Scuola Internazionale
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Geometry and Mathematical Physics

Log Calabi-Yau Geometry

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Contents

1 Introduction

Let us introduce the main subject of study of this course and some reasons why we study them.

Definition (Calabi-Yau manifold). A *Calabi-Yau manifold* X is a comapct connected complex manifold which is Kähler and whose canonical bundle $K_X = \bigwedge^n T_X^*$ is trivial.

Recall that $T_X = T_X^{1,0}$, the holomorphic tangent bundle.

Remark. Local sections of K_X are holomorphic forms of maximal degree, i.e. with respect to local coordinates $z_1, \dots, z_n$, they have the form

$$f(z_1, \dots, z_n)\mathrm{d}z_1 \wedge \dots \wedge \mathrm{d}z_n$$

where f is holomorphic. Thus X is CY if and only if there exists a global nowhere vanishing holomorphic n-form.

Remark. Recall that X being Kähler means that there exists a real nondegenerate closed 2-form ω of type $(1,1)$. One can think of them as being the same as projective, but has the advantage that being projective is not stable under deformation but being Kähler is.

motivation from differential geometry

Theorem 1.1 (Yau). *Let X be CY. Then for every Kähler class $\alpha \in H^2(X;\mathbb{R})$, i.e. every class that can be represented by a Kähler form ω, there exists a unique Kähler form ω_{RF} in the same class, such that the Ricci curvature of the Riemannian metric defined by ω_{RF} vanishes identically.*

Recall that given a Kähler form ω, the contraction with J, $\omega(\cdot, J\cdot)$ defines a Riemannian metric.

Remark. In fact the hypothesis can be weakened: it is enough that K_X is trivial as a C^∞-bundle, i.e. $c_1(X) = 0$.

motivation from deformation theory

Theorem 1.2 (Bogomolov-Tian-Todorov). *A CY manifold X has unobstructed deformations.*

Recall that any compact complex manifold X admits a *versal deformation space* S: there exists a family $p : \mathfrak{X} \to S \ni 0$ such that $\mathfrak{X}_0 \cong X$ and any other local family $\pi : \mathfrak{Y} \to T \ni 0$, $\mathfrak{Y}_0 \cong X$ is pulled back from p via $F : T \to S$. The map F is in general not unique (hence the name “versal” instead of “universal”), but dF_0 is unique.

Note that in general S is only a (very singular) complex analytic space. We say X has *unobstructed deformations* if S is a complex manifold.

One way to prove BTT is using the Ricci flat metric.

Deformation theory gives a description of the local properties of the moduli spaces of complex structures around a given one. On the other hand for global properties, we have

Theorem 1.3 (Schumacher)**.** *There exists a Hausdorff (i.e. separated) complex analytic space $\mathcal{M}$ such that $\mathcal{M}$ is a coarse moduli space parameterising pairs (X, α), where X is CY and $\alpha \in H^2(X)$ is Kähler.*

Again this is a result that can be proven using the Ricci flat metric.

motivation from enumerative geometry Let X be a smooth projective manifold. Fix $\beta \in H_2(X; \mathbb{Z})$ called "degree" and $g \in \mathbb{N}$ called "genus". Let $C \subseteq X$ be a smooth curve of genus g such that $[C] = \beta$. There is a scheme $\mathcal{C}$ which parameterises smooth degree β genus g curves lying on X. Then if $\mathcal{C}$ is smooth at the point corresponding to C then

$$\dim_c \mathcal{C} = \int_C c_1(X) + (\dim_{\mathbb{C}} X - 3)(1 - g).$$

In case X is CY, the first term on RHS vanishes. If furthermore $\dim X = 3$, the dimension is 0 so $\mathcal{C}$ is a collection of points. In reality $\mathcal{C}$ is not always smooth but the formula works if we replace LHS by the virtual dimension.

global Torelli theorem

Theorem 1.4. *Suppose X, X' are CY complex surfaces, i.e. K3 surface. Then $X \cong X'$ if and only if there exists a Hodge isometry $H^2(X; \mathbb{Z}) \cong H^2(X'; \mathbb{Z})$.*

mirror symmetry A version of mirror symmetry conjecture says that give X^n CY, there exists X° CY of the same dimension such that (open and closed) A and B-models are exchanged. In particular

$$h^{p,q}(X) = h^{n-p,q}(X^\circ).$$

Moreover there should be an effective way to construct X° from X.

Here "closed" is sometimes called classical mirror symmetry, and "open" called (the more modern) homological mirror symmetry. The word "effective" points us to the Gross-Siebert programme based on the conjectures of Strominger-Yau-Zaslow.

Remark. In one special case for us, X° is obtained from the versal family of X.

2 Log Calabi-Yau surface

Definition (Looijenga pair)**.** A *Looijenga pair* (Y, D) consists of a compact smooth complex surface Y and an anticanonical divisor $D \subseteq Y$, i.e. $D \in |-K_Y|$, such that D is singular nodal reduced (no multiplicity).

Example. Let $Y = \mathbb{P}^2$, $E \subseteq Y$ a smooth cubic curve. Then $E \in |-K_Y|$ since $K_Y = \mathcal{O}_{\mathbb{P}^2}(-3)$ (to prove this, consider the meromorphic 2-form $\Omega = \frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y}$ where $x = \frac{X_1}{X_0}, y = \frac{X_2}{X_0}$ with simple poles at $\{X_1 = 0\}, \{X_2 = 0\}, \{X_0 = 0\}$. Thus $K_Y = \mathcal{O}_{\mathbb{P}^2}(-L_0 - L_2 - L_3)$). However E is not singular, so (Y, E) is not a Looijenga pair. We can take its degeneration, for example $D = L_0 + L_1 + L_2$, or a conic plus a line, or a nodal toric, and obtain a Looijenga pair (Y, D).

