Scuola Internazionale
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Geometry and Mathematical Physics

# Topics in Homological Mirror Symmetry

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# Contents

# 1 Introduction

Initiated by Kontsevich. There are two geometries: on the A side there is a symplectic manifold $(M,\omega)$ of dimension $2n$. A submanifold $L \subseteq M$ is called Lagrangian if $\dim L = n$ and $\omega|_L = 0$ while on the B side there is an algebraic variety $X$, for example a projective variety. The link between these two sides is triangulated categories: for $X$ we can consider the bounded derived category of coherent sheaves $D^b(\mathrm{Coh}(X))$, and for $M$ we can associate its Fukaya category $D^\pi\mathcal{F}(M,\omega)$. Wehn $F(M,\omega)$ and $D^b(\mathrm{Coh}(X))$ are equivalent, we say $(M,\omega)$ and $X$ are *mirror partners* and homological mirror symmetry holds for this pair. There are many variations to this formalism, for example one many consider dg categories or $A_\infty$-categories instead of triangulated categories.

**Seidel's quartic surface theorem** First we introduce the *Novikov field*

$$\Lambda = \{c_0 q^{m_0} + c_1 q^{m_1} + \cdots : c_i \in \mathbb{C}, m_i \in \mathbb{R}, \lim_{k\to\infty} m_k = +\infty\}.$$

On the symplectic side let $M_p \subseteq \mathbb{CP}^3$ be a smooth quartic surface defined by a homogeneous quartic polynomial $p$, for example $p = x_0^4 + x_1^4 + x_2^4 + x_3^4$, equipped with $\omega_p$ the restriction of Fubini-Study form. Up to symplectomorphism $(M_p, \omega_p)$ is independent of $p$. We associate to $M$ the Fukaya category $D^\pi\mathcal{F}(M_p,\omega_p)$, a $\Lambda$-linear triangulated category.

On the algebraic side consider a particular quartic surface in $\mathbb{P}^3_\Lambda$ given by

$$Y = \{y_0 y_1 y_2 y_3 + q(y_0^4 + y_1^4 + y_2^4 + y_3^4) = 0\}.$$

Divide $Y$ by the automorphism group

$$\Gamma_{16} = \{\begin{pmatrix} \alpha_0 & & & \\ & \alpha_1 & & \\ & & \alpha_2 & \\ & & & \alpha_3 \end{pmatrix} : \alpha_i^4 = 1, \alpha_0\alpha_1\alpha_2\alpha_3 = 1\} \subseteq \mathrm{PSL}(4,\Lambda).$$

and take its minimal resolution $X$.We associate to $X$ its derived category $D^b\,\mathrm{Coh}(X)$, another $\Lambda$-linear triangualted category.

**Theorem 1.1** (Seidel)**.** *There is an equivalence of categories*

$$D^\pi\mathcal{F}(M,\omega) \cong \hat{\psi}^* D^b\,\mathrm{Coh}(X)$$

*where* $\hat{\psi} \in \mathrm{Aut}(\Lambda/\mathbb{C})$ *is the mirror map.*

Let's briefly mention what a Fukaya category is, and in subsequent lectures we will expound more on it. Firstly it is not a category in the classical sense but an $A_\infty$-category, meaning that associativity does not hold strictly but only up to coherent homotopy. The objects are Lagrangian submanifolds of $(M,\omega)$. Given $L_0, L_1$ meeting transverly, $\mathrm{Hom}(L_0, L_1)$ is given by $\Lambda$-span of intersections of $L_0$ and $L_1$. The composition

$$\mathrm{Hom}(L_0,L_1) \otimes \mathrm{Hom}(L_1,L_2) \to \mathrm{Hom}(L_0,L_2)$$

is given by counting pseudoholomorphic triangles with boundaries the Lagrangians and vertices the intersections. There could be infinitely many triangles with given boundary data. Nevertheless owing to Gromov compactness for each $C$ there are only finitely many triangles with area at most $C$. Thus weighing each triangle by $q^{\text{area}}$, we get a series that converges $q$-adically, thus an element of $\Lambda$.

**HMS for toris (Polishchuk-Zaskow, Seidel, Lekili-Perutz)** If we lower the dimension in the previous example by one, we will get cubic cruves in $\mathbb{CP}^2$, i.e. elliptic curves. On the symplectic side we take $T^2 = \mathbb{R}^2/\mathbb{Z}^2$ with standard symplectic form $\omega = \mathrm{d}p \wedge \mathrm{d}q$. Note that any line on the fundamental domain is Lagrangian, so gives an object in the Fukaya category. On the analytic side (as opposed to algebraic), take the Tate curve $\Lambda^*/q^{\mathbb{Z}}$ where $\Lambda^* = \mathbb{A}^1_\Lambda \setminus \{0\}$ and it is equipped with a ring of analytic functions, whose global section is

$$\mathcal{O}_{\mathrm{an}}(\Lambda^*) = \{\sum_{k=0}^\infty c_k q^{m_k} t_{n_k} : c_k \in \mathbb{C}, m_k \in \mathbb{R}, n_k \in \mathbb{Z}, \lim_{k\to\infty} m_k + C n_k = \infty \text{ for any } C\}.$$

This is an analytic analogue of elliptic curve over $\Lambda$: one can write an elliptic curve as the quotient of $\mathbb{C}$ by a lattice

$$E = \mathbb{C}/\langle 1, \tau\rangle$$

where $\operatorname{Im}\tau > 0$. Taking exponential and setting $q = e^{2\pi i\tau}$, we get a transcendental model

$$E = \mathbb{C}^*/q^{\mathbb{Z}}.$$

Homological mirror symmetry holds:

$$D^\pi \mathcal{F}(T^2, \omega) \cong D^b \operatorname{Coh}(\Lambda^*/q^{\mathbb{Z}}).$$

In the transcendental presentation of an elliptic curve $E$, sections of line bundles $\mathcal{L}$ pull back to theta functions. But a section is nothing more than a morphism $\mathcal{O}_E \to \mathcal{L}$ so this gives a mophism in the derived category. If we set up things correctly, structure constants would be values of theta functions (?).

Similar results hold for the nonarchimedean ellptic curve, in which case we have

$$\Theta_{n,k}(t) = \sum_{n\mathbb{Z}+k} q^{i^2/2n} t^i, n \geq 0, k \in \mathbb{Z}/n\mathbb{Z}$$

and it transforms like

$$\Theta_{n,k}(qt) = q^{-n/2} t^{-n} \Theta_{n,k}(t).$$

How can we see this on the symplectic side? We are requied to count the number of triangles with boundary and sides as follow. Note that however, we do not require the triangles to be *embedded*. Thus on the universal cover $\mathbb{R}^2$, we can find infinitely many triangles, indexed by $\mathbb{Z}$. Thus weighing each triangle by $q^{\text{area}}$, we obtain the structure constant

$$\sum_i q^{i^2/2} = \Theta_{1,0}(1).$$

Finally, here is an outline of the course:

- symplectic geometry,
- $A_\infty$-categories,
- Fukaya category,
- homological mirror symmetry in the torus case, which is the same as classification of $A_\infty$-structures.
- disk counting in Fano variety and if time permitting, wall-crossing.

# 2 Fukaya categories I

This chapter is about symplectic geometry underlying the theory Fukaya category.

> **Definition** (symplectic manifold)**.** A *symplectic manifold* is a pair $(M, \omega)$ where $M$ is a smooth manifold of dimension $2n$ and $\omega \in \Omega^2(M)$ a 2-form such that $\mathrm{d}\omega = 0$ and $\omega^n$ nowhere vanishing.

In other words $\omega^n \in \Omega^{2n}(M)$ is a volume form and thus determines an orientation. Another way to think about nondegeneracy is that $\omega \in \Omega^2(M) = \Gamma(M, \bigwedge^2 T^*M)$ induces an isomorphism $TM \to T^*M$ via $X \mapsto \omega(X, -)$.

