$\partial_n \circ \partial_{n+1} = 0.$$

Proof. Let $\sigma = \langle v_0, \dots, v_n \rangle$. By definition

$$\partial(\sigma) = \sum_{i=0}^{n} (-1)^i \langle v_0, \dots, \hat{v}_i, \dots, v_n \rangle$$

so

$$\begin{aligned}\partial \circ \partial(\sigma) &= \sum_{i,j<i} (-1)^i (-1)^j \langle v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots, v_n \rangle \\ &\quad + \sum_{i,j>i} (-1)^i (-1)^{j-1} \langle v_0, \dots, v_i, \dots, v_j, \dots, v_n \rangle \\ &= \sum_{i,j<i} (-1)^{i+j} \langle v_0, \dots, \hat{v}_j, \dots, \hat{v}_i, \dots v_n \rangle \\ &\quad - \sum_{i,j>i} (-1)^{i+j} \langle v_0, \dots, \hat{v}_i, \dots, \hat{v}_j, \dots, v_n \rangle\end{aligned}$$

□

Definition (homology group)**.** The *nth homology group* of K is

$$H_n(K) = Z_n(K)/B_n(K).$$

Remark. The homology we discuss in this course is simplicial homology, which has the advanatage that all the homology groups are finitely generated. Thus in principle, $H_n(K)$ can always be computed using linear algebra. But except in the following few demonstrative examples, as a man of culture you should avoid it as much as possible.

Example. Let K be the standard simplicial circle. The vertices of K are e_0, e_1, e_2. Thus

$$\begin{aligned} C_0(K) &= \langle e_0 \rangle \oplus \langle e_1 \rangle \oplus \langle e_2 \rangle \cong \mathbb{Z}^3 \\ C_1(K) &= \langle e_0, e_1 \rangle \oplus \langle e_1, e_2 \rangle \oplus \langle e_2, e_0 \rangle \cong \mathbb{Z}^3 \\ C_n(K) &= 0 \text{ for } n > 1 \end{aligned}$$

There is only one interesting boundary map $\partial = \partial_1 : C_1(K) \to C_0(K)$. Looking at the definitions, we can write down a matrix for ∂, in the bases we choose

$$\begin{pmatrix} -1 & 0 & 1 \\ 1 & -1 & 0 \\ 0 & 1 & -1 \end{pmatrix}$$

After a bit of work we can put in Smith normal form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

so $\operatorname{im} \partial_1 \cong \mathbb{Z}^2$ (as a direct summand). $\ker \partial_1 \cong \mathbb{Z}$. Thus

$$\begin{aligned} H_0(K) &= Z_0(K)/B_0(K) = C_0(K)/\operatorname{im} \partial_1 \cong \mathbb{Z}^3/\mathbb{Z}^2 \cong \mathbb{Z} \\ H_1(K) &= Z_1(K)/B_1(K) = \ker \partial_1/0 \cong \mathbb{Z} \\ H_n(K) &= 0 \text{ for } n > 1 \end{aligned}$$

The fact that $H_1(K) \cong Z$ is related to intuitive observation that there is a "hole" in the simplicial complex. Contrast this with the next example. (We'll interpret $H_0(K)$ shortly)

Example. Let L be the standard 2-simplex $K \cup \{\sigma_2\}$ where $\sigma_2 = \langle e_0, e_1, e_2 \rangle$. We have (nontrivial) chain groups

$$\begin{aligned} C_0(L) &= C_0(K) \\ C_1(L) &= C_1(K) \\ C_2(L) &= \langle \sigma_2 \rangle \end{aligned}$$

The boundary map ∂_1 is same as before and for ∂_2, which is

$$\partial_2(\sigma_2) = \langle e_0, e_1 \rangle + \langle e_1, e_2 \rangle + \langle e_2, e_0 \rangle$$

which has a particularly simple matrix $(1, 1, 1)$. In particular ∂_2 is injective so $\ker \partial_2 = 0$. We know $\operatorname{im} \partial_2 \subseteq \ker \partial_1 \cong Z$. But we can see that $\operatorname{im} \partial_2$ is a direct summand of $C_1(L)$ so $\operatorname{im} \partial_2 = \ker \partial_1$. Thus the homology groups are

$$\begin{aligned} H_0(L) &= H_0(K) \cong \mathbb{Z} \\ H_1(L) &= Z_1(L)/B_1(L) = \ker \partial_1/\operatorname{im} \partial_2 \cong 0 \\ H_2(L) &= Z_2(L)/B_2(L) = \ker \partial_2/0 \cong 0 \end{aligned}$$

Alas! The first homology group has been killed.

Lemma 5.2. *Let K be a simplicial complex. If d is the number of path components of $|K|$ then*

$$H_0(K) \cong \mathbb{Z}^d.$$

Proof. Let $\pi_0(K)$ be the set of path components of $|K|$. Let $\mathbb{Z}[\pi_0(K)] \cong \mathbb{Z}^{|\pi_0(K)|}$ be the free abelian group generated by $\pi_0(K)$. There is a natural map

$$\begin{aligned} q : C_0(K) &\to \mathbb{Z}[\pi_0(K)] \\ \langle v \rangle &\mapsto [v] \end{aligned}$$

Because there is a vertex in every component of $|K|$, q is surjective. Note that $B_0(K) \subseteq \ker q$: $B_0(K)$ is generated by elements $\langle v \rangle - \langle u \rangle$ where $\langle u, v \rangle$ is a 1-simplex of K. Since $\langle u \rangle$ and $\langle v \rangle$ are in the same path component, $q(\langle v \rangle - \langle u \rangle) = 0$ so indeed $B_0(K) \subseteq \ker q$.