The complement $U = Y \setminus D$ is noncompact CY: suppose $D = \mathrm{div}(s_d)$ where $s_D \in H^0(Y, K_Y^{-1})$, then $\Omega = s_D^{-1}$ is a nowhere vanishing holomorphic 2-form on U and extends to Y with simples poles along D. Note

$$H^0(Y, K_Y(D)) \cong H^0(Y, K_Y - K_Y) \cong H^0(Y, \mathcal{O}_Y) \cong \mathbb{C}$$

so Ω is unique up to scalar multiplication.

Definition (log Calabi-Yau surface)**.** The pair $(U = Y \setminus D, \Omega)$ constructed from a Looijgena pair (Y, D) is called a *log Calabi-Yau surface.*

Lemma 2.1. *In a Looijenga pair (Y, D), D is either an irreducible nodal curve, or a cycle of smooth rational curves, where being cycle means that $H_1(D; \mathbb{Z}) = \mathbb{Z}$.*

Proof. Recall that the arithmetic genus of a projective scheme D (over $\mathbb{C}$) of dimension 1 is

$$p_a = 1 - \chi(\mathcal{O}_D) = 1 - h^0(\mathcal{O}_D) + h^1(\mathcal{O}_D).$$

Recall also the adjunction formula says that for D a reduced divisor on a surface Y, there is an equality

$$2p_a - 2 = D \cdot (D + K_Y),$$

where the dot on RHS is the intersection product for divisors or more generally line bundles. Since we are on a surface, this is defined to be

$$L \cdot N = \int_Y c_1(L) \smile c_1(N)$$

for line bundles L, N and hence by divisor-line bundle correspondence

$$D_1 \cdot D_2 = \int_Y c_1(\mathcal{O}(D_1)) \smile c_1(\mathcal{O}(D_2))$$

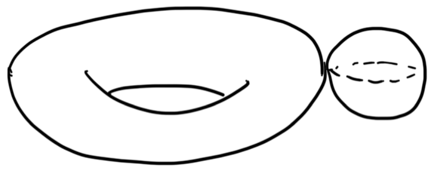
for divisors D_1, D_2.

Applying the adjunction to our case, $D + K_Y \sim 0$ so $p_a = 1$. There exists a family $f : \mathfrak{D} \to S \ni 0$ such that $\mathfrak{D}_0 \cong D$ and the generic fibre is smooth (analytic locally it is obvious that $\{xy = 0\}$ can be smoothed by $\{xy = \varepsilon\}$, then an argument generalises this globally). Arithmetic genus is constant in a flat family so

$$p_a(\mathfrak{D}_0) = p_a(\mathfrak{D}_s) = \text{topological genus of } \mathfrak{D}_s$$

so D_s is topologically a torus.

Now we turn the question around and ask how can a torus degenerate to a nodal curve. The only way is to "pinch" some S^1-generators. It can not be the the union of a cubic and and a $\mathbb{P}^1$ intersecting at a point (see diagram) as it does not have the correct arithmetic genus.



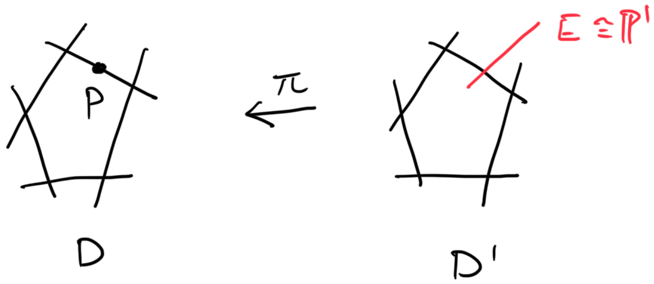
□

From now on we only consider Looijenga pairs (Y, D) where Y is projective and D is connected, as GHK do.

2.1 Blowup construction of Looijenga pairs

Let (Y, D) be a Looijenga pairs. We can get new ones by blowing up points. Let p be a smooth point of D and

$$(Y' = \mathrm{Bl}_p Y, D' = \text{proper transform of } D).$$



Suppose $D_0, \ldots, D_n$ are the components of D and $p \in D_i$, then D has component $D'_0, \ldots, D'_n$ with

$$(D'_j)^2 = \begin{cases} D_j^2 & j \neq i \\ (D_i)^2 - 1 & j = i \end{cases}$$

Lemma 2.2. *(Y', D') is a Looijenga pair, so $U' = Y' \setminus D'$ is a log CY surface.*

In general U' is not isomorphic to U.

Proof. We use the blowup formula: if $\pi : Y' \to Y$ is a blowup then

$$K_{Y'} \sim \pi^* K_Y + E$$

from which it follows

$$-K_{Y'} \sim \pi^*(-K_Y) - E \sim \pi^*(D) - E.$$

□

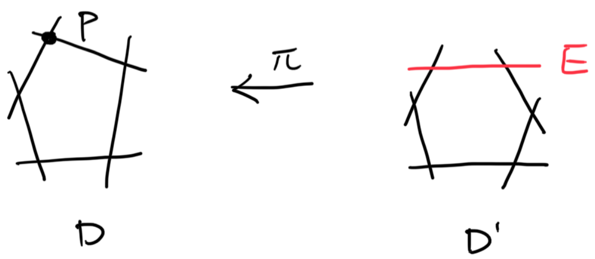
Example. The blowup $(\mathbb{P}^2, L_0 + L_1 + L_2)$ at finitely many smooth points is log CY.

Moving the position of the blowup centres changes the complex structure on the underlying smooth manifold of log CY. We will prove that roughly the positions give local coordinates on the space of complex structures on $\tilde{U}$.