**Example.**

1. Let $Q$ be a smooth manifold. Then the cotangent bundle $M = T^*Q$ is equipped with a canonical symplectic form. In local coordinates $(q_1, \dots, q_n)$ on $Q$ and $(p_1, \dots, p_n)$ dual coordinates on the fibre of $T^*Q$, we can define a 1-form
$$\Theta = \sum_{i=1}^n p_i \mathrm{d}q_i$$
which is invariantly defined. Then $\omega = \mathrm{d}\Theta = \sum \mathrm{d}p_i \wedge \mathrm{d}q_i$ is the symplectic form.

2. The first example is motivated by classical mechanics. $Q$ is the configuration space of a system and $T^*Q$ is the phase space. There is a function $H : T^*Q \to \mathbb{R}$ called Hamiltonian which measures total energy. Hamilton's equation says
$$\dot{q}_i = \frac{\partial H}{\partial p_i}, \dot{p}_i = -\frac{\partial H}{\partial q_i}.$$
A coordinate-free way to describe the equation is to say the vector field generated by $H$ satisfies the ODE
$$\omega(-, X_H) = \mathrm{d}H.$$

3. Any Kähler manifold is a symplectic manifold: one way to define a Kähler manifold is a complex manifold $M$ together with a Riemannian metric $g$ such that $\omega(X, Y) = g(X, iY)$ is closed. As the Riemannian metric is nondegenerate, $\omega$ is a symplectic form. For example $\mathbb{CP}^N$ with the Fubini-Study metric is symplectic, so is any complex submanifold $M \subset \mathbb{CP}^N$.

> **Definition** (Lagrangian submanifold)**.** Let $(M, \omega)$ be a symplectic manifold of dimension $2n$. A submanifold $L \subseteq M$ is *isotropic* if $\omega|_L = 0$. $L$ is *Lagrangian* if in addition $\dim L = n$.

**Example.**

1. In $T^*Q$ the zero section of $T^*Q$ regarded as a vector bundle over $Q$ is Lagrangian: in locally coordinates $p_i$'s are constant so $\omega$ vanishes.

2. Given $q \in Q$, the tangent space at $q$

$$T_q^* Q = \{(q, p) : p \in T_q^* Q\}$$

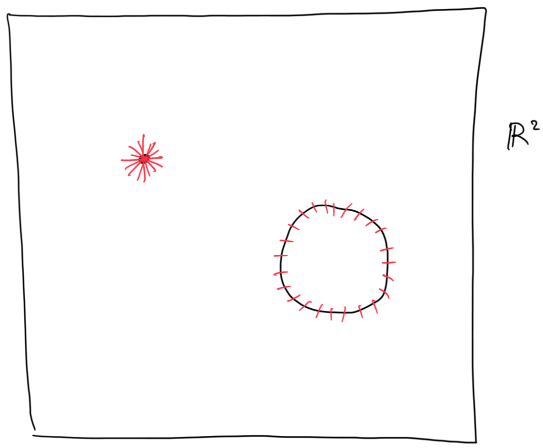
is Lagrangian.

3. A general procedure to produce Lagrangian submanifold is to take a smooth submanifold $N \subseteq Q$ and consider the conormal bundle

$$T_N^* Q = \{(q, p) : q \in N, \sum (p_i \mathrm{d} q_i)|_{T_q N} = 0\}$$

where $p_i$'s are interpreted as linear functions on tangent spaces on $Q$.

4. $T^*\mathbb{R}^2$, visualised as the zero section here and hence two missing dimensions "going out of the plane". Then we can visualise the cotangent space as a




point with hair.

5. On a surface $M$, any 2-form is closed for degree reason so any volume form defines a symplectic form. Furthermore any one dimensional submanifold is automatically isotropic. For example we can take $M = T^*S^1 \cong S^1 \times \mathbb{R}$. Let $q$ and $p$ be coordinates on $S^1$ and $\mathbb{R}$ respectively. Then $\omega = \mathrm{d}p \wedge \mathrm{d}q$ is a symplectic form and any submanifold is Lagrangian.

Slogan: Lagrangians are natural "boundary conditions".

## 2.1 Infinitesimal symplectic geometry

Linear algebra: let $V$ be a vector space of dimension $2n$ and $\omega \in \bigwedge^2 V^*$ e. $V$ can be modelled by $\mathbb{C}^n \cong \mathbb{R}^{2n}$ with the hermitian form

$$\langle z, w \rangle = \sum \overline{z}_i w_i$$

which can be split into real and imaginary part

$$\langle z, w \rangle = b(z, w) + i\omega(z, w)$$

wher $b$ and $\omega$ are $\mathbb{R}$-valued, $\mathbb{R}$-linear forms, with $b$ symmetric and skew symmetric. Then $(\mathbb{C}^n, \omega)$ is a model for symplectic vector spaces, in the sense that every $(V, \omega)$ is isomorphic to it for some $n$.

There are several automorphism groups by considering different structures on $\mathbb{C}^n$:

$$\begin{aligned} \mathrm{GL}(n, \mathbb{C}) &= \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^n) \\ \mathrm{Sp}(2n) &= \mathrm{Aut}_{\mathbb{R}}(\mathbb{C}^n, \omega) \\ \mathrm{U}(n) &= \mathrm{Aut}_{\mathbb{C}}(\mathbb{C}^n, b + i\omega) \end{aligned}$$

These three groups are homotopy equivalent (in fact, $\mathrm{U}(2)$ is a subgroup of the other two and the inclusions are homotopy equivalences). As a consequence, for $(M, \omega)$ symplectic, $(TM, \omega) \to M$ is a symplectic vector bundle with structure group $\mathrm{Sp}(2n)$, so characteristic classes of $\mathrm{Sp}(2n)$-bundles are the same as those for $\mathrm{U}(n)$-bundles, i.e. Chern classes.

A linear subspace $L \subseteq (\mathbb{C}^n, \omega)$ with dimension $n$ is *Lagrangian* if $\omega|_L = 0$. The subgroup $\mathrm{U}(n) \subseteq \mathrm{Sp}(2n)$ acts transitively on the set of Lagrangian subspaces, with the stabiliser of $\mathbb{R}^n \subseteq \mathbb{C}^n$ being $\mathrm{O}(n) \subseteq \mathrm{U}(n)$. The space of Lagrangian subspace is thus

$$\mathrm{LGr}(n) \cong \mathrm{U}(n)/\mathrm{O}(n),$$

the *Lagrangian Grassmannian*.

Just as characteristic classes of vector bundles come from cohomology of Grassmannians, cohomology of Lagrangian Grassmannian is related to the so-called Maslov classes. Using the homotopy exact sequence coming from

$$1 \longrightarrow \mathrm{O}(n) \longrightarrow \mathrm{U}(n) \longrightarrow \mathrm{LGr}(n) \longrightarrow 1$$

one finds $\pi_1(\mathrm{LGr}(n)) \cong \mathbb{Z}$, with the isomorphism induced by

$$\det{}^2 : \mathrm{U}(n)/\mathrm{O}(n) \to \mathrm{U}(1).$$

This gives the *Maslov class* $\mu \in H^1(\mathrm{LGr}(n); \mathbb{Z})$.

Globally, let $(M, \omega)$ and consider the symplectic bundle $(TM, \omega)$. There is a bundle version of Grassmannian $\mathrm{LGr}(M) \to M$ whose fibre over $x$ is $\mathrm{LGr}(T_x M)$. We would like to obtain a class $\mu_M \in H^1(\mathrm{LGr}(M); \mathbb{Z})$ that restricts to $\mu$ on each fibre. This is possible if $2c_1(M) = 0 \in H^2(M; \mathbb{Z})$.

**Remark.** The existence of Maslov class is equivalent to having a global determinant square map. Determinant is a section of the line bundle $\bigwedge^n T^*M$, so $\det{}^2$ is defined if and only if $(\bigwedge^n T^*M)^{\otimes 2}$ is trivial, if and only if $2c_1(M) = 0$.