Because $H_0(K) = Z_0(K)/B_0(K) = C_0(K)/B_0(K)$, q descends to a map

$$H_0(K) \to \mathbb{Z}[\pi_0(K)].$$

Left to check this is injective, i.e. $\ker q \subseteq B_0(K)$. Note that $\ker q$ is generated by terms of the form $\langle v \rangle - \langle u \rangle$ where $[u] = [v]$. By simplicial approximation, there exists a "simplicial path" from u to v

$$c = \langle v_0, v_1 \rangle + \langle v_1, v_2 \rangle + \dots + \langle v_{k-1}, v_k \rangle$$

where $v_0 = u, v_k = v$. But $\partial_1(c) = \langle v \rangle - \langle u \rangle \in B_0(K)$ as required. □

5.2 Chain complexes & chain homotopies

Definition (chain complex). A *chain complex* $C_\bullet$ is a sequence of abelian groups $(C_n)_{n \in \mathbb{Z}}$ with $C_n = 0$ for $n < 0$ and *boundary homomorphisms* $\partial_n : C_n \to C_{n-1}$ such that

$$\partial_{n-1} \circ \partial_n = 0$$

for all n.

A *chain map* $f_\bullet : C_\bullet \to D_\bullet$ is a sequence of homomorphisms $f_n : C_n \to D_n$ such that the following diagram commutes for all n:

$$\begin{tikzcd} C_n \arrow[r, "f_n"] \arrow[d, "\partial_n"] & D_n \arrow[d, "\partial_n"] \\ C_{n-1} \arrow[r, "f_{n-1}"] & D_{n-1} \end{tikzcd}$$

Note. Note that we suppress the notational distinction between the boundary homomorphisms of $C_\bullet$ and $D_\bullet$. This is a common practice as they have different domains and there is little room for confusion.

Definition (boundary, cycle, homology). If $C_\bullet$ is a chain complex, then define *boundaries* B_n and *cycles* Z_n

$$\begin{aligned} B_n(C_\bullet) &= \operatorname{im} \partial_{n+1} \leq C_n \\ Z_n(C_\bullet) &= \ker \partial_n \leq C_n \end{aligned}$$

The *nth homology* is defined as

$$H_n(C_\bullet) = Z_n(C_\bullet)/B_n(C_\bullet).$$

Lemma 5.3. *A chain map $f_\bullet : C_\bullet \to D_\bullet$ induces a well-defined homomorphism*

$$f_* : H_n(C_\bullet) \to H_n(D_\bullet)$$

for all n.

Proof. Trivial from commutativity of ∂ and f_n. □

Example. If K is a simplicial complex then $(C_n)_{n\in\mathbb{Z}}$ form a chain complex $C_\bullet(K)$.

Lemma 5.4. *A simplicial map $f : K \to L$ induces a chain map $f_\bullet : C_\bullet(K) \to C_\bullet(L)$ by*

$$\begin{aligned} f_n : C_n &\to D_n \\ \sigma &\mapsto \begin{cases} f(\sigma) & \textit{if } \dim f(\sigma) = \dim \sigma \\ 0 & \textit{if } \dim f(\sigma) < \dim \sigma \end{cases} \end{aligned}$$

Therefore f induces homomorphisms $f_* : H_n(K) \to H_n(L)$.

In other words, f_n does exactly what you would expect, and it simply forgets simplices that are "crushed down".

Proof. Easy verification. See online notes for details. □

Example. Retraction of a standard 2-simplex K to standard 1-simplex L.

Remark. The map is functorial, i.e. given simplicial maps $K \xrightarrow{f} L \xrightarrow{g} M$, $(g \circ f)_* = g_* \circ f_*$. In addition $(\mathrm{id}_K)_* = \mathrm{id}_{H_n(K)}$.

Definition (chain homotopy)**.** Let $f_\bullet, g_\bullet : C_\bullet \to D_\bullet$ be chain maps. A *chain homotopy* $h_\bullet$ between $f_\bullet$ and $g_\bullet$ is a sequence of homomorphisms $h_n : C_n \to D_{n+1}$ such that

$$g_n - f_n = \partial_{n+1} \circ h_n + h_{n-1} \circ \partial_n$$

for all n. Write $f_\bullet \simeq g_\bullet$ or $f_\bullet \simeq_{h_\bullet} g_\bullet$.

Lemma 5.5. *If $f_\bullet \simeq g_\bullet : C_\bullet \to D_\bullet$ then*

$$f_* = g_* : H_n(C_\bullet) \to H_n(D_\bullet)$$

for all n.

Proof. Consider $[c] \in H_n(C_\bullet)$ so $c \in Z_n(C_\bullet) = \ker \partial_n$. Then

$$g_n(c) - f_n(c) = \partial_{n+1} \circ h_n(c) + \underbrace{h_{n-1} \circ \partial_n(c)}_{=0} \in B_n(D_\bullet)$$

so

$$[g_n(c)] = [f_n(c)]$$

so $f_* = g_*$ as claimed. □

Example. Continuation of the previous example.

Definition (cone)**.** A simplicial complex K is a *cone* if there is a vertex x_0 such that for every $\tau \in K$ there exists $\sigma \in K$ such that $x_0 \in \sigma$ and $\tau \leq \sigma$.

Lemma 5.6. *If K is a cone then it has the same homology as a point, i.e.*

$$H_n(K) = \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n > 0 \end{cases}$$

Proof. Let x_0 be a point as in the definition of a cone. Consider

$$i : \{\langle x_0 \rangle\} \to K$$
$$r : K \to \{\langle x_0 \rangle\}$$

the obvious inclusion and retraction. Clearly $r \circ i = \mathrm{id}_{\{\langle x_0 \rangle\}}$ so $r_* \circ i_* = \mathrm{id}_{H_n(\{\langle x_0 \rangle\})}$ for all n. Thus left to show $i_\bullet \circ r_\bullet \simeq \mathrm{id}_{C_\bullet(K)}$ as if so then $i_* \circ r_* = \mathrm{id}_{H_n(K)}$ for all n so r_* is an isomorphism and the result follows.