In the limiting case where we blowup a node $p \in D$ (c.f. toric blowup in GHK), the statement and the proof of the lemma remain the same, but the proper transform is now

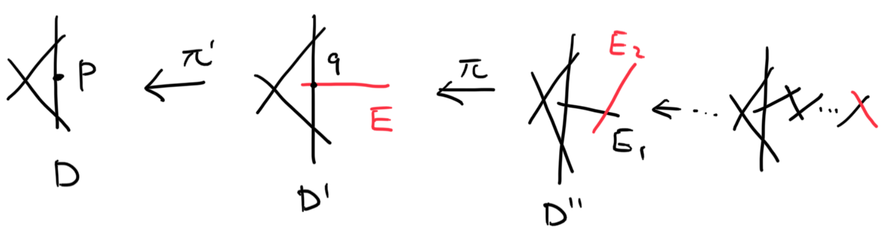
$$\pi^*(D) - E = (\pi^*(D))^{\text{red}}$$

since the multiplicity of E in $\pi^*(D)$ equals to the number of local branches at p. Moreover $\pi : Y' \setminus D' = U' \to Y \setminus D = U$ is an isomorphism, which implies that the complex structure on log CY stays the same under this operation.



On the other hand if we try to "blowup infinitely close points", formally blowup Y' at $q = D' \cap E$, we get a Looijegena pair (Y'', D''). In Y'' we have the slightly pathological rational curve E_2 which has self-intersection -2, $(E_2)^2 = -2$, and $E_2 \cap D'' = \emptyset$ (c.f. internal curves in GHK). If we keep blowing up the intersection points, we get a chain of rational curves but the self-intersection number doesn't decrease further, i.e.

$$(E_1')^2 = -1, (E_j')^2 = -2 \text{ for } j > 1.$$



These pathological cases are annoying so we want to remove them:

Definition (generic Looijenga pair)**.** A Looijenga pair (Y, D) is *generic* if there are no internal (-2)-curves.

2.2 Unobstructedness for log Calabi-Yau

Recall that Bogomolov-Tian-Todorov sas that a compact CY X is unobstructed, i.e. the versal deformation space is smooth. We would like to extend the result to noncompact log CY.

Example. In general, the deformation theory for noncompact manifolds can have pathologies. If U is smooth affine then the infinitesimal deformation vanishes: $H^1(U, T_U) = 0$ by Serre vanishing. However in general there exists nontrivial deformations of U. For example let U be a compact Riemann surface Σ with some finite number of points removed (proof: the sum of the points $\sum p_i$, regarded as a divisors, is positive. By Kodaira embedding theorem $\mathcal{O}(\sum p_i)$ is an ample line bundle. Suppose for a second it is very ample, then it defines an embedding so $\sum p_i = \Sigma \cap H$. Thus $U = \Sigma \setminus \{p_i\} \subseteq \mathbb{P}^N \setminus \{\text{hyperplane}\}$ so is affine. For the general case, exist $r > 0$ such that $\sum rp_i$ is very ample).

Remark. This works in general to show that if $D \subseteq Y$ is an ample divisor in a projective variety then $Y \setminus D$ is affine.

In general, moving the complex structure to obtain Σ' and removing the same points p_i's of the underlying smooth manifold. Then $\Sigma \setminus \{p_i\}$ and $\Sigma' \setminus \{p_i\}$ are not biholomorphic. To see this, note that smooth affine cubic is infinitesimally rigid. Perturb the coefficients of the affine cubic changes the j-invariant of the compactification. If $\Sigma \setminus \{p_i\}$ and $\Sigma' \setminus \{p_i\}$ were isomorphic, the isomorphism extends over the points so we would obtain isomorphic elliptic curves with different j-invariants.

The situation for log CY is better: there is a natural family $(\mathfrak{Y}, \mathfrak{D}) \to B$ of Looijenga pairs where $\mathfrak{Y} \to B$ is a holomorphic submersion and $\mathfrak{D} \subseteq \mathfrak{Y}$ is a divisor such that

$$(\mathfrak{Y}_b, \mathfrak{D}_b) \cong (Y_b, D_b)$$

for some Looijenga pair. We then obtain a family of log CY surfaces $(\mathfrak{U}, \Omega)$.

We consider deformation of a pair (U, Ω) where U is a surface and Ω is a nowhere vanishing holomorphic 2-form such that

- $H^1(U, \mathcal{O}_U) = 0$ (holds for example when U is affine),
- U and Ω are algebraic (always true for log CY).

Theorem 2.3 (Kaledin, Verbitsky)**.** *Deformations of* (U, Σ) *are unobstructed. Moreover the versal deformation space is a neighbourhood of* $0 \in H^2(U, \mathbb{C})$.

Theorem 2.4 (Kontseich-Katzarkov-Pantev, Iacono)**.** *Deformations of Looijenga pairs* (Y, D) *are unobstructed.*

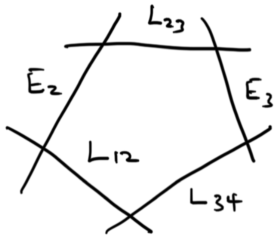
When is log CY affine? It is not always true that the complement $U = Y \backslash D$ for a Looijenga pair (Y, D) is affine.

One special important case is Y del Pezzo (i.e. $-K_Y$ ample). By previous argument U is affine if $-K_Y$ is ample.

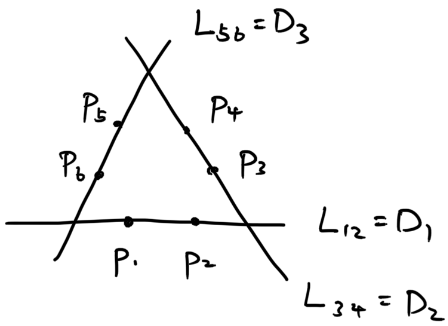
Example.

