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Geometry and Mathematical Physics

Topics in Mirror Symmetry

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1 Introduction

What is mirror symmetry? We will consider invariants of complex manifolds (M, J). This is what is called the B-model. Examples: Hodge structures, periods of differential forms, derived category of coherent sheaves. We will also be interested in invariants of symplectic manifolds (M, ω). This is the A-model. Examples: Gromov-Witten invariants (count number of holomorphic curves), Fukaya category.

example of Gromov-Witten invariants:

1. number of lines passing through two points in $\mathbb{P}^1$,
2. number of rational curves in $\mathbb{P}^2$ through the maximal number of points (Kontsevich)
3. number of rational curves of degree d on a quintic 3-fold.

For us complex/symplectic structures will appear as structures on Kähler manifolds.

Meta-mirror symmetry: there exists pairs of Kähler spaces (M, ω, J), $(\check{M}, \check{\omega}, \check{J})$ such that $A(M, \omega)$ correponds to $B(\check{M}, \check{J})$ and vice versa.

This has surprising consequences on the geometry of M and $\check{M}$:

- the Hodge diamonds of M and $\check{M}$ are "mirrors" of each other,
- closed string mirror symmetry: Gromov-Witten invariants of (M, ω) is related to periods of $(\check{M}, \check{J})$ and vice versa (up to some serious algebra).

In this course we will focus on open strings/homological mirror symmetry. It is conjectured by Kontsevich in 1994 and the statement is: if (M, ω, J) and $(\check{M}, \check{\omega}, \check{J})$ are mirror partners then there should be equivalences of categories $\mathbf{D^b Fuk}(M, \omega) \simeq \mathbf{D^b Coh}(\check{M}, \check{J})$ and $\mathbf{D^b Coh}(M, J) \simeq \mathbf{D^b Fuk}(\check{M}, \check{\omega})$.

Why is this good?

1. it is a fundemental statement. It should imply enumerative/closed string mirror symmetry
2. HMS is much more general: closed string mirror symmetry requires $M, \check{M}$ to be Calabi-Yau 3-folds.
3. **Fuk** and **Coh** are very rich invariants. HMS gives us a very useful dictionary.

One word on "much more general": HMS is an "" in the sense that it holds in a variety of settings, which have specific features. For example on the A-model side, If M is open ("exact") then we have sheaf-theoretic models of **Fuk**. If M is open we often have extra data: regular function $W : M \to \mathbb{C}$, or a skeleton. Then we have "wrapped" Fukaya category. On the B-model side, often M comes with $W : M \to \mathbb{C}$.

Part of the goal of this course will be explaining how to define $\mathbf{Fuk}(M, \omega)$ and $\mathbf{D^b Coh}(M, J)$. Before doing that we give a very informal picture of what they are and use it to explain a (small piece) of HMS for elliptic curves.

Definition. Let (M, ω) be a symplectic manifold. A submanifold $L \subseteq M$ is *Lagrangian* if $i^*\omega = 0$ (which implies $\dim L \leq \frac{1}{2} \dim M$) and $\dim L = \frac{1}{2} = \dim M$.

The objects of the *Fukaya category* $\mathbf{Fuk}(M, \omega)$ are Lagrangians $L \subseteq M$ satisfying some extra properties plus some extra structures, including:

1. pin structure,
2. grading,
3. datum of a unitary local system.

(We will ignore these extra structures for the time being.) If L_1, L_2 are Lagrangians that meet transversely then they will meet at finitely many points and we define

$$\operatorname{Hom}_{\mathbf{Fuk}}(L_1, L_2) = \bigoplus_{p \in L_1 \cap L_2} \mathbb{C}\langle p \rangle.$$

(We will see that Hom can be graded etc.) Composition of morphisms in the Fukaya category involves pseudo-holomorphic disks

$$\begin{aligned} \operatorname{Hom}(L_1, L_2) \otimes \operatorname{Hom}(L_2, L_3) &\to \operatorname{Hom}(L_1, L_3) \\ (q, r) &\mapsto \sum_D e^{\int_D u^*\omega} \cdot p \end{aligned}$$

summing over $u : D \to M$ pseudo-holomorphic triangle with vertices p, q, r.

In fact the Fukaya category is not an actual category, but an A_∞-category. Informally speaking, we have "higher compositions"

$$\mu_N : \operatorname{Hom}(L_1, L_2) \otimes \cdots \otimes \operatorname{Hom}(L_{N-1}, L_N) \to \operatorname{Hom}(L_1, L_N)$$

which is defined in terms of counts of pseudo-holomorphic polygons.

On the other hand, the derived category of M, $\mathbf{D^b Coh}(M, J)$ has objects bounded complexes of coherent sheaves.

1.1 Homological mirror symmetry for elliptic curves

A-side: E a symplectic torus, $\omega = A \mathrm{d}x \wedge \mathrm{d}y$ where $A > 0$, a b-field which is a class $b \in H^2(E, \mathbb{R})/H^2(E, \mathbb{Z})$. Call $\rho = iA + b$ the *complexified Kähler parameter*.

B-side: $\tau \in \mathbb{H}$ and $E_\tau = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\tau$. Let $q = \exp(2\pi i \tau)$. Then $E_\tau \cong \mathbb{C}^*/q^{\mathbb{Z}}$.

Statement: $E^q_\rho \leftrightarrow E^\rho_q$, where E^q_ρ is the elliptic curve with complex structure given by q and symplectic structure given by ρ.

Simplest example: $E^i_i \leftrightarrow E^i_i$. On the symplectic side, the objects of the Fukaya category are the smooth embedded Lagrangians, which do not bound disks. Note that for dimension reasons, every curve on the torus is Lagrangian. One can see that (up to Hamiltonian isotopy), we can reduce to geodesic Lagrangians, that is, images of straight lines in the unviersal cover with rational slope. These Lagrangians are identified by

1. the slope (p, q),

2. the intersection with x-axis.