We write down the following chain homotopy

$$\begin{aligned} h_n : C_n(K) &\to C_{n+1}(K) \\ \langle v_0, \dots, v_n \rangle &\mapsto \begin{cases} \langle x_0, v_0, \dots, v_n \rangle & x_0 \notin \sigma \\ 0 & x_0 \in \sigma \end{cases} \end{aligned}$$

Now check directly that

$$(\mathrm{id}_{C_n(K)} - i_n \circ r_n)(\sigma) = (\partial_{n+1} \circ h_n + h_{n-1} \circ \partial_n)(\sigma)$$

for all $\sigma \in K$. There are 4 cases depending on if $x_0 \in \sigma$ and if $n = 0$. We'll do the case $x_0 \in \sigma, n \neq 0$. The others are similar but easier. Let $\sigma = \langle v_0, \dots, v_n \rangle$ and suppose $x_0 = v_j$. Now

$$\begin{aligned} (\partial \circ h + h \circ \partial)(\sigma) &= h \circ \partial(\sigma) \\ &= h(\sum_{i=0}^n (-1)^i \langle v, \dots, \hat{v}_i, \dots, v_n \rangle) \\ &= (-1)^j \langle x_0, v_0, \dots, v_{j-1}, v_{j+1}, \dots, v_n \rangle \\ &= (-1)^j (-1)^j \langle v_0, \dots, v_{j-1}, x_0, v_{j+1}, \dots, v_n \rangle \\ &= \sigma \\ &= (\mathrm{id}_{C_n(K)} - i_n \circ r_n)(\sigma) \end{aligned}$$

□

5.3 Homology of the simplex and the sphere

Example. Let $K = \{\tau \leq \sigma_n\}$ where σ_n is the standard n-simplex. By considering any vertex, K is obviously a cone so by the lemma

$$H_n(K) \cong \begin{cases} \mathbb{Z} & n = 0 \\ 0 & n \geq 1 \end{cases}$$

Example. Let $L = \partial\sigma_n \subseteq K$ be the standard $(n-1)$-sphere where $n \geq 2$. In other words $L = K - \{\sigma_n\}$.

$$\begin{tikzcd}
0 = C_{n+1}(L) \arrow[r] & C_n(L) = 0 \arrow[r] \arrow[d, hook] & C_{n-1}(L) \arrow[r] \arrow[d, equal] & C_{n-2}(L) \arrow[r] \arrow[d, equal] & \cdots \arrow[r] & C_1(L) \arrow[r] \arrow[d, equal] & C_0(L) \arrow[r] \arrow[d, equal] & 0 \\
0 = C_{n+1}(K) \arrow[r] & C_n(K) = \langle \sigma_n \rangle \arrow[r] & C_{n-1}(K) \arrow[r] & C_{n-2}(K) \arrow[r] & \cdots \arrow[r] & C_1(K) \arrow[r] & C_0(K) \arrow[r] & 0
\end{tikzcd}$$

So for $k \leq n-2$, $H_k(L) = H_k(K)$. The only interesting case is $k = n-1$. Because $H_{n-1}(K) \cong 0$, $Z_{n-1}(K) = B_{n-1}(K)$ and similarly $Z_n(K) = B_n(K) = 0$ so ∂_n is injective. Because $B_{n-1}(L) \cong 0$,

$$\begin{aligned} H_{n-1}(L) = Z_{n-1}(L)/B_{n-1}(L) &\cong Z_{n-1}(L) \\ &= Z_{n-1}(K) \\ &= B_{n-1}(K) \\ &= \operatorname{im} \partial_n \\ &\cong C_n(K) \\ &\cong \mathbb{Z} \end{aligned}$$

so

$$H_k(L) \cong \begin{cases} \mathbb{Z} & k = 0, n-1 \\ 0 & k \geq 2 \end{cases}$$

Intuitively the homology groups detect $n-1$-dimensional holes (compare to D^n). It also gives something that π_1 fails to detect and gives us a way to differentiate S^{n-1} for different n. (?)

5.4 Continuous maps and homotopies

Question: if $\phi : |K| \to |L|$ is continuous, does it induce some kind of map $\phi_* : H_n(K) \to H_n(L)$? The obvious idea to take simplicial approximation $f : K^{(r)} \to L$ of ϕ and set $\phi_* = f_* : H_n(K^{(r)}) \to H_n(L)$. This brings two immediate problems: in general this r is not 1, and moreover ϕ_* may depend on the choice of f.

> **Definition** (continguous)**.** Two simplicial maps $f, g : K \to L$ are *contingu-ous* if for every $\sigma \in K$ there exists $\tau \in L$ such that $f(\sigma), g(\sigma) \leq \tau$.

Informally this is the homotopy of simplicial maps.

Remark. Look back at the proof of lemma 4.25, we proved that if f is a simplicial approxiamtion to ϕ and if $x \in \sigma, \phi(x) \in \mathring{\tau}$ then $f(\sigma) \leq \tau$. Therefore if f, g are both simplicial approximation to ϕ then they are contiguous.

Lemma 5.7. *If $f, g : K \to L$ are contiguous then*

$$f_* = g_* : H_n(K) \to H_n(L)$$

for all n.

Proof. Need to exhibit a chain homotopy between $f_\bullet$ and $g_\bullet$. Fix a total ordering $<$ on th vertices of K. Now for each simplex $\sigma \in K$ we can write it in a unqiue way $\sigma = \langle v_0, \dots, v_n \rangle$ such that

$$v_0 < v_1 < \dots < v_n.$$

Now define a chain homotopy

$$\begin{aligned} h_n : C_n(K) &\to C_{n+1}(L) \\ \langle v_0, \dots, v_n \rangle &\mapsto \sum_{j=0}^{n} (-1)^j \langle f(v_0), \dots, f(v_j), g(v_j), \dots, g(v_n) \rangle \end{aligned}$$

$\langle f(v_0), \dots, f(v_j), g(v_j), \dots, g(v_n) \rangle = 0$ if it is not an $(n+1)$-dimensional simplex.