1. $(\mathbb{P}^2, L_0 + L_1 + L_2)$.
2. Let $Y = \mathrm{Bl}_{p_1,p_2,p_3,p_4}(\mathbb{P}^2)$ be a degree 5 del Pezzo. Exercise: show
$$-K_Y = \pi^*\mathcal{O}_{\mathbb{P}^2}(3) - E_1 - E_2 - E_3 - E_4$$
is ample.

   Define L_{ij} to be the proper transform of the line ℓ_{ij} through p_i and p_j in $\mathbb{P}^2$ for $i \neq j$ and set $D = L_{12} + E_2 + L_{23} + E_3 + L_{34}$. It is anticanonical:
$$\begin{aligned} D &\sim \pi^*(\ell_{12}) - E_1 - E_2 + E_2 + \pi^*(\ell_{23}) - E_2 - E_3 + E_3 + \pi^*(\ell_{34}) - E_3 - E_4 \\ &\sim \pi^*(\ell_{12} + \ell_{23} + \ell_{34}) - \sum E_i \\ &\sim \pi^*\mathcal{O}_{\mathbb{P}^2}(3) - \sum E_i \\ &\sim -K_Y \end{aligned}$$



   It is also connected nodal. We will prove that (Y, D) is rigid.
3. Cubic surface in GHK: let $Y = \mathrm{Bl}_{p_1,\ldots,p_6}(\mathbb{P}^2)$ and let L_{ij} to be the proper transform of ℓ_{ij} as before. Set
$$D = L_{12} + L_{34} + L_{56},$$
"triangle of lines on cubic surface". Importantly (Y, D) is not rigid.



General criterion for affineness:

Theorem 2.5. *$U = Y \setminus D$ is affine if and only if (Y, D) is generic and $M_{ij} = [D_i \cdot D_j]$ is not a negative semidefinite matrix. We call this the* positive case.

Example.

1. If Y is not generic then there exists $E \subseteq Y$ such that $E \cong \mathbb{P}^1$ and $E \cap D = \emptyset$, so $\mathbb{P}^1 \subseteq U$.

2. On the degree 5 del Pezzo surface. Note that it is a cycle of (-1)-curves as
$$(L_{ij})^2 = (\pi^*(\ell_{ij}) - E_i - E_j)^2 = -1.$$
Then
$$M = \begin{pmatrix} -1 & 1 & 0 & 0 & 1 \\ 1 & -1 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 & 0 \\ 0 & 0 & 1 & -1 & 1 \\ 1 & 0 & 0 & 1 & -1 \end{pmatrix} \sim \begin{pmatrix} 1 & 0 \\ 0 & < 0 \end{pmatrix}$$
so from the general criterion $Y \setminus D$ is affine.

3. The matrix cubic surfaces is negative definite. By a version of Castelnuovo criterion, there exists a morphism $f : Y \to \widetilde{Y}$, $f(D) = p$ which is an isomorphism outside D. Here $\widetilde{Y}$ is projective and p is a cusp singularity. Then $Y \setminus D$ is the complement of a point on a projective surface $\widetilde{Y} \subseteq \mathbb{P}^N$. A generic hyperplane misses the singularity so U contains a projective surface so cannot be affine.

4. Consider the Looijgena pairs coming from rational elliptic surfaces. A rational elliptic surface is a smooth rational surface Y with a map $f : Y \to \mathbb{P}^1$ whose generic fibre is a smooth curve of genus 1. To show that they exist, choose two smooth cubics $E_1, E_2 \subseteq \mathbb{P}^2$. Their pencil, i.e. span in $H^0(\mathbb{P}^2, \mathcal{O}_{\mathbb{P}^2}(3))$ defines a rational map
$$\begin{aligned} \tilde{f} : \mathbb{P}^2 &- - > \mathbb{P}^1 \\ y &\mapsto [s_0(y) : s_1(y)] \end{aligned}$$
outside $E_1 \cap E_2$, which consists of nine points if we choose the cubics generically. The map can be resolved to get a morphism $f : \mathrm{Bl}_{p_1,\ldots,p_9} \mathbb{P}^2 \to \mathbb{P}^1$. By construction the generic fibre is a smooth elliptic curve, the proper transform of a smooth element E in the pencil, which is anticanonical by blowup formula:
$$\pi^*(E) - \sum D_i \sim -K_{\mathrm{Bl}_{p_1,\ldots,p_9} \mathbb{P}^2}.$$
The singular fibres are singular anticanonical nodal curves for generic E_1 and E_2. Then $(\mathrm{Bl}_{E_1 \cap E_2} \mathbb{P}^2, \text{singular fibre})$ is a Looijenga pair. The complement cannot be affine (because it contains an elliptic curve) and One can show M is negative semidefinite but not negative definite.

Exercise. Find a toric model for del Pezzo surface (hint: try to run the proof for existence of toric model).

3 Torelli theorem

Conjectured by R. Friedman and proved by GHK.

Let (Y, D) be a Looijenga pair. An *orientation* of D is a choice of generator for $H_1(D, \mathbb{Z})$. This is the same as a choice of cyclic order for the components $D_1, \dots, D_n$ of D.

An isomorphism $f : (Y, D) \to (Y', D')$ is an isomorphism $f : Y \to Y'$ such that $f(D_i) = D_i'$ and cyclic order is respected. This means D and D' are abstractly the same curve and f acts by cyclic permutation. With this in mind, sometimes we also use notations like (Y_1, D) and (Y_2, D), where the two divisors are isomorphic abstractly.