The basic idea of mirror symmetry in this setting is

- line bundles of degree d are sent to Lagrangians of slope $(1, d)$.
- more generally if V is a vector bundle on E then V is sent to a line of slope $\deg V / \operatorname{rk} V$.

We are interested in the following three line bundles: $\mathcal{O}, \mathcal{L} = \mathcal{O}(e), \mathcal{L}^2 = \mathcal{O}(2e)$ where $e \in E$ is the identity for the group structure.

Analytic approach to understand sections of $\mathcal{L}$. Consider $\pi : \mathbb{C}^* \to E \cong \mathbb{C}^*/q^{\mathbb{Z}}$. Every line bundle on $\mathbb{C}^*$ is trivial: the exponential exact sequence shows $H^1(\mathbb{C}^*, \mathcal{O}^*) \cong H^2(\mathbb{C}^*, \mathbb{Z}) = 0$. Thus giving a line bundle $\mathcal{L}$ on E is the same as giving a $\mathbb{Z}$-equivariant structure on $\pi^*\mathcal{L}/\mathbb{C}^*$ where $\pi^*\mathcal{L}$ is the trivial line bundle. That is, we nned to specify a $\mathbb{Z}$-action on $\mathbb{C}^\times\mathbb{C}^*$ which is compatible with the $\mathbb{Z}$-action on $\mathbb{C}^*$. It follows that every line bundle on E is determined by a holomorphic map $\phi : \mathbb{C}^* \to \mathbb{C}^*$ in such a way that $\mathbb{Z}$ acts on $\mathbb{C} \times \mathbb{C}^*$ via

$$1 \cdot (u, v) = (\phi(v)(u), qv).$$

It turns out that if we choose $\phi(v) = \exp(-\pi i \tau)v^{-1}$, the resulting line bundle is $\mathcal{L} = \mathcal{O}(e)$.

A section of $\mathcal{L}$ is the same as a function on $\mathbb{C}$ satifying

1. $\theta(z + 1) = \theta(z)$,
2. $\theta(z + \tau) = \phi(z)\theta(z)$.

Up to multiplication by a scalar, there is only one such function, called the Jacobi theta function

$$\theta(z) = \sum_{m \in \mathbb{Z}} \exp(2\pi i(\frac{m^2}{2}\tau + mz)).$$

Thus for $\mathcal{L} = \mathcal{O}(e)$, $H^0(E, \mathcal{L})$ is one-dimensional, spanned by θ.

Before proceeding further, we need to say something more about theta functions, so as to be able to describe canonical basis of sections of $\mathcal{L}^n$. We define

$$\theta[c', c''](\tau, z) = \sum_{m \in \mathbb{Z}} \exp\{2\pi i(\tau\frac{(m + c')^2}{2} + (m + c')(z + c''))\}.$$

Proposition 1.1. *$\theta[\frac{a}{n}, 0](n\tau, nz)$ is a basis of sections for $\mathcal{L}^n$, where $a \in \{0, 1, \dots n-1\}$.*

$$\begin{aligned} \mathbf{D^bCoh}(E) &\to \mathbf{Fuk}(E) \\ \mathcal{O} &\mapsto \Lambda_1 \\ \mathcal{L} &\mapsto \Lambda_2 \\ \mathcal{L}^2 &\mapsto \Lambda_3 \end{aligned}$$

where $\Lambda_1, \Lambda_2, \Lambda_3$ are the lines with slopes $0, 1, 2$ intersecting the x-axis at the origin. Let $e_1 = e_2$ be the origin and $e_3 = (\frac{1}{2}, 1)$. Then we have isomorphisms

$$\begin{aligned} \mathrm{Hom}(\mathcal{O}, \mathcal{L}) &\to \mathrm{Hom}(\Lambda_1, \Lambda_2) \\ \theta &\mapsto e_1 \\ \mathrm{Hom}(\mathcal{L}, \mathcal{L}^2) = \mathrm{Hom}(\mathcal{O}, \mathcal{L}) &\to \mathrm{Hom}(\Lambda_2, \Lambda_3) \\ \theta &\mapsto e_2 \\ \mathrm{Hom}(\mathcal{O}, \mathcal{L}^2) &\to \mathrm{Hom}(\Lambda_1, \Lambda_3) \\ \theta[0,0](2\tau, 2z) &\mapsto e_2 \\ \theta[\frac{1}{2}, 0](2\tau, 2z) &\mapsto e_3 \end{aligned}$$

We briefly verify that this assignment respects composition. In the Fukaya category, we have

$$\begin{aligned} \mathrm{Hom}(\Lambda_1, \Lambda_2) \otimes \mathrm{Hom}(\Lambda_2, \Lambda_3) &\to \mathrm{Hom}(\Lambda_1, \Lambda_3) \\ e_1 \otimes e_1 &\mapsto C_2 e_2 + C_3 e_3 \end{aligned}$$

where C_2 is the count (weighted by area) of "holomorphic triangles" with verticies e_1, e_1, e_2. We need to look at maps $\varphi : D \to E$ such taht D has three components. We pass to a lift of the universal cover of E. One such triangle has vertices $(0,0), (1,0)$ and $(2,2)$, with the lifts $\widetilde{\Lambda}_1, \widetilde{\Lambda}_2$ chosen to pass through the origin and $\widetilde{\Lambda}_3$ passing through $(1,0)$. Note that the map $\varphi\colon D \to \mathbb{C}$ whose image is this triangle "shifted to the right" defines the same triangle on E. Thus we fix thelift of e_1 to be the origin. Then the triangles have vertices $(0,0), (N,0), (2N, 2N)$ with $N \in \mathbb{Z}$. It has area $N^2 \geq A$, where A is the area of the torus (i.e. the symplectic form is $A\mathrm{d}x \wedge \mathrm{d}y$), which is 1 in our case. It follows that

$$C_2 = \sum_{N=-\infty}^{\infty} \exp(-2\pi N^2)$$

(note this defers from our preliminary definition of composition in Fukaya category by a constant $\exp(-2\pi)$.)