We can now check directly that this defines a chain homotopy. See online notes for details. □

Lemma 5.8. *Let K be a simplicial complex. A simplicial map $s : K' \to K$ is a simplicial approximation to the identity if and only if $s(\hat{\sigma})$ is a vertex of σ for all $\sigma \in K$. Futhermore such an s exists.*

Proof. In the setting, the definition of simplicial approximation tells us that

$$\mathrm{id}_{|K|}(\mathrm{St}_{K'}(\hat{\sigma})) = \mathring{\sigma} \subseteq \mathrm{St}_K(s(\hat{\sigma})),$$

which hapens if and only if $\sigma(\hat{\sigma})$ is a vertex of σ.

For all $\sigma \in K$, choose any vertex of σ and assign $s(\hat{\sigma})$ to it. A simplex of K is the form $\langle \hat{\sigma}_0, \dots, \hat{\sigma}_n \rangle$ such that $\sigma_0 \le \sigma_1 \le \dots \le \sigma_n$ so every $s(\hat{\sigma}_i)$ is a vertex of σ_n. Therefore $\langle s(\hat{\sigma}_0), \dots, s(\hat{\sigma}_n) \rangle$ is a face of σ_n, so s is a simplicial map. □

The choice of s induces a canonical homomorphism $s_* : H_n(K') \to H_n(K)$.

Proposition 5.9. *$s_* : H_n(K') \to H_n(K)$ is an isomorphism for all n.*

Proof. Postponed until Mayer-Vietoris sequence. □

Definition. Let $\alpha : |K| \to X$ be a triangulation we define

$$H_n(X) = H_n(K).$$

Let $\phi : X \to Y$ be continuous and $\alpha : |K| \to X, \beta : |K| \to Y$ be triangulations. Let $f : K^{(r)} \to L$ be a simplicial approximation to $\beta^{-1} \circ \phi \circ \alpha$. Using simplicial approximation to the identity, we identify $H_n(K^{(r)}) = H_n(K)$ for all r. Now set

$$\phi_* = f_* : H_n(X) = H_n(K) = H_n(K^{(r)}) \to H_n(L) = H_n(Y).$$

By results we have proven, ϕ_* is independent of the choice of simplicial approximation. However, we want something stronger: we want homology to be a homotopy invariant.

Theorem 5.10. *If X, Y are triangulable spaces and $\phi, \psi : X \to Y$ are homotopic. Then*
$$\phi_* = \psi_*.$$

Non-examinable. Sketch of proof. Let $\alpha : |K| \to X, \beta : |L| \to Y$. By hypothesis

$$\beta^{-1} \circ \phi \circ \alpha \simeq \beta^{-1} \circ \psi \circ \alpha : |K| \to |L|$$

are homotopic. Let $\Psi : |K| \times I \to |L|$ be such a homotopy. By example sheet 3 Q9 $|K| \times I \cong |M|$ for some simplicial complex M, such that the "top" and "bottom" $K_0, K_1 \cong K$ and embeds in M via $i : K \to M, j : K \to M$, and for all $\sigma \in K$ there exists $M_\sigma \subseteq M$ such that $|M_\sigma| \cong \sigma \times I$. Note that

$$|\partial M_\sigma| = (\sigma \times \{0\}) \cup (\sigma \times \{1\}) \cup M_{\partial\sigma}.$$

Define a chain homotopy

$$\begin{aligned} h_n : C_n(K^{(r)}) &\to C_{n+1}(M^{(r)}) \\ \sigma &\mapsto \sum \cdots \end{aligned}$$

Interpreting the equation about ∂M_σ as a statement about oriented simplices,

$$\partial \circ h(\sigma) = j(\sigma) - i(\sigma) - h \circ \partial(\sigma)$$

so
$$j(\sigma) - i(\sigma) = \partial \circ h(\sigma) + h \circ \partial(\sigma).$$

Note that $F \circ i$ is a simplicial approximation to $\phi = \Phi \circ i$ and $F \circ j$ is a simplicial approximation to $\tau = \Phi \circ j$. Thus

$$F \circ j(\sigma) - F \circ i(\sigma) = F \circ \partial \circ h(\sigma) + F \circ h \circ \partial(\sigma) = \partial \circ (F \circ h)(\sigma) + (F \circ h) \circ \partial(\sigma)$$

so $F \circ h$ is a homotopy between $F \circ i$ and $F \circ j$. Thus

$$\phi_* = (F \circ i)_* = (F \circ j)_* = \psi_*.$$

□

Lemma 5.11. *If X, Y, Z are triangulable and $X \xrightarrow{\phi} Y \xrightarrow{\psi} Z$ then*

$$(\psi \circ \phi)_* = \psi_* \circ \phi_*.$$

Also $(\mathrm{id}_X)_* = \mathrm{id}_{H_n(X)}$.

Proof. Omitted. □

Corollary 5.12. *If X, Y are triangulable and $\phi : X \to Y$ is a homotopy equivalence then $\phi_* : H_n(X) \to H_n(Y)$ is an isomorphism for all n.*

In other words, homology is a homotopy invariance.

6 Homology calculations

6.1 Homology of spheres and applications

Example. $S^{n-1} \cong |L|$ where L is the standard simplicial $(n-1)$-sphere. Therefore

$$H_k(S^{n-1}) \cong H_k(L) = \begin{cases} \mathbb{Z} & k = 0, n-1 \\ 0 & \text{otherwise} \end{cases}$$

Two implications: first since

$$H_{n-1}(S^{n-1}) \cong \mathbb{Z} \ncong 0 \cong H_{n-1}(*)$$

S^{n-1} is not contractible. Secondly since

$$H_{n-1}(S^{n-1}) \cong \mathbb{Z} \ncong 0 \cong H_{n-1}(S^{m-1})$$

for $m \neq n$ we see $S^{n-1} \not\simeq S^{m-1}$ unless $n = m$.