Recall that for a compact CY surface X, its isomorphism type is determined by $H^2(X; \mathbb{C})$ together with

- its decomposition $H^2(X; \mathbb{C}) \cong H^{2,0}(X) \oplus H^{1,1}(X) \oplus H^{0,2}(X)$, a weight 2 Hodge structure,
- and the intersection form.

What about our noncompact CY surface $U = Y \setminus D$? One possiblity is to work out the analogue of Hodge structure of noncompact manifolds, which is very complicated. Friedman came up with a different idea. He replaced $H^2(X; \mathbb{C})$ with a sublattice inside $\mathrm{Pic}(Y)$.

Definition (period point)**.** Given a Looijenga pair (Y, D), define

$$D^\perp = \{L \in \mathrm{Pic}(Y) : \int_{D_i} c_1(L|_{D_i}) = L \cdot D_i = 0 \text{ for all } i\}.$$

The *period point* of (Y, D) is the element of $\mathrm{Hom}(D^\perp, \mathrm{Pic}^0(D))$ given by $L \mapsto L_{D_i}$.

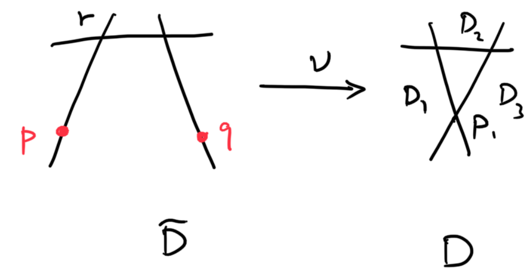
Remark. $\mathrm{Pic}(Y), \mathrm{Pic}(D)$ and $\mathrm{Pic}^0(D)$ are algebraic groups. In particular we will prove $\mathrm{Pic}(Y)$ is a free abelian group of finite rank. For example

$$\mathrm{Pic}(\mathbb{P}^2) = \mathbb{Z}\langle \mathcal{O}_{\mathbb{P}^2}(1)\rangle$$
$$\mathrm{Pic}(\mathrm{Bl}_p Y) \cong \mathrm{Pic}(Y) \oplus \mathbb{Z}\langle \mathcal{O}_{\mathrm{Bl}_p Y}(E)\rangle$$

so for example the Picard group of the degree 5 del Pezzo is free of rank 5. On the other hand

Lemma 3.1. *$\mathrm{Pic}^0(D)$ is isomorphic to $\mathbb{C}^*$ as an algebraic group.*

Proof. Consider $\nu : \widetilde{D} \to D$, the partial normalisation at a node $p_1 \in D$ with p, q the fibres over p_1.



There is a map $\nu^* : \mathrm{Pic}^0(D) \to \mathrm{Pic}^0(\widetilde{D})$. Claim there is an exact sequence

$$0 \longrightarrow \mathbb{C}^* \xrightarrow{\eta} \mathrm{Pic}^0(D) \xrightarrow{\nu^*} \mathrm{Pic}^0(\widetilde{D}) \longrightarrow 0$$

Claim $\mathrm{Pic}^0(\widetilde{D})$ is trivial: given $L \in \mathrm{Pic}^0(\widetilde{D})$ then $L \cdot \widetilde{D}_i = 0$ (restriction or intersection?). But each component of $\widetilde{D}_i$ is rational, so $L|_{\widetilde{D}_i}$ is trivial. As $\widetilde{D}$ is a chain of $\mathbb{P}^1$, L is trivial.

Claim for each $\lambda \in \mathbb{C}^*$, there exists a rational function f_λ on $\widetilde{D}$ such that $f_\lambda(p) = \lambda, f_\lambda(q) = 1$. Assuming this, note $p, q \notin \mathrm{div}(f_\lambda)$, so descends to a line bundle on $\mathrm{Pic}^0(D)$.

Exercise. Show the sequence is exact.

Intuitively, this says that the only obstruction for an element of $\mathrm{Pic}^0(D)$ to be trivial is the ratio of the value at p and q.

Existence of f_λ: choose $f_\lambda|_{D_1}$ to be of the form $\frac{az+b}{cz+d}$ for suitable coefficients such that $f_\lambda(p) = \lambda, f_\lambda(r) = 1$ extend as a constant on other components. □

Remark. What people usually mean by the word "period" for Ω a holomorphic volume form on a compact surface X (?) are the integrals

$$\int_C \Omega$$

for C a real cycle on X. Torelli's theorem says that the additional structures on $H^2(X;\mathbb{C})$ is equivalent to the *period map*

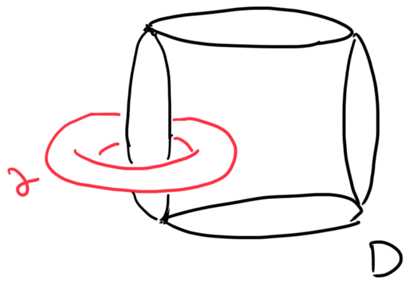
$$\int_{(-)} \Omega : H_2(X;\mathbb{Z}) \to \mathbb{C}.$$

See Huybrechts, Lectures on K3 surfaces.

In our noncompact case there is an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow H_2(U;\mathbb{Z}) \longrightarrow D^\perp \longrightarrow 0$$

where $\mathbb{Z}$ is generated by γ, the class of some 2-torus on Y (see diagram).