Now suppose $2c_1(M) = 0$ and a global $\mu_M \in H^1(\mathrm{LGr}(M); \mathbb{Z})$ is chosen. Let $L \subseteq M$ be a Lagrangian. Then for $x \in L$, $T_x L$ is a Lagrangian subspace of the fibre $\mathrm{LGr}(T_x M)$, so $s_L = \{T_x L\}$ determines a section of $\mathrm{LGr}(M)$ over $L$. The pullback

$$s_L^*(\mu_M) \in H^1(L; \mathbb{Z})$$

is called the Maslov class of $L$.

**Exercise.** Compute the Maslov class of a closed curve in $\mathbb{C} \subseteq T^*\mathbb{C}$.

We also have simpler cohomology forms.

**Definition.** $(M, \omega)$ is *exact* if $[\omega] = 0 \in H^2(M; \mathbb{R})$, i.e. $\omega = \mathrm{d}\Theta$ for some $\Theta \in \Omega^1(M)$.

**Example.** By definition $(T^*Q, \omega)$ is exact.

**Proposition 2.1.** *If $M$ is compact then $\omega$ cannot be exact.*

*Proof.* If $\omega = \mathrm{d}\Theta$ then $\omega^n = \mathrm{d}(\Theta \wedge \omega^{n-1})$. As $M$ is compact

$$0 < \mathrm{vol}(M) = \int_M \omega^n = \int_M \mathrm{d}(\Theta \wedge \omega^{n-1}) = 0.$$

Absurd. □

Thus if we want to study "symplectically simple" manifolds we are bound to leave compact manifolds behind.

**Definition** (exact Lagrangian submanifold)**.** Suppose $(M, \omega)$ is exact and choose $\Theta$ such that $\mathrm{d}\Theta = \omega$, which is called a *Liouville 1-form.* Then given $L \subseteq M$ Lagrangian, $\Theta|_L$ defines a closed 1-form on $L$. $L$ is *exact* if $[\Theta|_L] = 0 \in H^1(L; \mathbb{R})$.

Motivation for Fukaya category: intersection theory for Lagrangian submanifolds. Suppose $L_1, L_2$ are two compact Lagrangian submanifolds of $(M, \omega)$. Note $\omega^n$ induces orientations on $L_1, L_2$ (or orientability?). Choose orientations. Assume $L_1 \pitchfork L_2$ so the signed count (sign determined by if $T_xL_1 \oplus T_xL_2 = T_xM$ is a decomposition compatible with the oreintations) of intersection points, denoted by $L_1 \cdot L_2$, is well-defined. Equivalently we can take the classes $[L_1], [L_2] \in H_n(M)$ and compute their intersection product using Poincaré duality.

The goal is to categorify intersection theory of Lagrangian submanifolds, say we want $L_1 \cdot L_2 = \chi(CF(L_1, L_2))$ for some chain complex $CF(L_1, L_2)$. As a first step, let $\mathbb{K}$ be a field and consider

$$CF(L_1, L_2) = \bigoplus_{x \in L_1 \cap L_2} \mathbb{K}x$$

endowed with a $\mathbb{Z}/2$ grading by sign of intersection. Then

$$\chi(CF(L_1, L_2)) := \dim CF^{\mathrm{even}}(L_1, L_2) - \dim CF^{\mathrm{odd}}(L_1, L_2)$$

is the intersection number $L_1 \cdot L_2$.

In favourable cases, there is an algebraic structure on the vector spaces $\{CF(L_i, L_j)\}$ called *$A_\infty$-category.*

**Example.** Recall $M = T^*Q$ is an exact symplectic manifold with Liouville 1-form $\Theta = \sum p_i \mathrm{d}q_i$. Let $L_1$ be the zero section. Given $f : Q \to \mathbb{R}$, let $L_2$ be the graph of $\mathrm{d}f \in \Omega^1(Q) = \Gamma(Q, T^*Q)$

**Proposition 2.2.** *$L_2$ is an exact Lagrangian in $T^*Q$.*

*Proof.* $\mathrm{d}f : Q \to T^*Q$ satisfies

$$(\mathrm{d}f)^*\Theta = \mathrm{d}f,$$

an exact 1-form so $(\mathrm{d}f)^*\omega = 0$. $\square$

Then

$$L_1 \cap L_2 = \{q \in Q : \mathrm{d}f_q = 0\} = \mathrm{Crit}(f).$$

The hypothesis $L_1 \pitchfork L_2$ is equivalent to saying the critical points of $f$ are nondegenerate, i.e. $f$ is Morse. Locally near a critical point there is a coordinate in which $f$ is given by $q_1^2 + q_2^2 + \cdots + q_r^2 - q_{r+1}^2 - \dots q_n^2$.

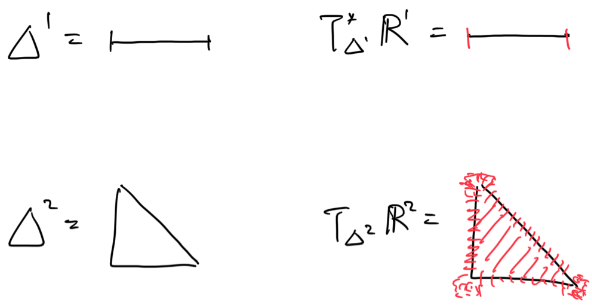
(For example if $f = \frac{1}{2}q^2$ then $\mathrm{d}f = q\mathrm{d}q$ has graph (...), whose intersection is $\{q = 0\}$, the critical points of $f$.)

The underlying vector space is $CF(L_1, L_2) = \bigoplus_{x \in \mathrm{Crit(f)}} \mathbb{K}x$. We know from Morse theory that it is the underlying vector space for the Morse complex whose cohomology is $H^*(Q)$. Thus Fukaya category (in full fledged form) absorbs Morse theory.

Suppose we want to maximise $f : \Delta^n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : x_i \geq 0, \sum x_i \leq 1\} \to \mathbb{R}$. Look at the derivative and look at the boundary. Note that $\Delta^n$ is a union of locally closed smooth manifolds $\{S_\alpha\}$.

singular subsets of $T^*\mathbb{R}^n$: $T^*_{\Delta^n}\mathbb{R}^n$, the "conormal bundle" of $\Delta^n$. Then

$$T^*_{\Delta^n}\mathbb{R}^n = \bigcup_\alpha T^*_{S_\alpha}\mathbb{R}^n.$$




Given $f : \Delta^n \to \mathbb{R}$ differentiable, the maximum value of $f$ is attained at some point of $\pi(\mathrm{graph}(\mathrm{d}f) \cap T^*_{\Delta^n}\mathbb{R}^n)$

Question for thinking: what are Lagrange multipliers about? (Answer: adding slack variables for a more uniform phrasing of the problem)

Now given a bunch of intersection points, we would like to define morphisms between them. A natural candidate would be surfaces with the given boundary. We have to however pick a special class of surfaces so that the moduli space of such maps would be "compact". Pioneered by Gromov in 1985 is the theory of "holomorphic maps" from Riemann surfaces to symplectic manifolds.

If $(M, J)$ is an almost complex manifold then $TM$ carries the structure of a $\mathbb{C}$-vector bundle.

The condition for a smooth map $f : (M_1, J_1) \to (M_2, J_2)$ between two almost complex manifolds to be pseudoholomorphic,

$$Df \circ J_1 = J_2 \circ Df,$$

is in general overdetermined and has no solution. One way is to impose integrability condition, requiring the Nijnhuis tensor to vanish, thus recovering the theory of complex manifolds. Another way, realised by Gromov, is to require $M_1$ to have complex dimension 1 (for dimension reason $M_1$ is automatically a complex manifold).

Given a symplectic manifold $(M, \omega)$, the space of almost complex structures compatible with $\omega$ is contractible.

**Example.** Looking at maps $\mathbb{CP}^1 \to (M, \omega, J)$ gives Gromov-Witten theory. More importantly in Fukaya category, we look at Riemann surfaces with boundary to $M$ such that the boundaries are mapped to a configuration of Lagrangian submanifolds.