Theorem 6.1 (invariance of domain)**.** *If $\mathbb{R}^m \cong \mathbb{R}^n$ then $m = n$.*

Proof. Suppose $\phi : \mathbb{R}^m \to \mathbb{R}^n$ is a homeomorphism. wlog $\phi(0) = 0$. This induces a homeomorphism $\mathbb{R}^m \setminus \{0\} \cong \mathbb{R}^n \setminus \{0\}$. But they are homotopy equivalent to S^{m-1} and S^{n-1} respectively. $m = n$. ☐

Theorem 6.2 (Brouwer fixed point theorem)**.** *Let D^n be the closed n-dimensional disk. Then any continuous map $\phi : D^n \to D^n$ has a fixed point.*

Proof. Identical to the 2 dimensional case, substituting H_{n-1} for π_1. ☐

6.2 Mayer-Vietoris theorem

Definition ((short) exact sequence)**.** A sequence of homomorphism of abelian groups

$$\cdots \longrightarrow A_{i+1} \xrightarrow{f_{i+1}} A_i \xrightarrow{f_i} A_{i-1} \longrightarrow \cdots$$

is *exact* at A_i if $\ker f_i = \operatorname{im} f_{i+1}$. The sequence is *exact* if it is exact at every A_i.

A *short exact sequence* is one of the form

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

Example.

1. $A \xrightarrow{f} B \to 0$ is exact at B if and only if f is surjective.

2. $0 \to A \xrightarrow{f} B$ is exact at A if and only if f is injective.

3. A very short exact sequence
$$0 \longrightarrow A \xrightarrow{f} B \longrightarrow 0$$
is an isomorphism $f : A \to B$.

Theorem 6.3 (Mayer-Vietoris)**.** *Let $K = L \cup M$ with $N = L \cap M$ be simplicial complexes. Consider the inclusion maps*

$$\begin{tikzcd} N \arrow[r, hook, "i"] \arrow[d, hook, "j"] & L \arrow[d, hook, "\ell"] \\ M \arrow[r, hook, "m"] & K \end{tikzcd}$$

then there exists $\delta_* : H_n(K) \to H_{n-1}(N)$ making this sequence exact.

$$\begin{tikzcd} & & \cdots \arrow[r] & H_{n+2}(K) \arrow[dlll, "\delta_*"] \\ H_{n+1}(N) \arrow[r, "i_*\oplus j_*"] & H_{n+1}(L) \oplus H_{n+1}(M) \arrow[r, "\ell_*-m_*"] & H_{n+1}(K) \arrow[dll, "\delta_*"] \\ H_n(N) \arrow[r, "i_*\oplus j_*"] & H_n(L) \oplus H_n(M) \arrow[r, "\ell_*-m_*"] & H_n(K) \arrow[dll, "\delta_*"] \\ H_{n-1}(N) \arrow[r, "i_*\oplus j_*"] & H_{n-1}(L) \oplus H_{n-1}(M) \arrow[r, "\ell_*-m_*"] & H_{n-1}(K) \arrow[dll, "\delta_*"] \\ H_{n-1}(N) \arrow[r] & \cdots \end{tikzcd}$$

The core of the theorem is a result in homological algebra.

Definition (exact chain map)**.** A sequence of chain maps
$$A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet$$
is *exact* at $B_\bullet$ if
$$A_n \xrightarrow{f_n} B_n \xrightarrow{g_n} C_n$$
is exact at B_n for all $n \in \mathbb{Z}$.

Lemma 6.4 (snake lemma)**.** *Let*
$$0 \longrightarrow A_\bullet \xrightarrow{f_\bullet} B_\bullet \xrightarrow{g_\bullet} C_\bullet \longrightarrow 0$$
be a short exact sequence of chain complexes. For any $n \in \mathbb{Z}$ there is a

homomorphism $\delta_* : H_{n+1}(C_\bullet) \to H_n(A_\bullet)$ *such that*

$$\begin{tikzcd}
 & \cdots \arrow[r, "g_*"] & H_{n+1}(C_\bullet) \arrow[dll, "\delta_*"] \\
H_n(A_\bullet) \arrow[r, "f_*"] & H_n(B_\bullet) \arrow[r, "g_*"] & H_n(C_\bullet) \arrow[dll, "\delta_*"] \\
H_{n-1}(A_\bullet) \arrow[r, "f_*"] & \cdots &
\end{tikzcd}$$

Proof. Consider the massive commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
0 \arrow[r] & A_{n+1} \arrow[r, "f_{n+1}"] \arrow[d, "\partial_{n+1}"] & B_{n+1} \arrow[r, "g_{n+1}"] \arrow[d, "\partial_{n+1}"] & C_{n+1} \arrow[r] \arrow[d, "\partial_{n+1}"] & 0 \\
0 \arrow[r] & A_n \arrow[r, "f_n"] \arrow[d, "\partial_n"] & B_n \arrow[r, "g_n"] \arrow[d, "\partial_n"] & C_n \arrow[r] \arrow[d, "\partial_n"] & 0 \\
0 \arrow[r] & A_{n-1} \arrow[r, "f_{n-1}"] \arrow[d, "\partial_{n-1}"] & B_{n-1} \arrow[r, "g_{n-1}"] \arrow[d, "\partial_{n-1}"] & C_{n-1} \arrow[r] \arrow[d, "\partial_{n+1}"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

Let's construct the map

$$\begin{aligned} \delta_* : H_{n+1}(C_\bullet) &\to H_n(A_\bullet) \\ [x] &\mapsto ? \end{aligned}$$

with $x \in Z_{n+1}(C_\bullet)$. Since g_{n+1} is surjective, $x = g_{n+1}(y)$ for some $y \in B_{n+1}$. Consider $\partial_{n+1}(y)$, by commutativity

$$g_n \circ \partial_{n+1}(y) = \partial_{n+1} \circ g_n(y) = \partial_{n+1}(x) = 0$$

as $x \in Z_{n+1}(C_\bullet)$. Thus $\partial_{n+1}(y) \in \ker g_n = \operatorname{im} f_n$. Thus exists $z \in A_n$ such that $f_n(z) = \partial_{n+1}(y)$.