Similarly

$$C_3 = \sum_{N=-\infty}^{\infty} \exp(-\pi(N - \frac{1}{2})N).$$

What we get is in fact $C_2 = \theta[0,0](i2A, 0), C_3 = \theta[\frac{1}{2}, 0](i2A, 0)$.

For the A-side we quote the following algebraic result:

Proposition 1.2 (addition formula for theta functions)**.**

$$\theta(z) \cdot \theta(z) = \theta[0,0](i2A,0)\theta[0,0](2\tau,2z) + \theta[\frac{1}{2},0](i2A,0)\theta[\frac{1}{2},0](2\tau,2z).$$

From this we conclude that that the compositions in the two categories match. Thus by HMS, the classical addition formula for theta functions come from counting triangles.

2 A_∞-spaces

We need to define A_∞-categories (because the Fukaya category has this structure). We will take the historical path and satrt from A_∞-spaces, followed by A_∞-algebras, and then A_∞-categories.

The premier example of an A_∞-space is the loop space

$$\Omega_* M = \{\alpha : [0,1] \to M : \alpha(0) = \alpha(1) = *\}.$$

In fact connected A_∞-spaces are equivalent to $\Omega_* M$ where M is simply connected. This is part of homotopy theory concerning operads and recognition principles for (infinite) loop spaces.

The basic idea is the following: X is an A_∞-space if it has a multiplication $m_2 : X \times X \to X$ which is associative up to homotopy, and such that all higher associative laws only held up to homotopy.

For example for $X = \Omega_* M$, composition is clearly not associative, but is associative up to homotopy: let $m_2 : X \times X \to X$ be composition. Then there is a map

$$m_3 : [0,1] \times X \times X \times X \to X$$

such that

$$\begin{aligned} m_3(0, x_1, x_2, x_3) &= m_2(x_3, m_2(x_2, x_1)) \\ m_3(1, x_1, x_2, x_3) &= m_2(m_2(x_3, x_2), x_1) \end{aligned}$$

Now if we have four loops, there are many ways to put brackets between them and they should all be equivalent. This is

We denote the boundary of the pentagon by K_4. Using m_2, m_3, we can write down a map

$$\overline{K}_4 \times X \times X \times X \times X \to X.$$

For example on the

The point is that if $X = \Omega_* M$ we can extend this map on all the pentagon $\overline{K}_4$, namely we can construct a map m_4 such that

$$\begin{array}{ccc} \partial\overline{K}_4 \times X \times X \times X \times X & \longrightarrow & X \\ \downarrow & & \| \\ \overline{K}_4 \times X \times X \times X \times X & \longrightarrow & X \end{array}$$

Stasheff in the 1960s constructed an infinite sequence of *associahedra* in such a way that the vertices of the associahedron $\overline{K}_N$ corresponds to the various ways to bracket $x_N \cdots x_1$. The maximal proper faces of $\overline{K}_N$ correspond to choosing one single bracket, e.g.

$$x_N \cdots x_{p+q+1}(x_{p+q} \cdots x_{q+1}) x_q \cdots x_1$$

and are isomorphic to $\overline{K}_{N-p+1} \times \overline{K}_p$.

Definition (A_∞-space)**.** An A_∞*-space* X is a space X equipped with maps

$$m_N : \overline{K}_N \times X \times \cdots \times X \to X$$

which are compatible with restrictions to faces.

(For those who are familiar with the language of operads, another name for the A_∞-operad is E_1-operad. We have an infinite hierarchy $E_1, E_2, \ldots, E_\infty$. E_2 is also called the "little disk" operad. The A_∞-operad is a (fibrant) resolutiono of the associative operad.)

We shall give two alternative descrptions of the $\overline{K}_N$ as moduli spaces.

metric ribbon graphs

Definition (metric ribbon graph)**.** A *metric ribbon graph* is a connected tree with finitely many vertices and edges, plus a a cylic ordering of eedges around each vertex, and a length in $(0, \infty]$ assigned to each edge.

Definition. $\overline{K}_N$ is the moduli space of metric ribbon trees with $(N+1)$ external vertices (i.e. 1-valent vertex) such that each edge connected to an external vertex has infinite length.

$K_N \subseteq \overline{K}_N$ is the moduli space of metric ribbon trees such that all internal edges have finite length.

Example. $\overline{K}_2$ is just a point since all segments have infinite length. For $\overline{K}_3$,

The combinatorial type of the tree gives us a *cubical decomposition* of $\overline{K}_N$, namely the cubical cells come from assigning lengths in $[0, \infty]$ to the internal edges:

What we are doing is "topologising" associativity. That is, we should think that we are labelling incoming verticies with "arguments". Then given any 3-valent ribbon tree there is only one way to put brackets around incoming vertices.

Note that metric ribbon trees also allow us to beter understand the boundary structure of $\overline{K}_N$. Placing a single bracket, such as the following

$$x_N \cdots x_{p+q+1}(x_{p+q} \cdots x_{p+1})x_p \cdots x_1,$$

coreesponds to a face $\overline{K}_{N-p+1} \times \overline{K}_p$. From the point-of-vew of trees this corresponds to letting the length of one internal edge to be infinite.

moduli space of marked points Recall that $\overline{M}_{0,N+1}$ is the compactified moduli space of $N+1$ curves on $\mathbb{P}^1$.

$M_{0,4} \cong \mathbb{P}^1 \setminus \{0, 1, \infty\}$ and $\overline{M}_{0,4} \cong \mathbb{P}^1$, where the family over 0 is two copies of $\mathbb{P}^1$ with a nodal singularity (similar for 1 and ∞).

In general $\overline{M}_{0,N}$ parameterises stable rational curves, i.e. trees of rational curves, with nodal singularities, such that on each component there are at least 3 special points (namely singularities and markings).

To recover $\overline{K}_N$ from $\overline{M}_{0,N}$, we do the following

1. consider the real locus of $M_{0,N+1}$. This corresponds to curves such that the marked points lie on the equator.