The third map is by the identification

$$D^\perp \subseteq \mathrm{Pic}(Y) \subseteq H^2(Y;\mathbb{Z}) \cong H_2(Y;\mathbb{Z})$$

where the second inclusion is by the first Chern class and the isomorphism is by Poincaré duality. The claim of GHK is that the period point $\phi_{(Y,D)} \in \mathrm{Hom}(D^\perp, \mathrm{Pic}^0(D))$ is equivalent to the map

$$\int_{(-)} \Omega : H_2(U;\mathbb{Z}) \to \mathbb{C}$$

with normalisation $\int_\gamma \Omega = 1$.

A Torellis type theorem is one of the form "periods determine isomorphism types".

Theorem 3.2 (Torelli theorem for Looijenga pairs). *Let (Y_1, D) and (Y_2, D) be Looijenga pairs. Suppose there exists $\mu : \mathrm{Pic}(Y_1) \xrightarrow{\cong} \mathrm{Pic}(Y_2)$ isomorphism of free abelian groups preserving intersection pairings, such that*

1. *$\mu([D_i]) = [D_i]$ for all i,*
2. *$\phi_{Y_2} \circ \mu = \phi_{Y_1}$,*
3. *$\mu(\Delta_{Y_1}) = \Delta_{Y_2}$, where Δ_{Y_i} is the set of classes of internal (-2)-curves,*
4. *for certain subcones (containing Kähler cones), we have $\mu(C_1^{++}) = C_2^{++}$,*

then $\mu = f^*$ for some isomorphism $f : (Y_2, D) \to (Y_1, D)$. In fact the converse is also true.

Definition. Given a Looijenga pair (Y, D), the cone $C^+ \subseteq \mathrm{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{R}$ is defined to be the connected component of $\{x : x^2 > 0\}$ containing Kähler classes (equivalently, ample classes).

Define

$$\widetilde{M} = \{E \in \mathrm{Pic}(Y) : E^2 = -1, K_Y \cdot E = -1, E \cdot H > 0 \text{ for some ample } H\}$$

and define

$$C^{++} = \{x \in C^+ : x \cdot E \geq 0 \text{ for } E \in \widetilde{M}\}.$$

For example in the blowup construction of Looijenga pairs the class of the exceptional curve is an element of $\widetilde{M}$.

Exercise. Compute the "number of moduli" for degree 5 del Pezzo (which we claim to be 0) and the cubic surface.

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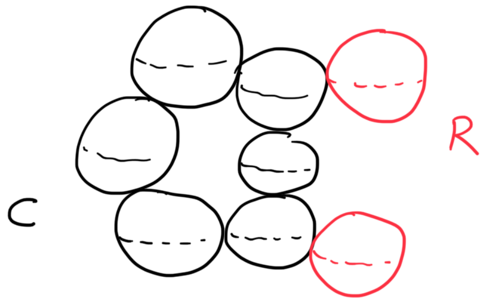


Correction (to prove D is a cycle of rational smooth components): we could have $D = C + R$ where C has arithmetic genus 1 and R is a union of rational curves $p_a(R) = 0$. But we can rule it out by adjunction:

$$2p_a(C) - 2 = C \cdot (C + K).$$

By assumption $C + R \sim -K$ so

$$0 = C \cdot (C - C - R) = -C \cdot R < 0$$

if $R \neq \emptyset$, absurd.

Example.

1. Degree 5 del Pezzo: consider the period point $\phi_{(Y,D)} \in \mathrm{Hom}(D^\perp, \mathbb{C}^*$. To compute $D^\perp$, note that $\mathrm{Pic}(Y) = \mathbb{Z}H \oplus \bigoplus_{i=1}^4 \mathbb{Z}E_i$ has rank 5. For any proper transform of line L_{ij},

   $$L \cdot (L_{ij}) = L \cdot (H - E_i - E_j), L \cdot E_k = -e_k.$$

   After some linear algebra $D^\perp = 0$ so as far as periods are concerned (i.e. modulo the technical conditions) that all degree 5 del Pezzo surfaces with this boundary are isomorphic.

2. Cubic surface: $\mathrm{Pic}(Y) = \mathbb{Z}H \oplus \bigoplus_{i=1}^6 \mathbb{Z}E_i$. Same calculation shows that $D^\perp$ has rank 4. Naïvely one expects that the versal deformation space is 4 dimensional.

   Check that there is a 4 dimensional space of periods achieved by cubic surfaces. The cubic surface (Y, D) is obtained by the blowup construction applied to $(\mathbb{P}^2, \ell_1 + \ell_2 + \ell_3)$. From this description, if we have an isomorphism of Looijenga pairs of cubic surfaces $(Y, D) \to (Y', D')$ then there exists an automorphism φ of $\mathbb{P}^2$ preserving the three lines such that $\varphi(p_i) = p_i'$. Thus we have a family of complex structure on (Y, D) of dimension is

   $$\#\{p_i\} - \dim(\text{automorphism preserving three lines}) = 6 - 2 = 4.$$

We can also check directly that a 4 dimensional family of periods is realised. We have some classes in $D^\perp$ given by $\{H - E_1 - E_i - E_j\}$ where $i \in \{3,4\}, j \in \{5,6\}$. As an exercise, these classes give 4 independent period points, so there is a 4 dimensional family of periods points inside $(\mathbb{C}^*)^4$.

Remark. Compare this with the result of Kaledin-Verbitsky: if (Y, Ω) is affine algebraic holomorphic symplectic then there is a smooth versal family given by open neighbourhoods of $[\Omega] \in H^2_{\mathrm{dR}}(U, \mathbb{C})$.

We have an exact sequence

$$0 \longrightarrow \mathbb{Z} \longrightarrow H_2(U, \mathbb{Z}) \longrightarrow D^\perp \longrightarrow 0$$

The dimensions of the versal family of (Y, D) (from Torelli theorem of GHK) and (U, Ω) (from KV) match.