We would like to check that $z \in Z_n(A_\bullet)$. Consider $\partial_n(z)$,

$$f_{n-1} \circ \partial_n(z) = \partial_n \circ f_n(z) = \partial_n \circ \partial_{n+1}(y) = 0.$$

But f_{n-1} is injective so $\partial_n(z) = 0$, and thus $z \in Z_n(A_\bullet)$. Thus let $\delta_*([x]) = [z]$.

We have to check this is well-defined and δ_* is a homomorphism. Finally we have to check exactness at $H_n(A_\bullet), H_n(B_\bullet), H_n(C_\bullet)$. This is just a tedious exercise in diagram chasing. □

Proof of Mayer-Vietoris. By the snake lemma it suffices to check that the following is an exact sequence of chain complex.

$$0 \longrightarrow C_\bullet(N) \xrightarrow{i_\bullet \oplus j_\bullet} C_\bullet(L) \oplus C_\bullet(M) \xrightarrow{\ell_\bullet - m_\bullet} C_\bullet(K) \longrightarrow 0.$$

- exactness at $C_n(N)$: i_n induces $C_n(N)$ as a direct summand in $C_n(L)$ and similar for j_n. Thus $i_n \oplus j_n$ is injective.
- exactness at $C_n(K)$: consider $c \in C_n(K)$, have $c = c_L + c_M$ where c_L, c_M are "supported" in L and M respectively. To make it precise, this means that they are respective images of ℓ_n and m_n, i.e. there exists $b_L \in C_n(L), b_M \in C_n(M)$ such that $c_L = \ell_n(b_L), c_M = m_n(b_M)$. Thus
$$c = \ell_n(b_L) - m_n(-b_M) = (\ell_n - m_n)(b_L, -b_M).$$
- exactness at $C_n(L) \oplus C_n(M)$: $(b_L, b_M) \in \ker(\ell_n - m_n)$ if and only if each simplex σ that appears in b_L also appears in b_M with the same coefficients, if and only if $(b_L, b_M) \in \operatorname{im}(i_n \oplus j_n)$.

$\square$

Lemma 6.5 (five lemma)**.** *Suppose the following diagram is commutative and the rows are exact:*

$$\begin{tikzcd}
A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & D \arrow[r] \arrow[d, "\delta"] & E \arrow[d, "\varepsilon"] \\
A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & D' \arrow[r] & E'
\end{tikzcd}$$

if $\alpha, \beta, \delta, \varepsilon$ are isomorphisms then so is γ.

Proof. Example sheet 4. $\square$

Recall that we claimed before given a barycentric subdivision K' of a simplicial complex K, the induced map $s_* : H_n(K') \to H_n(K)$ is an isomorphism.

Proof. Induction on the number of simplices of K. If $K = \{*\}$ then $K = K'$ so obviously true. For inductive step, choose $\sigma \in K$ with maximal dimension. Let

$$\begin{aligned}
L &= K - \{\sigma\} \\
M &= \{\tau \leq \sigma : \tau \in K\} \\
N &= L \cap M = \partial\sigma
\end{aligned}$$

L, N have fewer simplices than K. Because M, M' are both cones, $s_* : H_n(M') \to H_n(M)$ is an isomorphism. Look at Mayer-Vietoris for $K = L \cup_N M, K' = L' \cup_{N'} M'$.

$$\begin{tikzcd}
H_{n+1}(N') \arrow[r] \arrow[d, "s_*"] & H_{n+1}(L') \oplus H_{n+1}(M') \arrow[r] \arrow[d, "s_* \oplus s_*"] & H_{n+1}(K') \arrow[r] \arrow[d, "s_*"] & H_n(N') \arrow[r] \arrow[d, "s_*"] & H_n(L') \oplus H_n(M') \arrow[d, "s_* \oplus s_*"] \\
H_{n+1}(N) \arrow[r] & H_{n+1}(L) \oplus H_{n+1}(M) \arrow[r] & H_{n+1}(K) \arrow[r] & H_n(N) \arrow[r] & H_n(L) \oplus H_n(M)
\end{tikzcd}$$

so the five lemma finishes the job. $\square$

6.3 Homology of compact surfaces

Recall that classification of compact surfaces says there are two classes of surfaces there are two classes: Σ_g and S_g. We are going to use Mayer-Vietoris to compute their homologies. Before that we have to know the homology of Γ_r.

Example. $\Gamma_r = \bigvee_{i=1}^r S^1$. $\Gamma_0 = *, \Gamma_1 \cong S^1$ and their homologies are known. Use standard simplicial 1-sphere. There is a slight issue here that the way we set up simplicial complex makes it difficult to glue things. Instead we are going to use abstract simplicial complex.

Let K be such that $|K| \cong \Gamma_r$, $L, M \subseteq K$ be such that $|L| \cong \Gamma_{r-1}, |M| \cong S^1$, and $K = L \cup_N M$ where $N = \{v_0\}$. By Mayer-Vietoris,

$$\begin{tikzcd}
H_1(N) \arrow[r] & H_1(L) \oplus H_1(M) \arrow[r] & H_1(K) \arrow[dll, "\delta_*"] \\
H_0(N) \arrow[r] & H_0(L) \oplus H_0(M) \arrow[r] & H_0(K) \arrow[dll] \\
0
\end{tikzcd}$$

which is

$$\begin{tikzcd}
0 \arrow[r] & H_1(\Gamma_{r-1}) \oplus \mathbb{Z} \arrow[r] & H_1(\Gamma_r) \arrow[dll, "\delta_*"] \\
\mathbb{Z} \arrow[r] & \mathbb{Z} \oplus \mathbb{Z} \arrow[r] & \mathbb{Z} \arrow[dll] \\
0
\end{tikzcd}$$

and unfortunately we have to get out hands dirty to understand δ_*. The map $H_0(N) \to H_0(L) \oplus H_0(M)$ sends generator to generators so is injective and thus by exactness at $H_1(K)$, $\operatorname{im} \delta_* = 0$. Thus we have a very short exact sequence

$$0 \longrightarrow H_1(\Gamma_{r-1}) \oplus \mathbb{Z} \longrightarrow H_1(\Gamma_r) \xrightarrow{\delta_*} 0$$

so $H_1(\Gamma_r) \cong \Gamma_1(\Gamma_{r-1}) \oplus \mathbb{Z}$. Thus we conclude that

$$H_n(\Gamma_r) = \begin{cases} \mathbb{Z} & n = 0 \\ \mathbb{Z}^r & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

Note that $H_1(\Gamma_r) \cong \langle[\alpha_1]\rangle \oplus \cdots \oplus \langle[\alpha_r]\rangle$ where $[\gamma_i]$ is a generator for $H_1(i\text{th circle})$.