**Fukaya category** Let $(M, \omega)$ be a symplectic manifold and let $L_1, L_2, L_3, \ldots$ be Lagrangian submanifolds, which are going to be objects of the category. The chain complexes, assuming $L_1 \pitchfork L_2$, of the form

$$CF(L_1, L_2) = \bigoplus_{x \in L_1 \cap L_2} \mathbb{K}x$$

are the morphisms. The geometric objects that connect the intersection points are pseudoholomorphic curves. To this end pick a compatible almost complex structure $J$ on $(M, \omega)$. Consider $(\Sigma, j)$ a Riemann surface and $u : \Sigma \to M$ a pseudoholomorphic map. If $\Sigma$ has a boundary $\partial\Sigma$ we require $u : \partial\Sigma \to L \subseteq M$ for some $L$ Lagrangian (if it has multiple boundary components we require this for each component). There are many choices of the topology of $\Sigma$, for example the Riemann sphere and the genus $g$ surface. For us the most important cases are the *closed* disk $D^2$ and punctured disks $D^2 \setminus \{\text{pt}\}$.

Now suppose $P^0$ is a bouned simply connected polygonal domain in $\mathbb{C}$. By Riemann mapping theorem, there is a biholomorphic map $u : D^\circ \to P^\circ$. How does this map look like? It turns out there is a formula (Schwarz-Christoffel formula): suppose the interior angles of $P$ are $\alpha_1\pi, \alpha_2\pi, \ldots, \alpha_n\pi$ where $\alpha_i \in (0, 2]$. Then

$$u(w) = A \int_0^w \prod_{k=1}^n (w - w_k)^{\alpha_k - 1} \mathrm{d}w + B$$

for some constants $w_k, A, B$. These $w_k$'s are actually on the boundary of the disk (note that the $w_k$'s are determined by $P$ up to $\mathrm{Aut}(D)$). The formula cannot possibly extend to a biholomorphism to all of $D$, as at $w_k$'s the boundary of the disk is not mapped conformally to the polygon. The boundary-punctured disk $D \setminus \{w_1, \ldots, w_n\}$ is the natural domain of $u$ on which it is a biholomorphism. Then the map $u : D \to P \subseteq \mathbb{C}$, is an example of a *pseudoholomorphic polygon*: $u$ is a biholomorphism on $D \setminus \{w_1, \ldots, w_n\}$ and extend continuously to $w_k$'s, such that $u(w_k)$'s are the intersection of the Lagrangian, and the arc between $w_k$ and $w_{k+1}$ are mapped to a Lagrangian. Stated formally,

**Definition** (pseudoholomorphic polygon)**.** Consider $L_0, \dots, L_n$ metting trasversally at $x_i \in L_{i-1} \cap L_i$. Let $(D, \{w_0, \dots, w_k\})$ be a disk with $n+1$ boundary punctures. A *pseudoholomorphic polygon* in $M$ with boundary data $\{L_i, x_i\}$ is a map $u : D \setminus \{w_0, \dots, w_n\} \to M$ such that

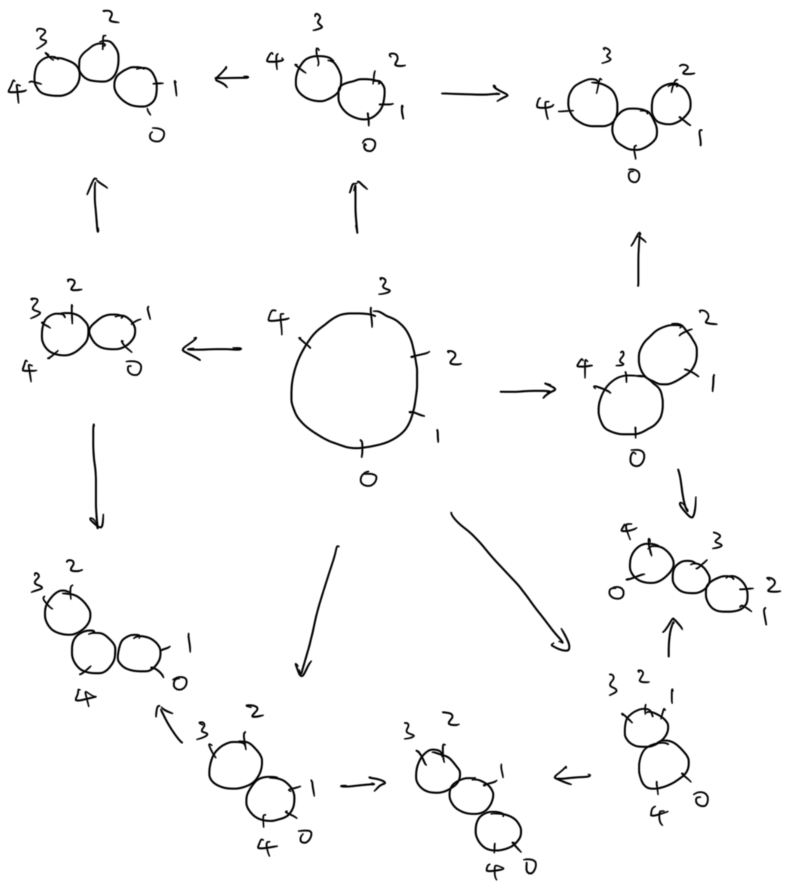
- $u$ is pseudoholomorphic on $D^\circ$,
- $\lim_{w \to w_k} u(w) = x_k$,
- $u$ maps boundary arc between $w_i$ and $w_{i+1}$ to $L_i$.

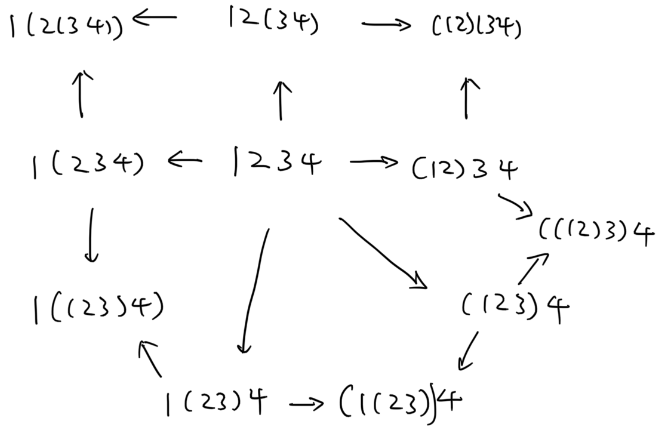
We want to count the number of such polygons (so such pseudoholomorphic maps modulo $\mathrm{PSL}(2, \mathbb{R})$), or more generally construct the moduli space of these maps. In case of good boundary conditions, we get a nice finite dimensional moduli space.

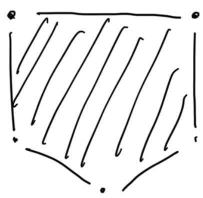
We assume that we do obtain a good moduli space, and not only that, it is zero-dimensional so the count is simply the number of components. Finally we disregard all orientation issue by counting modulo 2.

Let's take a minute to look at the space $\mathcal{R}^{n+1}$ given by disks with $n+1$ boundary points modulo $\mathrm{Aut}(D)$. $\mathcal{R}^3$ is a point since $\mathrm{Aut}(D)$ acts simply transitively: for any $\{w_0, w_1, w_2\}$ there is a unique automorphism taking them to any prescribed configuration. $\mathcal{R}^4$ is an open interval, since once we fixed $w_0, w_1, w_2$ then $w_3$ is free to lie anywhere between $w_2$ and $w_0$. More generally $\mathcal{R}^n \cong B^{n-3}$ for $n \geq 3$. On the other hand $\mathcal{R}^2 = [*/\mathbb{R}]$.

**degeneration of domain** We denote by $\overline{\mathcal{R}^{n+1}}$ the stable curve compactification of $\mathcal{R}^{n+1}$. When two points come together, a new component is created. Note that the new configuration of 4 boundary points is rigid so $\overline{\mathcal{R}^4}$ is the closed interval.