Basic ingredients for Torelli

Theorem 3.3. *Let (Y, D) be a Looijenga pair. Then exists a toric blowup (Y', D') of (Y, D) such that (Y', D') has a toric model $\pi : (Y', D') \to (\overline{Y}, \overline{D})$. In other words it is obtained by blowup construction starting from a toric Looijenga pair $(\overline{Y}, \overline{D})$.*

Toric blowup means blowing up nodes of D, possibly several times. Recall that this induces an isomorphism of the associated log CY surfaces. By a *toric surface* $\overline{Y}$ we mean a compact smooth surface $\overline{Y}$ which admits an effective $(\mathbb{C}^*)^2$-action with a dense open orbit. A *toric structure* is a choice of such an action and dense open orbit.

Example.

1. $\mathbb{P}^2$ has a toric structure such that the dense open orbit is the complement of three lines.

2. Hirzebruch surfaces: pick $e \geq 0$. Let $\overline{Y} = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(e))$. Let F_1, F_2 be two distinct fibres of the fibration $p : \overline{Y} \to \mathbb{P}^1$ and let C_0, C_∞ be the 0-section and the section at infinity. Then

$$\overline{Y} \setminus \{C_0 + F_1 + F_2 + C_\infty\} \cong (\mathbb{C}^*)^2$$

   equivariantly.

   For example for $e = 0$, $\overline{Y} \cong \mathbb{P}^1 \cong \mathbb{P}^1$. For $e = 1$, $\overline{Y} \cong \mathrm{Bl}_p \mathbb{P}^2$ (to see it is ruled, take a line $\ell \subseteq \mathbb{P}^2$ that does not pass through p. Then we have a projection $\mathbb{P}^2 \setminus \{p\} \to \ell \cong \mathbb{P}^1$. Blowing up at p replaces p with the projectivised normal bundle, so this rational map $\mathbb{P}^2 - > \mathbb{P}^1$ is resolved to $p : \mathrm{Bl}_p \mathbb{P}^2 \to \mathbb{P}^1$ with fibres isomorphic to $\mathbb{P}^1$. As an exerice, show $\mathrm{Bl}_p \mathbb{P}^2 \cong \mathbb{P}(\mathcal{O} \oplus \mathcal{O}(1)))$.

For more on Hirzebruch surfaces see Beauville.

Theorem 3.4. *Suppose $\overline{Y}$ is a toric surface (which for us is always smooth and compact). Then it can be obtained from $\mathbb{P}^2$ or $\mathbb{F}_e$ by blowing up torus fixed points (possibly repeatedly and infinitely nearby).*

For a proof see Fulton.

Note. Given $\overline{Y}$ toric, we get the dense orbit $(\mathbb{C}^*)^2$. Set $\overline{D} = \overline{Y} \setminus U$. By toric geometry $\overline{D}$ is a union of reduced irreducible components $\overline{D}_i$ preserved by torus action and $\overline{D}_i \cong \mathbb{P}^1$. Moreover $\overline{D}$ is an anticanonical divisor. An informal way to see why $\overline{D}$ is anticanonical: $(\mathbb{C}^*)^2$ has a holomorphic 2-form $\Omega = \frac{\mathrm{d}x}{x} \wedge \frac{\mathrm{d}y}{y}$. It extends to a meromorphic section of $K_{\overline{Y}}$ with simple poles along D.

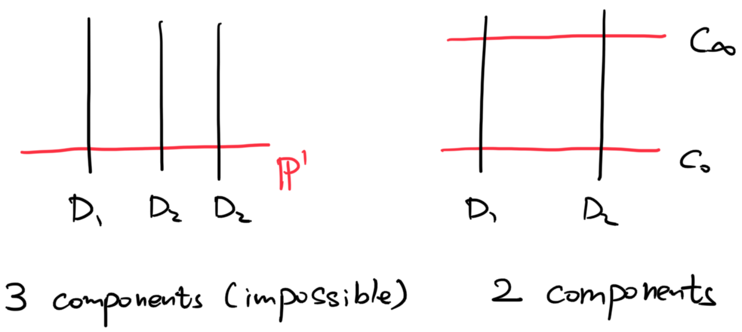
proof of obtaining (Y, D) from toric. Consider the two basic operations on a Loojienga pair (Y, D):

1. let $E \subseteq Y$ be a (-1)-curve (smooth rational), $E \nsubseteq D$. Then defined a new Looijenga pair (Y', D') where Y' is the blowdown of E and D' is the image of D.
2. replace (Y, D) with a toric blowup (Y', D').

Fact: the theorem is true for (Y, D) if and only if it is true for (Y', D') defined in 1 and 2.

Now we hit the problem with a big hammer: from classification of projective surfaces, since $-K_Y$ is effective, after a finite number of operations 1 and 2, we end up with a new Y which is either $\mathbb{P}^2$ or a Hirzebruch surface. In case $Y \cong \mathbb{P}^2$, by blowing up a node of D (i.e. operation 2) we reduce to the case of Hirzebruch surface.

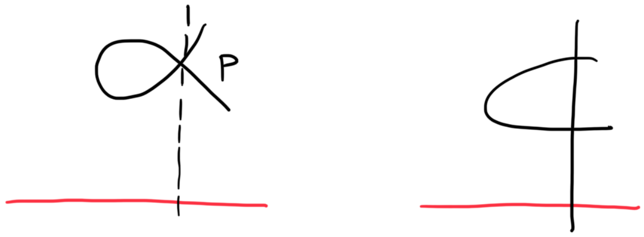
Argue on the number of components of D contained in the fibres of $q : \mathbb{F}_e \to \mathbb{P}^1$: it cannot be greater than or equal to 2 since we kow D must be a cycle.