Remark. Note that this proof does not assume anything other path-connectedness. Thus $\delta_* : H_1(K) \to H_0(N)$ is always zero as long as the intersection N is connected. Then we don't have to worry about the last row of Mayer-Vietoris.

Example. $\Sigma_g = \Gamma_{2g} \cup_{\rho_g} D^2$ where

$$\rho_g(1) = \alpha_1 \beta_1 \alpha_1^{-1} \beta_1^{-1} \cdots \alpha_g \beta_g \alpha_g^{-1} \beta_g^{-1}.$$

Just as mapping cylinder, to compute the homology it's convenient to introduce a new space

$$\Sigma_g^* = \Gamma_{2g} \cup_{2g} (S^1 \times [0,1])$$

with $(x,0) \sim \rho_g(x)$ for all $x \in S^1$. This is just Σ_g with D^2 removed. Note that Σ_g^* deformation retracts to Γ_{2g}.

A bit of technical work: we can triangulate Σ_g^* by triangulating Γ_{2g} then taking a simplicial approximation to ρ_g and triangulating $S^1 \times I$ in a way compactible with that.

Next note that

$$\Sigma_g = \Sigma_g^* \cup_i D^2$$

where $i : S^1 \to \Sigma_g^*$ is the map identifying ∂D^2 with $S^1 \times I$. Choose a triangulation of D^2 compatible with the induced triangulation of its boundary.

Back to the actual computation.

$$\begin{tikzcd}
H_2(S^1) \arrow[r] & H_2(\Sigma_g^*) \oplus H_2(D^2) \arrow[r] & H_2(\Sigma_g) \arrow[dll, rounded corners] \\
H_1(S^1) \arrow[r] & H_1(\Sigma_g^*) \oplus H_1(D^2) \arrow[r] & H_1(\Sigma_g) \arrow[dll, rounded corners] \\
0 & &
\end{tikzcd}$$

note that by the previous remark we don't have to write down the last row. Fill in information we already knew,

$$0 \longrightarrow 0 \longrightarrow H_2(\Sigma_g) \longrightarrow \mathbb{Z} \xrightarrow{i_*} \mathbb{Z}^{2g} \longrightarrow H_1(\Sigma_g) \longrightarrow 0$$

To figure out the two unknown groups we need to understand i_*, which induced by ρ_g. But since homology groups are abelian, we have

$$i_*(1) = (\gamma_g)_*(1) = [\alpha_1] + [\beta_1] - [\alpha_1] - [\beta_1] + \cdots + [\alpha_g] + [\beta_g] - [\alpha_g] - [\beta_j] = 0$$

So we split that into two exact sequences

$$0 \longrightarrow H_2(\Sigma_g) \longrightarrow \mathbb{Z} \longrightarrow 0 \quad 0 \longrightarrow \mathbb{Z}^{2g} \longrightarrow H_2(\Sigma_g) \longrightarrow 0$$

so

$$H_n(\Sigma_g) \cong \begin{cases} \mathbb{Z} & n = 0, 2 \\ \mathbb{Z}^{2g} & n = 1 \\ 0 & \text{otherwise} \end{cases}$$

which in particular implies that they are pairwise non-homotopy equivalent.

Example. $S_g = \Gamma_{g+1} \cup_{\sigma_g} D^2$ where

$$\sigma_g(1) = \alpha_0^2 \alpha_1^2 \cdots \alpha_g^2.$$

By almost the same argument, we obtain a Mayer-Vietoris sequence

$$0 \longrightarrow H_2(S_g) \xrightarrow{\delta_*} \mathbb{Z} \xrightarrow{i_*} \mathbb{Z}^{g+1} \longrightarrow H_1(S_g) \longrightarrow 0$$

where this times i_* is induced by σ_g, and

$$i_* = (\gamma_g)_*(1) = 2[\alpha_0] + 2[\alpha_1] + \cdots + 2[\alpha_g]$$

which is injective so $\delta_* = 0$. Thus $H_2(S_g) = 0, H_1(S_g) \cong \mathbb{Z}^g \oplus (\mathbb{Z}/2\mathbb{Z})$.

6.4 Rational homology and Euler characteristic

Basically homology with coefficients in $\mathbb{Q}$.

Definition (rational chain)**.** Let K be a simplicial complex. The vector space of *rational n-chain* $C_n(K;\mathbb{Q})$ is defined as the vector space over $\mathbb{Q}$ with basis the n-simplices of K.

A typical element thus has the form $\sum_i \lambda_i \sigma_i$ where $\lambda_i \in \mathbb{Q}$ and σ_i's are n-simplices. We can define the boundary map $\partial_n : C_n(K;\mathbb{Q}) \to C_n(K;\mathbb{Q})$ as before and the condition $\partial \circ \partial = 0$ is satisfied so we have another homology theory.