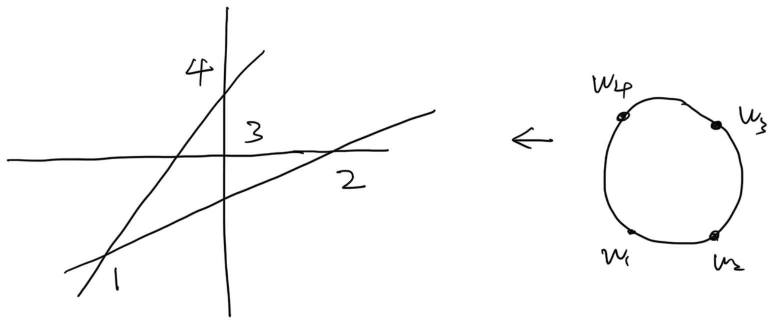


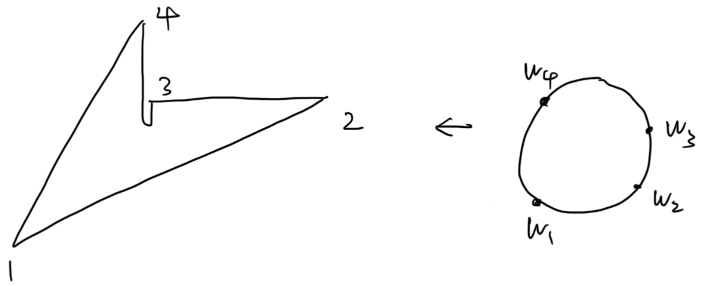


It turns out that the polytope $\overline{\mathcal{R}^{n+1}}$ has been studied before, under the name *Stasheff associahedron*, which arises in coherently associative structures. They can also be described as ways to parenthesising the string of numbers from 1 to $n$.

Geometrically,

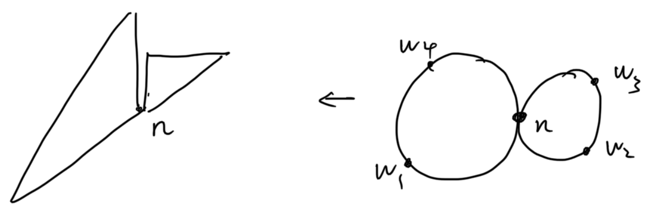



The map extends biholomorphically to the $2\pi$-angle (as the exponent is 2) so we omit it.

"$w_2$ and $w_3$ gets closer"

Pushing to the limit: a node $n$ forms




**Associativity up to homotopy** This leads to $A_\infty$-algebra and $A_\infty$-category. Recall that for $\mathbb{K}$ a field, an associative $\mathbb{K}$-algebra is a $\mathbb{K}$-vector space $A$ together with multiplication $m : A \otimes A \to A$ and identity $e : \mathcal{K} \to A$, satisfying the associativiy axiom (among others)

$$m(a, m(b, c)) = m(m(a, b), c).$$

What if $A$ is a chain complex $(A = \bigoplus_{n\in\mathbb{Z}} A^n, \mathrm{d})$? One possibility is to require $m$ to satisfy the associativity axiom and be a chain map, i.e.

$$\mathrm{d}(m(a, b)) = m(\mathrm{d}a, b) + (-1)^{|a|} m(a, \mathrm{d}b).$$

This gives a differential graded algebra. An example is the de Rham complex of a manifold together with exterior derivative and wedge product.

However dictating something to be equal goes against the principle of homotopy theory. Instead we can require $m : A \otimes A \to A$ to be a chain map of degree 0 and require the associator

$$\mathrm{Assoc} = m(m(-, -), -) - m(-, m(-, -)) : A^{\otimes 3} \to A$$

to be homotopic to 0, i.e. there eixsits $P : A^{\otimes 3} \to A[-1]$ such that

$$\mathrm{d}P + P\mathrm{d} = \mathrm{Assoc}.$$

Note that we will get a strictly associative algebra structure on the homology of $A$.

Pushing this idea further, we can define higher associator $\mathrm{Assoc}_4 : A^{\otimes 4} \to A[-1]$ by comparing different combinations of $P$ and $m$, namely the terms

$$P(m(w, x), y, z), P(w, m(x, y), z), P(w, x, m(y, z)), m(P(w, x, y), z), m(w, P(x, y, z))$$

and adding them up to a sign. Note that the five terms correspond to the five codimension one faces of the pentagon. We postulate the existence of a higher nullhomotopy $Q : A^{\otimes 4} \to A[-2]$ of $\mathrm{Assoc}_4$.

We adopt the sign convention in Seidel's paper FCPLT.

**Definition** ($A_\infty$-algebra)**.** An $A_\infty$*-algebra* consists of a graded vector space $A = \bigoplus_{n\in\mathbb{Z}} A^n$ and maps $\mu^d : A^{\otimes d} \to A[2-d]$ for $d \geq 1$ satisfying $A_\infty$-equations: for all $d \geq 1$, for all $a_1, a_2, \dots, a_d \in A$,

$$\sum_{e,i} (-1)^* \mu^{d-e+1}(a_d, \dots, a_{i+e+1}, \mu^e(a_{i+e}, \dots, a_{i+1}), a_i, \dots, a_1) = 0$$

where $* = |a_1| + |a_2| + \cdots + |a_i| - i$.

This formalises our previous discussion as follow: up to a sign $\mathrm{d} = \mu^1, m = \mu^2, P = \mu^3, Q = \mu^4$ etc.

**Definition** ($A_\infty$-category)**.** An $A_\infty$-category $\mathbf{A}$ consists of a collection of objects and for each pair of objects $X, Y$, a graded vector space $\mathrm{Hom}(X, Y)$. For each sequence $X_0, X_1, \dots, X_d$ there is a map

$$\mu^d : \mathrm{Hom}(X_{d-1}, X_d) \otimes \cdots \otimes \mathrm{Hom}(X_0, X_1) \to \mathrm{Hom}(X_0, X_d)[2-d]$$

such that the $A_\infty$-equations hold.

Returning to geometry, for the Fukaya category of a symplecticmanifold $(M, \omega)$,

$$\mu^d : CF(L_{d-1}, L_d) \otimes \cdots \otimes CF(L_0, L_1) \to CF(L_0, L_d)[2-d]$$

counts psoduoholomorphic polygons. There are a few remaining issues:

- what happens when $d = 1$?
- why do $A_\infty$-equations hold?
- clarifying degree and orientation issues.

## 2.2 Grading on Fukaya category

Recall that for $(V, \omega)$ a symplectic vector space, the Lagrangian Grassmannian $\mathrm{LGr}(V)$ is the moduli space of symplectic subspaces of $V$. It is isomorphic to $\mathrm{U}(n)/\mathrm{O}(n)$ with fundamental group $\mathbb{Z}$, witnessed by $\det^2 : \mathrm{U}(n)/\mathrm{O}(n) \to S^1$. Form the pullback with respect to the universal cover of $S^1$,

$$\begin{tikzcd} \mathrm{LGr}^{\#}(V) \arrow[r] \arrow[d] & \mathbb{R} \arrow[d] \\ \mathrm{LGr}(V) \arrow[r] & S^1 \end{tikzcd}$$

If we think of $\det^2$ as the "phase" of a Lagrangian then points in $\mathrm{LGr}^{\#}(V)$ are Lagrangian together with a lift of their phases to $\mathbb{R}$.

To globalise the construction, we assume $2c_1(M) = 0$. This implies that the existence $\alpha : \mathrm{LGr}(TM) \to S^1$ equivalent to $\det^2$ on each fibre. Now let $L \subseteq M$ be Lagrangian. The Maslov class $\mu_L \in H^1(L; \mathbb{Z})$ is the pullback of $\alpha$ along the section of $\mathrm{LGr}(TM)$ determined by $L$.