2. Now $K_N \subseteq M^{\mathbb{R}}_{0,N+1}$ is the subset corresponding to rational curves with marked points on the equators which occur in cyclic order.

3. Finally $\overline{K}_N$ is the closure of K_N inside $\overline{M}^{\mathbb{R}}_{0,N+1}$.

For example, $\overline{K}_3$ is the closure of the negative reals inside $\overline{M}_{0,4}$.

We can interpret $\overline{K}_N$ as a moduli space of holomorphic disks (plus bubbling on the boundary) with cyclically ordered marked points on the boundary.

3 A_∞-algebras and A_∞-categories

Fix a ground field k.

> **Definition** (A_∞-algebra)**.** A (non-unital) A_∞*-algebra* is a $\mathbb{Z}$-graded vector space $A = \bigoplus_{p\in\mathbb{Z}} A^p$ with graded k-linear maps $m_d : A^{\otimes A} \to A$ of degree $2-d$ for $d \geq 1$ such that
>
> $$\sum_{\substack{1\leq p\leq d \\ 0\leq q\leq d-p}} (-1)^{|a_1|+\dots|a_q|-q} m_{d-p+1}(a_d, \dots, a_{p+q+1}, m_p(a_{p+q}, \dots, a_{q+1}), a_q, \dots, a_1) = 0.$$

Remark. If X an A_∞-space and $C_*(X)$ is its cellular chains then $C_*(X)$ has a natural structure of an A_∞-algebra: we need to take $C_*(\overline{K}_N)$ with the polyhedral cellular decomposition. The top cell of $C_*(\overline{K}_N)$ labels an operation of degree $N-2$. Indeed the $N-2$ degree part of $C_*(\overline{K}_N)$ is one-dimensional and hence determines a degree $N-2$ map

$$C_*(\overline{K}_N) \otimes C_*(X)^{\otimes N} \to C_*(X).$$

The compatibility condition is encoded in the cellular decomposition of $\overline{K}_N$.

Now we try to understand more concretely the axioms of A_∞-algebras. For $d=1$, $\deg m_1 = 1$ and the axiom says

$$m_1(m_1(a_1)) = 0$$

so m_1 is a differential.

For $d=2$, $\deg m_2 = 0$ so

$$\deg(m_2(a_2, a_1)) = \deg(a_2) + \deg(a_1).$$

The axiom becomes

$$m_2(a_2, m_1(a_1)) + (-1)^{\deg a_1 - 1} m_2(m_1(a_2), a_1) + m_1(m_2(a_2, a_1)) = 0.$$

This means that (up to sign adjustments) m_2 satisfies the graded Leibniz rule

$$\partial(a_2 \cdot a_1) = -a_2 \cdot \partial(e_1) + (-1)^{\deg a_1 - 1} \partial(a_2) \cdot a_1.$$

(read d as m_1 and $\cdot$ as m_2)

For $d=3$ we get

$$\begin{aligned} 0 &= m_3(a_3, a_2, m_1(a_1)) + (-1)^{|a_1|-1} m_3(a_3, m_1(a_2), a_1) \\ &\quad + (-1)^{|a_1|+|a_2|-2} m_3(m_1(a_3), a_2, a_1) + m_2(a_3, m_2(a_2, a_1)) \\ &\quad + (-1)^{|a_1|-1} m_2(m_2(a_3, a_2), a_1) + m_1(m_3(a_3, a_2, a_1)). \end{aligned}$$

Note that the fourth and fifth term are

$$(-1)^{|a_2|} a_3 \cdot (a_2 \cdot a_1) - (-1)^{|a_2|} (a_3 \cdot a_2) \cdot a_1$$

so m_3 is measuring the failure of associativity of m_2.

Let use make some obervations:

1. if $m_k = 0$ for all $k \geq 3$ then an A_∞-algebra is the same as an associative dg algebra.

2. if $m_1(e_1) = m_1(e_2) = m_1(e_3) = 0$ (i.e. if a_1, a_2, a_3 are closed) then the A_∞-condition becomes

$$a_3 \cdot (a_2 \cdot a_1) - (a_3 \cdot a_2) \cdot a_1 = \pm d(m_3(a_3, a_2, a_1)).$$

   Thus on $H^*(A)$ the multiplication is associative.

Definition (morphism of A_∞-algebras, quasiisomorphism)**.** A *morphism of A_∞-algebras* $f : A \to B$ is a family $f_d : A^{\otimes d} \to B$ of graded maps of degree $1-d$ satisfying certain conditions.

A morphism of A_∞-algebras is a *quasiisomorphism* if f_1 is a quasiisomorphism.

Remark. If $f_m = 0$ for all $m \geq 2$ then we get for all d,

$$m_d^B(f_1(a_d), \ldots, f_1(a_1)) = f_1(m_d^A(a_d, \ldots, a_1)).$$

such a morphism is called *strict*.

Definition (A_∞-category)**.** A *(non-unital) A_∞-category* **A** consists of

1. a set of objects,
2. for two objects A and B, a chain complex $\mathrm{Hom}_{\mathbf{A}}(A, B)$,
3. $$m_d : \mathrm{Hom}(X_{d-1}, X_d) \otimes \cdots \mathrm{Hom}(X_0, X_1) \to \mathrm{Hom}(X_0, X_d)$$

   of degree $d-1$

Remark. An A_∞-category is not an ordinary category: composition of morphisms is not associative!

Definition (functor between A_∞-categories)**.** A functor between A_∞-categories **A** and **B** is given by the following data:

1. a map F from objects of **A** to objects of **B**,
2. for all $d \geq 0$, maps

$$F_d : \mathrm{Hom}_A(X_{d-1}, X_d) \otimes \cdots \otimes \mathrm{Hom}_A(X_0, X_1) \to \mathrm{Hom}_B(FX_0, FX_d)$$

   satisfying certain axioms.