Definition (rational homology)**.** Let K be a simplicial complex. We define

$$\begin{aligned} Z_n(K;\mathbb{Q}) &= \ker \partial_n \\ B_n(K;\mathbb{Q}) &= \operatorname{im} \partial_{n+1} \\ H_n(K;\mathbb{Q}) &= Z_n(K;\mathbb{Q})/B_n(K;\mathbb{Q}) \end{aligned}$$

If $\alpha : |K| \to X$ is a triangulation then define

$$H_n(X;\mathbb{Q}) = H_n(K;\mathbb{Q}).$$

Rational homology encodes slightly less data but has the advantage of easier to compute (as $\mathbb{Q}$ is a field). More specifically, it simply forgets the torsion elements:

Lemma 6.6. *Let K be a simplicial complex. If $H_n(K) \cong \mathbb{Z}^k \oplus F$ where F is a finite group then*

$$H_n(K;\mathbb{Q}) \cong \mathbb{Q}^k.$$

Proof. An exercise in commutative algebra. See online notes. ☐

Example. For all n,

$$H_n(\mathbb{R}P^2;\mathbb{Q}) \cong H_n(*;\mathbb{Q}).$$

Definition (Euler characteristic)**.** Let X be a triangulable space and $\alpha : |K| \to X$ a triangulation. Then the *Euler characteristic* of X is

$$\chi(X) = \chi(K) = \sum_{n\in\mathbb{Z}} (-1)^n \dim_{\mathbb{Q}} H_n(K;\mathbb{Q}).$$

Note that this is obviously a topological invariant.

Given the way rational homology is defined, the nice thing about $\chi(K)$ is that it is *really* easy to compute.

Lemma 6.7. *Let K be a simplicial complex. Then*

$$\chi(K) = \sum_{n\in\mathbb{Z}} (-1)^n \#\{n\text{-simplicies in } K\}.$$

In particular if K is 2-dimensional, with V, E, F the number of 0, 1 and

2-simplices then
$$\chi(K) = V - E + F.$$

Proof. The number of n-simplex is not a natural algebraic object so instead we use $\dim_{\mathbb{Q}} C_n(K;\mathbb{Q})$. Next, recall that we are working with vector spaces so we can apply rank-nullity theorem to

$$\begin{aligned} Z_n(K;\mathbb{Q}) &\twoheadrightarrow H_n(K;\mathbb{Q}) \\ \partial_n : C_n &\to B_{n-1} \end{aligned}$$

to get

$$\begin{aligned} \dim Z_n &= \dim H_n + \dim B_n \\ \dim C_n &= \dim B_{n-1} + \dim Z_n \end{aligned}$$

Now

$$\begin{aligned} \sum_{n\in\mathbb{Z}} (-1)^n \dim_{\mathbb{Q}} C_n &= \sum_{n\in\mathbb{Z}} (-1)^n \dim B_{n-1} + \sum_{n\in\mathbb{Z}} (-1)^n \dim Z_n \\ &= -\sum_{n\in\mathbb{Z}} (-1)^n \dim B_n + \sum_{n\in\mathbb{Z}} (-1)^n \dim Z_n \\ &= \sum_{n\in\mathbb{Z}} (-1)^n (\dim Z_n - \dim B_n) \\ &= \sum_{n\in\mathbb{Z}} (-1)^n \dim H_n \\ &= \chi(K) \end{aligned}$$

□

Note.

$$\begin{aligned} \chi(\Sigma_g) &= 2 - 2g \\ \chi(S_g) &= 1 - g \end{aligned}$$

6.5 Lefschetz fixed-point theorem

Definition (Lefschetz number)**.** Let $\phi : X \to X$ be a continuous map of triangulable space X. The *Lefschetz number* of ϕ is

$$L(\phi) = \sum_{n\in\mathbb{Z}} (-1)^n \operatorname{tr}(\phi_* : H_n(X;\mathbb{Q}) \to H_n(X;\mathbb{Q})).$$

Note. $L(\mathrm{id}_X) = \chi(X)$ so Lefschetz number generalises Euler characteristic.

Lemma 6.8. *If $f : K \to K$ is a simplicial map then*

$$L(|f|) = \sum_{n\in\mathbb{Z}} (-1)^n \operatorname{tr}(f_* : C_n(K;\mathbb{Q}) \to C_n(K;\mathbb{Q})).$$

Proof. Consider the following commutative diagrams of linear maps with exact rows:

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, "\alpha"] & B \arrow[r] \arrow[d, "\beta"] & C \arrow[r] \arrow[d, "\gamma"] & 0 \\ 0 \arrow[r] & A' \arrow[r] & B' \arrow[r] & C' \arrow[r] & 0 \end{tikzcd}$$

By linear algebra we can find bases for B, B' such that the matrix for β have has the form

$$\begin{pmatrix} \gamma & * \\ 0 & \alpha \end{pmatrix}$$

so in particular

$$\operatorname{tr}\beta = \operatorname{tr}\gamma + \operatorname{tr}\alpha.$$

What is left is an application of rank-nullity theorem, similar to the proof of Euler characteristic. □

Theorem 6.9 (Lefschetz fixed point theorem)**.** *Let X be a triangulable space and $\phi : X \to X$ a continuous map. If $L(\phi) \neq 0$ then ϕ has a fixed point.*

Non-examinable, sketch. Suppose ϕ has no fixed point. Since X is compact, exists $\delta > 0$ such that for all $x \in X$, $\|x - \phi(x)\| > \delta$. Now choose a simplicial approximation K of X with $\operatorname{mesh}(K) < \frac{\delta}{2}$. Let $f : K^{(r)} \to K$ be a simplicial approximation to ϕ. Note that if $v \in \sigma \in \sigma$ then $f(v) \notin \sigma$. Let $\iota_n : C_n(K; \mathbb{Q}) \to C_n(K; \mathbb{Q})$ be the map inducing canonical isomorphism of homology groups. For any n-simplex $\sigma \in K$, $\iota_n(\sigma)$ is supported on simplices contained in σ. Then $f_n \circ \iota_n$ takes every simplex of K off itself, i.e. $f_n \circ \iota_n(\sigma)$ does not contain σ. Now

$$\begin{aligned} L(\phi) &= \sum_{n \in \mathbb{Z}} (-1)^n \operatorname{tr}(f_n \circ \iota_n : C_n(K; \mathbb{Q}) \to C_n(K; \mathbb{Q})) \\ &= \sum_{n \in \mathbb{Z}} (-1)^n \cdot 0 \\ &= 0 \end{aligned}$$

□

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