Now if $\mu_L = 0$ then there is a further lifting $\alpha_L^{\#} : L \to \mathrm{LGr}^{\#}(TM)$

$$\begin{tikzcd} & \mathrm{LGr}^{\#}(TM) \arrow[r] \arrow[d] & \mathbb{R} \arrow[d] \\ & \mathrm{LGr}^{\#}(TM) \arrow[r, "\alpha"] \arrow[d] & S^1 \\ L \arrow[r] \arrow[ur] \arrow[uur, dotted, "\alpha_L^{\#}"] & M & \end{tikzcd}$$

The pair $(L, \alpha_L^{\#})$ is called a *graded Lagrangian submanifold*.

Our aim is to give intersections of $L_1$ and $L_2$ a grading, provided $L_1, L_2$ are graded. Let $(\lambda_0, \lambda_1)$ be two paths $[0,1] \to \mathrm{LGr}(V)$. Assume they are generic in the sense that

$$\lambda_0(s) \cap \lambda_1(s) = \{0\}$$

for all but finitely many $s$. For $s$ where $\lambda_0(s)$ and $\lambda_1(s)$ are not transverse, we define the *crossing form* at $s$ to be the quadratic form on $\lambda_0(s) \cap \lambda_1(s)$

$$v \mapsto -\frac{\mathrm{d}}{\mathrm{d}r}|_{r=s} \omega(\phi_{0,r,s}(v), \phi_{1,r,s}(v))$$

where $\phi_{k,r,s} : \lambda_k(s) \to \lambda_k(r)$ is a linear isomorphism for $|r - s|$ small and $\phi_{k,s,s} = \mathrm{id}$.

Set $\mathcal{P}^{-1}\,\mathrm{LGr}(V)$ be the space of paths $\lambda : [0,1] \to \mathrm{LGr}(V)$ such that $\lambda(0) \pitchfork \lambda(1)$ and $(\lambda, \lambda(1))$ has negative definite crossing form at $s = 1$. For such a path define

$$I(\lambda) = \sum_{0<s<1} \mathrm{sgn}(q_{\lambda,\lambda_{(1)}}(s)).$$

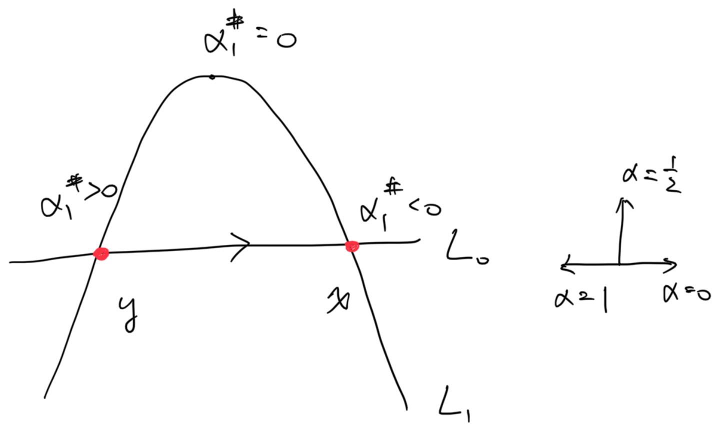
Let $\Lambda_0^\#, \Lambda_1^\# \in \mathrm{LGr}^\#(V)$ such that $\Lambda_0^\# \pitchfork \Lambda_1^\#$ (meaning the underlying subspaces are transverse).

> **Definition.** The *absolute index* $i(\Lambda_0^\#, \Lambda_1^\#) = I(\pi_0\lambda^\#)$ where $\lambda^\#(0) = \Lambda_0^\#, \lambda^\#(1) = \Lambda_1$ and $\pi_0\Lambda^\# \in \mathcal{P}^-\,\mathrm{LGr}(V)$.

**Example.** one dimensional eg

Let $L_0^\#, L_1^\#$ be two graded Lagrangian submanifolds that intersect transversally at $p$. Then we define the index at $p$ to be

$$i(p) = i(T_pL_0^\#, T_pL_1^\#).$$




grading: $\alpha_0^\# = 0$

$\alpha_1^\# \in (-1, 1)$

At $x$, $\alpha_1^\#(x) < 0$ so $i(x) = 0$

At $y$, $\alpha_1^\#(y) > 0$ so $i(y) = 1$.

**Example.**

Moral: while the index at $x$ and $y$ depends on the choice of the grading, the *difference* between them is always 1.

**Remark.** Dimension of moduli spaces of pseudoholomorphic curves with Lagrangian boundary conditions. Consider all maps $u : (\Sigma, \partial\Sigma) \to M$ such that the boundary is mapped to a given Lagrangian $L$. What should the virtual dimension of the moduli space? As $\Sigma$ is a noncompact Riemann surface, $u^*TM \to \Sigma$ can be trivialised. Then $u^*TL$ gives a map $\partial\Sigma \to \mathbb{C}^n$, giving a loop in $\mathrm{LGr}(\mathbb{C}^n)$. By Riemann-Roch type theorem

$$vdimM_u = n\chi(\Sigma) + \mu.$$

Moral: without the assumption $2c_1(M) = 0$ the dimension may depend on factors other than the Lagrangian.

Now assume $2c_1(M) = 0$ and $L_0^\#, \dots, L_d^\#$ are graded Lagrangians. Choose $y_i \in L_{i-1} \cap L_i$. Now consider the moduli space of $(d+1)$-gons with boundary conditions (complex structure on domain allowed to vary). Then it has virtual dimension

$$i(y_0) - \sum_{j=1}^{d} i(y_j) + (d-2)$$

where $d-2$ is the dimension of $\mathcal{R}^{d+1}$. Now the virtual dimension depends only on the indices of the intersections.

The operations on Fukaya category counts virtual dimension 0 polygons, so need

$$i(y_0) = \sum_{j=1}^{d} i(y_j) + (2-d).$$

In other words we obtain a degree 0 homogeneous map

$$\mu^d : CF(L_{d-1}^\#, L_d^\#) \otimes \cdots \otimes CF(L_0^\#, L_1^\#) \to CF(L_0^\#, L_d^\#)[2-d].$$

Now we address another issue: need to orient the moduli space. Pin structure in $L_i$ (easier: orientation and spin structure).

## 2.3 $A_\infty$-equations

Look at one dimensional moduli spaces, i.e.

$$i(y_0) - \sum i(y_j) + (d-2) = 1.$$

Compactify it and look at the boundary. There is a map $\overline{\mathcal{M}}_{d+1} \to \overline{\mathcal{R}}_{d+1}$ obtained by taking the domain and stabilising it.

Missed a lecture on 23/02/2022

## 2.4 Examples of FLoer cohomology and Fukaya categories

We will mostly restrict to real dimension 2 as we are able to visualise them. Consider $M = T^*S^1 = S^1 \times \mathbb{R}$ with symplectic form $\omega = \mathrm{d}q \wedge \mathrm{d}p$, where $q$ and $p$ are coordinates on $S^1$ and $\mathbb{R}$ respectively, so $\omega = \mathrm{d}\theta$ where $\theta = p\mathrm{d}q$. Any one dimensional submanifold is Lagrangian. The zero section $L$ is exact. The naive way to compute the Floer homology

$$HF(L, L) = H^*(CF(L, L), \mu^1)$$

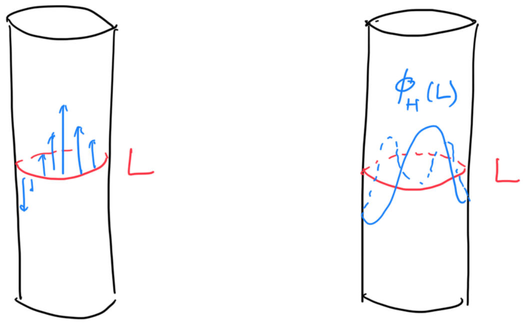
fails because $L$ is not transverse to itself. The strategy is to take two copies of $L$ and perturb one by a Hamiltonian diffeomorphism. Choose a function

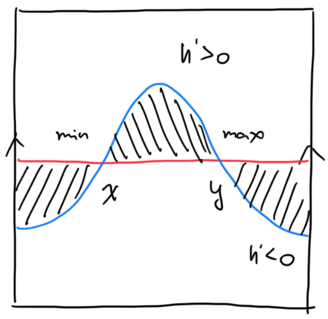
$H : M \to \mathbb{R}$ such that $H(q, p) = h(q)$ for some $h : S^1 \to \mathbb{R}$. The vector field $X_H$ is defined by

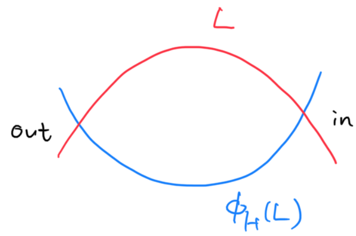
$$\omega(X_H, -) = \mathrm{d}H = h'(q)\mathrm{d}q,$$

so $X_H = h'(q)\frac{\partial}{\partial p}$. Let $\phi_H$ be the time 1-flow of $X_H$. Then $\phi_H(L) \cap L$ are the critical points of $h$, and an intersection $q$ is transverse if and only if $h''(q) \neq 0$, i.e. $q$ a nondegenerate critical point. Since we know Floer homology is invariant under Hamiltonian flow, we have reduced the problem to computing $HF(\phi_H(L), L)$.