We call a functor *strict* if $F_d = 0$ for all $d > 2$

Remark. A_∞-categories are related to several other category. A dg-category can be regarded as an A_∞-category, and conversely if $\mu_d = 0$ for $d \geq 3$ then an A_∞-category is the same as a dg-category. If **A** is an A_∞-category, we denote by $\mathbf{Ho}(\mathbf{A})$ the (ordinary) category with the same objects and with homs given by $H^0(\mathrm{Hom}_{\mathbf{A}}(X, Y))$.

Remark. In fact the theory of dg-categories is equivalent to the theory of A_∞-categories: every A_∞-category is equivalent (as an A_∞-category) to a dg-category. Some of the key words are strictification, homological perturbation lemma, minimal models.

1. Every A_∞-algebra $\mathbf{A}$ is equivalent to an A_∞-algebra with trivial differential (supported on $H^*(A)$).
2. In particular if A is a dg-algebra, there is a unique A_∞-algebra structure on $H^*(A)$ such that there is an equivalence of A_∞-algebras $A \to H^*(A)$.
3. The same statement holds for A_∞-categories.

In particular we can get an A_∞-model of the derived category of a scheme X: we find an enhancement of $\mathbf{D^bCoh}(X)$ to a dg-category. Then we use the homological perturbation method to induce an A_∞-structure on $\mathbf{D^bCoh}(X)$.

This explains how to interpret Kontsevich's HMS conjecture: if X and $\check{X}$ are mirror pairs then

$$\mathbf{Fuk}(X) \cong \mathbf{D^bCoh}(\check{X}), \mathbf{Fuk}(\check{X}) \cong \mathbf{D^bCoh}(X)$$

are equivalence of A_∞-categories.

4 Floer cohomology

We want to associate a cohomology theory to pairs of Lagrangians L_1, L_2 in a symplectic manifold M. There is a chain complex $CF^*(L_1, L_2)$ called *Floer complex* and we denote by $HF^*(L_1, L_2)$ its cohomology with the following properties:

1. $HF^*(L_0, L_1) \cong HF^*(L_0, \psi(L_1))$ where ψ is a Hamiltonian symplectomorphism.
2. If L_0 intersects L_1 transversely then
$$CF^*(L_0, L_1) \cong \bigoplus_{p \in L_0 \cap L_1} K\langle p \rangle$$
3. $HF^*(L_0, L_0) \cong H^*(L_0)$, the singular cohomology of L_0 with suitable coefficients.

Floer's original motivation was to prove Arnold's conjecture, which says that if L is a Lagrangian and ψ a Hamiltonian isotopy such that L intersects $\psi(L)$ transversely then
$$|\psi(L) \cap L| \geq \sum \dim H^i(L; \mathbb{Z}/2).$$
Note that the existence of Floer cohomology with properties 1, 2, 3 immediately implies the conjecture, since $CF^*(L, \psi(L)) = CF^*(L, L)$, and the dimension of a chain complex is certainly going to be larger than the sum of dimensions of the cohomologies.

Theorem 4.1 (Floer). *Assume that the area of any disk in M with boundary in L vanishes and that L intersects $\psi(L)$ transversely then*
$$|\psi(L) \cap L| \geq \sum \dim H^i(L; \mathbb{Z}/2).$$

Before proceeding, let us explain some concepts in symplectic geometry, such as Hamiltonian flow, symplectomorphism etc. Let M be a symplectic manifold and $f \in C^\infty(M)$. Then there exists a unique vector field X_f such that
$$\iota_{X_f}\omega = \mathrm{d}f.$$
The flow on M generated by X_f is called a *Hamiltonian flow*. If M is compact, we get a 1-parameter group of diffeomorphism by integrating X_f. These diffeomorphisms are all symplectomorphisms, i.e. they preserve ω.

Example. Let $X = T * M$ equipped with the symplectic form $\omega = -\mathrm{d}\theta$ where θ is the tautological 1-form. Now any function $f \in C^\infty(M)$ gives rise to a function on X. Let $\Gamma_{\mathrm{d}f} \subseteq X$ be the graph of $\mathrm{d}f$. Then the Hamiltonian flow generated by f is just Hamiltonian by $\Gamma_{\mathrm{d}f}$.

For example let $X = T^*S^1$. Take f to be a bump function on S^1. Then $\Gamma_{\mathrm{d}f}$ is the equation with a "squiggly" part. We can now displace via this flow any Lagrangian in T^*S^1.

Let us see why it is necessary that L does not bound any positive area disk for Arnold's conjecture to hold. Take a disc on T^*S^1. Then it can be displaced away from itself.

To deefine Floer complex and Floer differential using *Novikov coefficients*. Fix a ground field k.

Definition (Novikov ring)**.** The *Novikov ring* is given by

$$\Lambda_0 = \{\sum_{i=0}^{\infty} a_i T^{\lambda_i} : a_i \in k, \lambda_i \in \mathbb{R}_{\geq 0}, \lim_{i\to\infty} \lambda_i = \infty\}.$$

The *Novikov field* is

$$\Lambda = \{\sum_{i=0}^{\infty} a_i T^{\lambda_i} : a_i \in k, \lambda_i \in \mathbb{R}, \lim_{i\to\infty} \lambda_i = \infty\}.$$

Definition (Floer complex)**.** The *Floer complex* (as an ungraded vector space) is the free Λ-module generated by the intersection points of L_1 and L_2, proved that they intersect transversely.