Suppose $h$ has a unique maximum and minimum, for example the projection of a circle to a line on the same plane. Then $CF(\phi_F(L), L) = \mathbb{K}\langle x, y\rangle$.



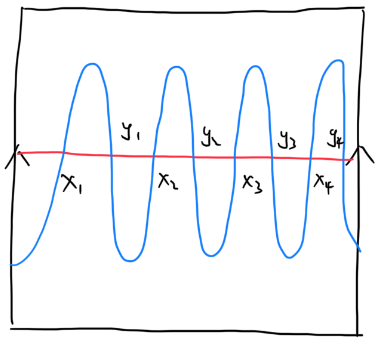

$\mu^1(x) = y$ bottom strip

$\pm y$ top strip

$\mu^1(y) = 0$

An explanation for the plus-minus sign: the cheap way is to use a field of characteristic 2. Alternatively, we can work out the orientation of the moduli space. The answer turns out to be negative. Thus $\mu^1 = 0$ so $HF(\phi_H(L), L) = \mathbb{K}(x, y)$. If we work out the degree we'll find $\deg x = 0, \deg y = 1$ so the Floer homology is $H^*(S^1; \mathbb{K})$. In fact $CF(\phi_H(L), L)$ is isomorphic to the Morse complex.

Had we chosen a different $H$, say of the form



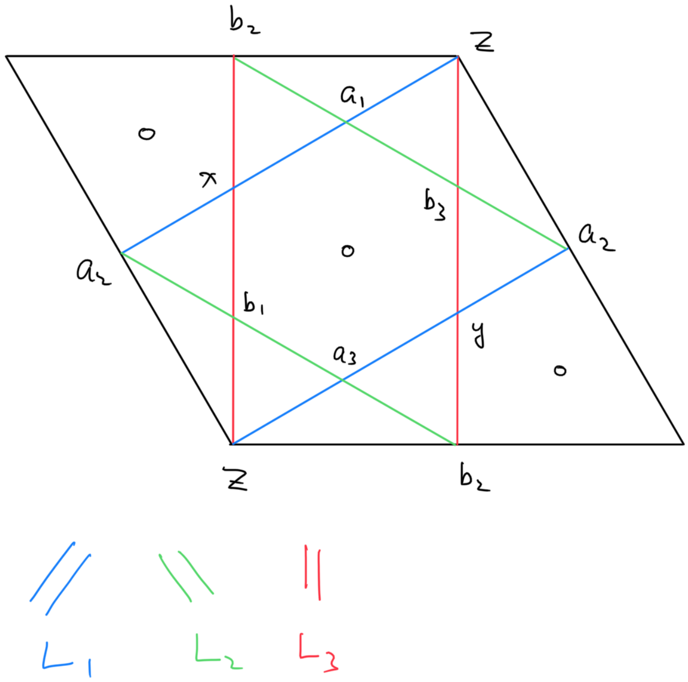

then

$$\begin{aligned}
\mu^1(x_1) &= y_1 - y_4 \\
\mu^1(x_2) &= y_2 - y_1 \\
\mu^1(x_3) &= y_3 - y_2 \\
\mu^1(x_4) &= y_4 - y_3 \\
\mu^1(y_i) &= 0
\end{aligned}$$

so

$$HF^* = \begin{cases} \langle x_1 + x_2 + x_3 + x_4 \rangle & * = 0 \\ \frac{\langle y_1, y_2, y_3, y_4 \rangle}{\langle y_{i+1} - y_i \rangle} \cong \mathbb{K}\langle [y_i] \rangle & * = 1 \end{cases}$$

Let us do another example where the cup product is nontrivial. Take $M$ to be the torus minus three points.




$CF(L_1, L_2) = \langle a_1, a_2, a_3 \rangle$ $\quad$ $\mu^1 = 0$ on all three.

$CF(L_2, L_3) = \langle b_1, b_2, b_3 \rangle$ $\quad$ Possible to grade s.t.

$CF(L_1, L_3) = \langle x, y, z \rangle$ $\quad$ all morphisms have deg 0

Up to a sign,

$\mu^2(b_1, a_1) = 0 \qquad \mu^2(b_2, a_1) = x \qquad \mu^2(b_3, a_1) = z$

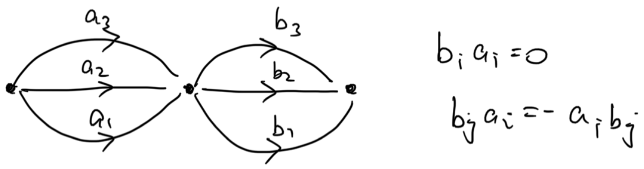
$\mu^2(b_1, a_2) = x \qquad \mu^2(b_2, a_2) = 0 \qquad \mu^2(b_3, a_2) = y$

$\mu^2(b_1, a_3) = z \qquad \mu^2(b_2, a_3) = y \qquad \mu^2(b_3, a_3) = 0$

Define the directed Fukaya category of $(L_1, L_2, L_3)$, $\mathbf{A}$, to have objects $L_i$ and

$$\mathrm{Hom}(L_i, L_j) = \begin{cases} \mathbb{K}\,\mathrm{id}_{L_i} & i = j \\ HF(L_i, L_j) & i < j \\ 0 & i > j \end{cases}$$

and composition given by $\mu^2$. This can be represented by a quiver with relation (diagram). Beilinson quiver for $\mathbb{P}^2_{\mathbb{K}}$. Belinson's theorem: for $A$ the path algebra of the following quiver with realtion,



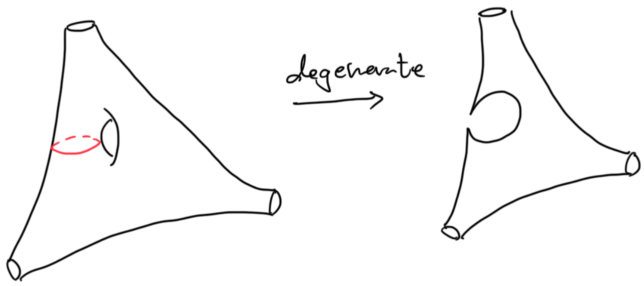

$$D^b(fgA - mod) \cong D^b(\mathrm{Coh}(\mathbb{P}^2_{\mathbb{K}})).$$

In fact there is an embedding

$$\begin{aligned} \mathbf{A} &\hookrightarrow \mathrm{Coh}\,\mathbb{P}^2_{\mathbb{K}} \\ L_1 &\mapsto \mathcal{O} \\ L_2 &\mapsto T \otimes \mathcal{O}(-1) \\ L_3 &\mapsto \mathcal{O}(1) \end{aligned}$$

This is a mversion fo homological mirror symmetry for $\mathbb{P}^2$. c.f. Seidel, More about vanishing cycles and mutations, Auroux-Katrarkov-Orlov. **A** is called the "category of vanishing cycles" and is regarded as a presentation of Fukaya-Seidel cateogry.

In Landau-Ginzburg model, the mioor of $M$ is $\tilde{M} = (\mathbb{C}^*)^2$ with the function $W(u,v) = u + v + \frac{1}{uv}$. Then $W^{-1}(0)$ is the torus with three points removed. $W$ defined a map $\tilde{M} \to \mathbb{C}$ whose generic fibre is a torus minus three points. $W$ has critical points, which are called of Lefschetz type.




## 2.5 Generalisation to noncompact Lagrangian

Recall that Floer homology is a categorification of intersection number, which makes sense when at least one of them is compact. However, as the figure shows, if both $L_1$ and $L_2$ are noncompact then the intersection number is not homotopy invariant.

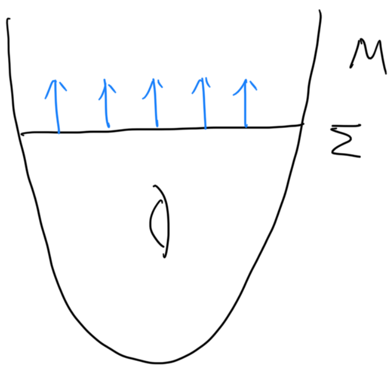
Intersection number of noncompact submfds are not homotopy invariant.

We add perturbations at infinity and we will get infinite dimensional Floer homology, so they do not come from intersection.

We have an answer for Liouville manifolds. Recall that for $(M, \omega)$ a symplectic manifold, a Liouville form is $\theta$ such that $\mathrm{d}\theta = \omega$. Define a vector field $Z$ by $\iota_Z \omega = \theta$. Then

$$\mathcal{L}_Z \omega = \mathrm{d}\iota_Z \omega + \iota_Z \mathrm{d}\omega = \mathrm{d}\theta = \omega,$$

i.e. $\omega$ is invariant under the flow of $Z$. Thus $\exp(tZ)_* \omega = e^t \omega$. In a *Liouville manifold* we require the flow of $Z$ to be complete. We also require the existence of a hypersurface $\Sigma$ of $M$ on which $Z$ points outward. Assume $Z \pitchfork \Sigma$. Let $\alpha = \theta|_\Sigma \in \Omega^1(\Sigma)$.




**Lemma 2.3.** *$\alpha$ is a contact 1-form on $\Sigma$, i.e. $\alpha \wedge (\mathrm{d}\alpha)^{n-1} > 0$.*

*Proof.* We know $\omega^n > 0$ on $M$. As $Z$ is transverse to $\Sigma$, $\iota_Z(\omega^n) > 0$ on $\Sigma$. As $\iota_Z$ is a derivation of degree $-1$,

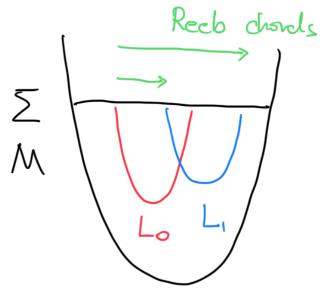
$$\iota_Z(\omega^n) = n(\iota_Z \omega) \wedge \omega^{n-1} = n\theta \wedge (\mathrm{d}\theta)^{n-1} > 0.$$

□

On the contact manifold $(\Sigma, \alpha)$, there is a Reeb vector field $R$ satisfying $\alpha(R) = 1, \iota_R \mathrm{d}\alpha = 0$. Inside $T\Sigma$ there is a distribution $\xi = \ker \alpha$ and $\mathrm{d}\alpha|_\xi$ is symplectic. We can consider the Reeb flow generated by $R$.

**Example.** Let $M = T^*Q, \theta = \sum p_i \mathrm{d}q_i, Z = \sum p_i \frac{\partial}{\partial p_i}$. Let $\Sigma$ be the unit sphere bundle in $T^*M$ with respect to a chosen metric $g$ on $Q$. Can easily check $Z$ is transverse to $\Sigma$. Then $R$ is the geodesic flow vector field on $\Sigma$.

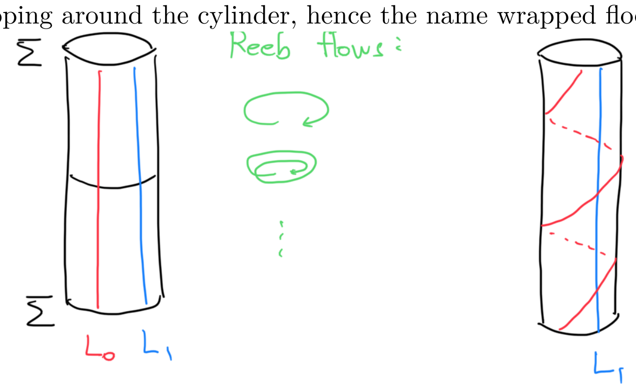
Wrapper Floer cohomology: generated by intersection points $L_0 \cap L_1$ and Reeb chords starting on $L_0$ and ending on $L_1$ (think of Reeb chords as intersections at infinity).




Choose a Hamiltonian $H$ "quadratic at infinity" and define

$$HW(L_0, L_1) = HF(\phi_H(L_0), L_1).$$

**Example.** There is no internal intersection point but infinitely many Reeb flows. Equivalently, one may visualise the Reeb flows dragging $L_0$ around and wrapping around the cylinder, hence the name wrapped floer cohomology.




> **Theorem 2.4** (Abouzaid)**.** *Let $Q$ be a closed connected manifold with $w_2(Q) = 0$. Let $q = Q$ and $L = T^*_q Q$. Then*
>
> $$CW^*(L, L) \cong C_{-*}(\Omega_q Q).$$

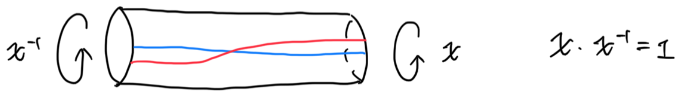
The Reeb chords from $L$ to $L$ are geodesics starting at $q$ and ending at $q$, which are generators of LHS. On the loop space there is a functional which measures the energy of the loop. By Morse theory, the homology of the loops space is encoded in the critical points of the functional, which are closed geodesics. Note that as loop space has a product given by composition, this gives wrapped cochain a product.

$L = T_q^*Q$ generates the wrapped Fukaya category, and $W(T^*Q)$ can be identified as the category of $C_{-*}(\Omega_q Q)$-modules, which is the same as local systems on $Q$.

**Example.** For $Q = S^1$ and $L$ a cotangent fibre as above,

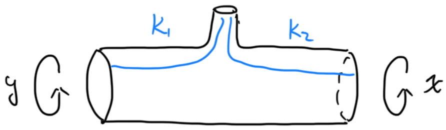
$$CW^*(L,L) = C_{-*}(\Omega S^1) = C_{-*}(\mathbb{Z}) = \mathbb{K}[\mathbb{Z}] = \mathbb{K}[x, x^{-1}]$$

One can verify that the product is the same as that given by "triangle composition" defined previously.




$W(T^*S^1)$ is the same as $\mathbb{K}[x, x^{-1}]$-module, i.e. $\mathrm{Coh}(\mathbb{A}^1 \setminus \{0\})$. This can be regarded as a mirror symmetry statement. Note that since $\mathbb{A}^1 \setminus \{0\}$ is non-proper, the Hom space of sheaves might be infinite dimensional, which is the original aim of generalising Floer homology to Lagrangians with infinitely many intersections.

**Example** (pair of pants)**.** If we do the computation we will find the same underlying vector space but a different product structure: $x$ and $y$ alone behave like polynomials but $xy = 0$. The triangle evincing $x \cdot x^{-1} = 1$ is no longer present in the pair of pants due to the puncture.




From the viewpoint of mirror symmetry, this suggest that the pair of pants is related to $\{xy = 0\} \subseteq \mathbb{A}^1$. In fact $W(P) \cong \mathrm{Coh}(\{xy = 0\})$. Let $L \in W(P)$ be the object corresponding to the structure sheaf. $K_1$ and $K_2$ corespondes the the structure sheaf of the two components.

# Index