Note that by Darboux theorem, locally every symplectic manifold is symplectomorphic to an open subset of $\mathbb{R}^{2N} = \mathbb{C}^N$ with the standard symplectic form. Thus we can say that L_1 and $L - 2$ mett transervely at p if around p,

$$l_1 \cup L_2 \cong \mathbb{R}^N \cup i\mathbb{R}^N.$$

Next, we want to define differentials on $CF(L_1, L_2)$. Equip M with an almost complex J compatible with ω, that is $\omega(-, J-)$ is a Riemannian metric. Such an almost complex structure always exists by a result of Gromov, and in fact there is a contractible space of choices. Floer differential $\partial : CF(L_0, L_1) \to CF(L_0, L_1)$ counts pseudoholomorphic strips with boundary on L_0, L_2. The coefficient of an intersection point q in $\partial(p)$ is obtained by considering maps

$$\begin{aligned} u : \mathbb{R} \times [0,1] &\to M \\ (s,t) &\mapsto u(s,t) \end{aligned}$$

such that u satsifes

1. the Cauchy-Riemann equation

   $$\mathrm{d}u \cdot i = J\mathrm{d}u$$

   or equivalently

   $$\frac{\partial u}{\partial s} + J(u)\frac{\partial u}{\partial t} = 0.$$

2. boundary condition $u(s,0) \in L_0, u(s,1) \in L_1$ for all $s \in \mathbb{R}$, $\lim_{s\to\infty}(s,t) = p, \lim_{s\to-\infty} u(s,t) = q$.

3. the energy is finite:

   $$E(u) = \int u^*\omega < \infty.$$

By Riemann mapping theorem, the domain biholormophic to a disk $\mathbb{D}$ with two marked points x_1, x_2 on the boundary. We consider maps $\varphi : \mathbb{D} \to M$ which are compatible with the complex structure such that

$$\varphi(\partial\mathbb{D}_+) \subseteq L_0, \varphi(\partial\mathbb{D}_-) = L_1, \varphi(x_1) = p, \varphi(x_2) = q$$

which have finite energy.

Note that such a map defines a class in $\pi_2(M, L_0 \cup L_1)$. Let $\widehat{M}(p, q, [u], J)$ be the space of solutions satifying the above three conditions and with homotopy class $[u]$. It has an $\mathbb{R}$-action by translation in the s-direction. We dnote by $M(p, q, [u], J)$ the orbit space.

Remark. $\widehat{M}(p, q, [u], J)$ is a smooth compact oriented manifold of dimension equal to the *Maslov index* of $[u]$ (which will be defined in a minute), proved that various conditions are satisfied.

$M(p, q, [u], J)$ is (the space of solutions of) a Fredholm problem. We have the Atiyah-Singer index theorem that allows us to compute the index from the topology of p, q, L_0, L_1. The index is the dimension of a smooth moduli space if the cokernel of the Fredholm operator satisfies certain conditions. Otherwise the space of solutions has a derived structure and the index computes the virtual dimension.

By the above discussion, if $[u]$ has index 1 then $M(p, q, [u], J)$ is a compact oriented 0-manifold so has a signed count. Temporary definition of differential:

$$\partial(p) = \sum_{q \in L_0 \cap L_1, [u] \text{ index } 1} \#M(p, q, [u], J) T^{\omega([u])} q.$$

Theorem 4.2. *If*

$$[\omega] \cdot \pi_2(M, L_0) = 0, [\omega] \cdot \pi_2(M, L_1) = 0,$$

(i.e. L_0, L_1 do not bound disks of a positive area), $\operatorname{char} k = 2, L_0, L_1$ *are oritned and have a spin structure then ∂ is well-defined and $\partial^2 = 0$. Futher up to isomorphism $HF(L_0, L_1)$ is independent of the chosen almost complex structure J, and is invariant under Hamiltonian isotopy.*

The Floer differential is well-defined by Gromov compactness, according to which given an energy bound E_0 there are only finitely many homotopy classes $[u]$ such that $\omega([u]) \leq E_0$ and $M(p, q, [u], J)$ is non-empty.

Maslov index Let $(\mathbb{R}^{2N}, \omega)$ be a symplectic vector space. We denote by $LGr(N)$ the "Lagrangian Grassmannian", the space parameterising Lagrangian N-planes. It is a fact that $LGr(N) = \mathrm{U}(N)/\mathrm{O}(N)$ (consider $\mathbb{R}^N \hookrightarrow \mathbb{R}^{2N} = \mathbb{C}^N$, then $\mathbb{R}^N$ is Lagrangian. $\mathrm{U}(N)$ acts transitively on N-planes with isotopy group $\mathrm{O}(N)$). With this identification we have a map

$$\det{}^2 : LGr(N) \to \mathrm{U}(1) = S^1.$$

This map induces an isomorphism of fundamental groups, and the index of a Lagrangian will be the image in $\mathbb{Z}$ under this map.

Given two *transverse* Lagrangian planes λ_0, λ_1 in $\mathbb{C}^N$, there exists $A \in \mathrm{Sp}(2N, \mathbb{R})$ which maps λ_0 to $\mathbb{R}^m$ and λ_1 to $i\mathbb{R}^m$. Then we have a distinguished homotopy class of paths in $LGr(N)$ connecting λ_0 and λ_1, represented by the *canonical short path*

$$\lambda_t = A^{-1}((e^{-i\pi t/2})\mathbb{R}^N), t \in [0, 1].$$

Since $\mathbb{R} \times [0, 1]$ is a contrac u^*TM is a trivial symplectic vector bundle. Fix a trivialisation $u^*TM \cong D \times \mathbb{R}^{2n}$. Now for every $s \in \mathbb{R}$, $u(s, 1) \in L_1$ so $u(s, 1)^*TL_1$ gives a family of Lagrangian subspaces inside $u(s, 1)^*TM \cong \mathbb{R}^{2N}$. Thus we obtain a path in $LGr(N)$. Similar for $u(s, 0)$. in order to get a loop, we need to fill in the shaded region. We do this via canonical short path, using that fact that T_pL_1 and T_pL_2 (resp. T_qL_1 and T_qL_2) are transverse.

Definition (Maslov class)**.** The *Maslov class* of u is the class in

$$\pi_1(LGr(N)) \cong \pi_1(S^1) \cong \mathbb{Z}$$

of the loop

$$\begin{array}{ccc} T_pL_1 & \longleftarrow & T_qL_1 \\ \downarrow & & \uparrow \\ T_pL_2 & \longrightarrow & T_qL_2 \end{array}$$

Example. Let $M = \mathbb{R}^2$. Then $LGr(1) = \mathbb{P}^1_{\mathbb{R}} \cong S^1$. It is easy to see that the Maslov class of u is 1. The tangent line to $L_1, L - 0$ go through all points of $\mathbb{P}^1_{\mathbb{R}}$ exactly once.

Let us make some comments on the Maslov index 1 constraint in the definition of Floer differential.

1. We are counting disks discarding multiple covers.
2. The Maslov index 1 constraint also imposes *direction*: the image above seems to suggest p and q are symmetric. But Maslov class 1 specifies whether the disk gives a differential from p to q or vice versa.

grading on Floer complex For this we need some extra assumptions on L_1, L_2 and M:

1. $2c_1(T_{\mathbb{C}}M) = 0$ (aside: the stronger condition $c_1(T_{\mathbb{C}}M) = 0$ is called Calabi-Yau). That is, we have a nowhere vanishing section θ of $(\bigwedge^m T_{\mathbb{C}}M)^{\otimes 2}$.
2. Once we fix θ, we can define for every Lagrangian $L \subseteq M$ *phase*

$$\begin{aligned} \varphi_L : L &\to S^1 \\ p &\mapsto \arg(\theta(T_pL, T_pL)) \end{aligned}$$

3. The "Maslov class of L" is $[\varphi_L] \in [L, S^1] = H^1(L; \mathbb{Z})$.

The grading on $CF^*(L_1, L_2)$ requires L_1 and L_2 to have vanishing Maslov class. The point is that in order to define $CF^*(L_1, L_2)$ as a graded complex, L_1 and L_2 need to be *graded Lagrangian.* A graded Lagrangian $\widetilde{L}$ is a Lagrangian L with vanishing Maslov class together with a lift to $\mathbb{R}$

$$\begin{array}{ccc} & & \mathbb{R} \\ & \nearrow & \downarrow \\ L & \xrightarrow{\varphi_L} & S^1 \end{array}$$

If we fix graded lifts $\widetilde{L}_0, \widetilde{L}_1$, we get a $\mathbb{Z}$-grading (changing the graded lifts changes the grading of CF^* but not the even/odd components).

What is the degree of $p \in L_0 \cap L_1$? We fix graded lifts $\widetilde{L}_0, \widetilde{L}_1$. We are going to associate to p a loop in $LGr(T_pM)$ as follows:

1. choose any path $\widetilde{\alpha}$ between $\widetilde{T_pL_0}$ and $\widetilde{T_pL_1}$ in $\widetilde{LGr(T_pM)}$, where $\widetilde{LGr(T_pM)}$ is the universal cover.
2. $\alpha = \pi(\widetilde{\alpha})$ defines a path from T_pL_0 to T_pL_1. We close this path by canonical short path.
3. We have associated to p a loop in $LGr(T_pM)$. Its image in $\pi_1(S^1)$ is the degree of p.

It is easy to see that if u is a strip connecting p and q with indedex 1 then

$$\deg q = \deg p + 1.$$

Thus the Floer differential has degree 1.

Remark. If we do not want to work with graded Lagrangians, we can work with $\mathbb{Z}/2$-grading on the Floer complex. Then we can get away with L_0, L_1 being oriented as in Floer's theorem (but still some extra assumptions on $L_0, L-1$ are needed for ∂ to be well-defined and for $\partial^2 = 0$, see those in Floer's theorem).

We are now ready to explain why $b^2 = 0$. The coefficient of q in $\partial^2 p$ is

$$\sum_{r \in L_0 \cap L_2} \#M(p, q, [u'], J) \cdot \#M(r, q, [u''], J) T^{\omega([u']) + \omega([u''])}.$$

We can break this up as follow. Set

$$[u] = [u'] + [u''] \in \pi_2(M, L_1 \cup L_2)$$

with index 2 and we can write the sum as

$$\sum_{[u]:\mathrm{ind}([u])=2} \Big(\sum_{r \in L_0 \cap L_1} \cdots \Big).$$

We are going to show the term in the parenthesis is 0. By Gromov compactness the moduli space $M(p, q, [u], J)$ can be compactified and the boundary points are of the foolowing three types:

1. broken strips,

2. disk bubbling,

3. sphere bubbdling.

The key point is to exclude 2 and 3 from occurring. For instance this is the case if $[\omega] \cdot \pi_2(M, L_i) = 0$, which is precisely the assumption we made. Now the structure of the boundary of $\overline{M}(p, q, [u], J)$ can be described explicitly:

$$M(p, q, [u], J) = \coprod_{\substack{r \in L_0 \cap L_1 \\ [u]=[u']+[u'']}} M(p, r, [u'], J) \times M(r, q, [u''], J).$$

Now the term in parenthesis is

$$\sum_{p \in \partial \overline{M}} (-1)^{\mathrm{sgn}} p$$

but the signed count of boundary points of a compact 1-dimensional manifold is 0, and we get the desired result.

Remark. For those who know GW-invariants, the relationship between $\overline{M}(p, q, [u], J)$ and $\overline{k}_N$ is analogous to that between $\overline{M}_{0,N}(X, [\beta])$ and $\overline{M}_{0,N}$.

To see $CF^*(L, L) = H^*(L)$, one uses Morse cohomology. Consider T^*S^1 with $L = S^1$, the zero section. $\varphi(L)$ is obtained by displaceing L via Hamiltonian isotopy induced by a map $f : S^1 \to \mathbb{R}$. $\varphi(L) = \Gamma_{\mathrm{d}f}$. Since L and $\varphi(L)$ meet transversely,

$$CDF^*(L, \varphi(L)) \cong \Lambda(p, q).$$

$M(p, q, J)$ has two points. Thus

$$\begin{aligned} \partial q &= 0 \\ \partial &= T^{A_1} q + T^{A_2} q \quad \text{if } A_1 = A_2 \end{aligned}$$

(we assume Λ has characteristic 2 to avoid keeping track of signes. But the areas are indeed the same since $L = \Gamma_{\mathrm{d}f}$ and by fundamental theorem of calculus. This means that in fact $\partial = 0$ and therefore

$$HF^*(L, \varphi(L)) = CF^*(L, \varphi(L)) \cong H^*_{\mathrm{sing}}(L, \Lambda).$$

4.1 Fukaya category

The Fukaya category **Fuk**(M) of a symplectic manifold M is an A_∞-category such that

- the objects are Lagrangian satisfying extra properties/structures . L is "unobstructed" (i.e. $[\omega] \cdot \pi_2(M, L) = 0$), is oriented, has a spin structure, grading and is equipped with U(1)-local system etc.
- the morphisms are given by $CF^*(L_1, L_2)$.
- have all A_∞ higher products

(To make the Fukaya category well-defined we need to choose consistent perturbation data (for every pair of Lagrangians L_1, L_2 we need to chooose a Hamiltonian isotopy to make them transverse).)

In fact for the sake of HMS the Fukaya category needs to be further enlarged. The point is that we want to be able to take the cones of morphisms. We need to pass to the trianglulated envelop of the Fukaya category. This is the same as adding to the Fukaya category all finite limits/colimits. There is a universal way to add cones (and in fact all limits/colimtis) to an A_∞-category, namely taking twisted complexes. We do not get *all* limits/colimits this way, and we need to pass to the split closure (i.e. the Karoubi envelop).

5 Aside: special Lagrangians

Let (X, ω, J) be a smooth compact Kähler manifold. X is *Calabi-Yau* if the canonical bundle $K_X = \bigwedge^n T^*X$ is trivial. If X is Calabi-Yau, let Ω be a nowhere vanishing section of K_X. If $L \subseteq X$ is a Lagrangian submanifold then $\Omega|_L = \psi \cdot \mathrm{vol}_L$ for some $\psi \in C^\infty(L, \mathbb{C}^*)$. Indeed $\dim_{\mathbb{R}} L = \dim_{\mathbb{C}} X$ so $\Omega|_L$ is a complex-valued top form on L.

Definition (special Lagrangian)**.** $L \subseteq X$ is *special Lagrangian* if eigenvalues of ψ is constant.

In the first example we claimed that lines with rational slopes are special Lagrangians on a Kähler torus. Why is this the case? On a torus, a section of the canonical is just $\mathrm{d}z = \mathrm{d}x + i\mathrm{d}y$. On the line $x = \frac{n}{m}y$, we can choose parameterisation $x \mapsto (x, \frac{m}{n}y)$ so the pullback to the line is $(1 + i\frac{m}{n})\mathrm{d}x$. Clearly eigenvalues of $1 + i\frac{m}{n}$ are constant.

For the other direction, we only sketch the argument. In each Hamiltonian isotopy class of Lagrangian submanifolds there is at most one special Lagrangian (there are many interesting analytic properties. For example they are volume minimising). Given any Lagrangian submanifold in a Riemann surface, there is a Hamiltonian isotopy sending it to a geodesic. In the case of a torus, this means that every Lagrangian can be isotoped to line.

Reference: Dominic Joyce, Conjectures on Bridgeland stability for Fukaya categories of Calabi-Yau manifolds, special Lagrangians.

Thomas, Yau, Special Lagraingian and stable bundles and mean curvature flow.

6 Aside: additional structure on objects of Fukaya category

Why are objects in the Fukaya category Lagrangian submanifolds *plus* the data of ($U(N)$-local system (plus other data)? The first answer is that we don't necessarily have to: there exist variants of the Fukaya category where

1. Lagrangians do not carry a local system,
2. or Lagrangians carry a $\mathrm{GL}(N, \mathbb{C})$-local system.

The variant of the Fukaya category where we consider Lagrangians with $U(N)$-local system is needed for HMS and this is especially clear from the SYZ viewpoint. According to SYZ, if X and $\check{X}$ are mirrors then we habe dual speical Lagrangian fibrations: if $\pi : X \to B$ and $\check{\pi} : \check{X} \to B$ are fibrations, then for $b \in B$, $\check{X}_b = \check{\pi}^{-1}(b)$ is the dual torus to X_b in the sense that $\check{X}_b$ is the moduli space of $U(1)$-local systems on X_b.

Example. As an example of a fibration in special Lagrangian tori, we can consider the moment map $p : X_\Sigma \to \Delta_\Sigma \subseteq M_\mathbb{R}$ where X_Σ is a toric variety and Δ_Σ is its moment polytope.

Let's see SYZ in the elliptic curve case ($\psi = \rho = i$). What are the dual SYZ fibrations in this case? Note that E is self-mirror, but we denote its dual by $\check{E}$ and note $\mathbf{Fuk}(E) \simeq \mathbf{D^bCoh}(\check{E})$. The special Lagrangian fibres of the SYZ fibrations are objects of the Fukaya category. $L_p = \pi^{-1}(p)$ plus a choice of $U(1)$-local system determines a point (say x) in $\check{E}$: indeed $\check{L}_p = \check{\pi}^{-1}(p)$ is the moduli psace of $U(1)$-local systems on L_p. Thus HMS sends the object (L_p, U) to $k(x) \in \mathbf{D^bCoh}(\check{E})$, where $k(x)$ is a skyscraper sheaf at x.

7 Aside: closed and open string mirror symmetry

Closed string mirror symmetry: $QH^*(X) \cong HH^*(\check{X})$. Open string mirror symmetry: $\mathbf{Fuk}(X) \cong \mathbf{D^b}(\check{X})$. Taking Hochshild cohomology and using PSS isomorphism that $QH^*(X) \cong HH^*(\mathbf{Fuk}(X))$, this seems to say that in principle open string mirror symmetry implies closed string mirror symmetry. However working this implication out is hard. As of today, these issues should be solved by work of Ganatra-Perutz-Sheridan.

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