2 components are possible. To make it into a cycle we need to have two sections of the ruling q. One checks that $D_1 + C_\infty + D_2 + C_0$ is anticanonical: we use the formula

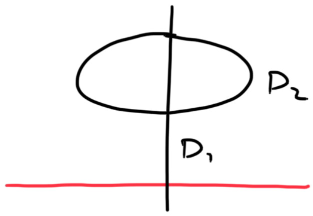
$$-K_{\mathbb{F}_e} = 2C_0 + (e+2)f$$

where f is the class of the fibre. In fact any section gives a toric structure (Y, D) on $\mathbb{F}_e$.



0 component can be transformed into 1 component

0 component: let F be the fibre through the node p. Let Y' be the blowup of the node p followed by blowdown of the the proper transform of F (note that F is a fibre so has self-intersection 0, so its proper transform has self-intersection -1) and D' the union of proper transform of D and the image of the exceptional divisor. After the transformation we will have a new Hirzebruch surface (this is not obvious) (exercise: it is $\mathbb{F}_{e'}$ where $e' = e + 1$ generically (if p does not lie on the zero section) and $e' = e - 1$ if it does) and the number of fibres in D is now 1.



1 component

Thus we are reduced to the case of 1 component. This is manifestly non-toric and the previous trick does not increase the number of fibres. What we need is to use

$$D_1 + D_2 \sim f + D_2 \sim -K_Y.$$

By the formula for the canonical class of Hirzebruch surface, $D_2 \sim 2C_0+(e+1)f$. Its intersection with the zero section is

$$C_0 \cdot D_2 = C_0 \cdot (2C_0 + (e+1)f) = -2d + (e+1) = -e + 1.$$

On the other hand $C_0 \not\subseteq D_2$ so this number must be nonnegative, which constrains e to be 0 or 1.

$e = 0$: $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. Now we know we are stuck because we are looking at the wrong ruling q. By looking at the other projection q' we are reduced to the case of 0 components.

$e = 1$: exercise. □

Remark 22/03: Since we start with some Y with $-K_Y$ effective, we can contract (-1)-curves until we reach $\mathbb{P}^2$ or $\mathbb{F}_e$ by classification theory of surfaces. There are two possibilities for the behaviour of the boundary with respect to

$\pi : (Y, D) \to (Y', D')$: either $E \nsubseteq D$ so $D' = \pi_* D$ or D' is obtained from D by contracting E. Then as an exercise one can show if there exists a toric model for (Y', D') then there exists one for (Y, D).

Secondly, we also want to go back: we might later need to blowup a node of D (i.e. toric blowup). Then the theorem holds for (Y, D) if and only if (Y', D') does.

End of remark.

Heuristics for Torelli theorem using toric model: suppose we have a toric model $\pi : (Y, D) \to (\overline{Y}, \overline{D})$ which is generic (recall that this means that there is no (-2)-internal curves or we do not blowup infinitely close points in the toric model). Suppose we have exceptional curves $\{E_{ij}\}$ mapped to D_i. By blowup formula

$$\operatorname{Pic}(Y) \cong \operatorname{Pic}(\overline{Y}) \oplus \bigoplus_{i,j} \mathbb{Z}E_{ij}.$$

We will see in a second that the Picard group of a toric variety is very well understood. In addition we have a period point $\phi_{(Y,D)} \in \operatorname{Hom}(D^\perp, \mathbb{C}^*)$ (we will give a *fixed* identification $\mathbb{C}^* \cong \operatorname{Pic}^0(D)$ to get rid of the ambiguity in the ordering of the components of D). Claim we can reconstruct (Y, D) from the abstract lattice $\operatorname{Pic}(Y)$ together with the period point: consider $[E_{ij}] - [E_{ik}] \in D^\perp$, $j \neq k$.

1. the intersection pairing on $\operatorname{Pic}(Y)$ corresponds to divisors E_{ij}, E_{ik} mapping to D_i.

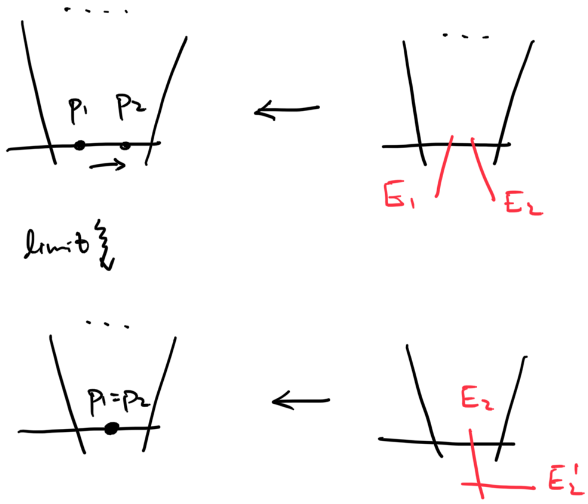
2. so the period map gives $\mathcal{O}_Y(E_{ij} - E_{ik})|_D = \mathcal{O}_D(p_{ij} - p_{ik}) = \lambda \in \operatorname{Pic}^0(D) = \mathbb{C}^*$. Note that this line bundle is supported on D_i.

3. recall the identification of $\mathbb{C}^*$ and $\operatorname{Pic}^0(D)$ via partial normalisation, we have $p_{ij} - p_{ik} = \operatorname{div}(f_\lambda)$ on $\widetilde{D}_i$. As a rational function on $\mathbb{P}^1$, f_λ has three independent parameters. We have imposed two independent conditions. Thus the divisor $\operatorname{div}(f_\lambda)$ is specified by λ up to a reference point. If we prescribe p_{ik}, the p_{ij} is uniquely determined by the period λ.

In conclusion, fixing a reference centre $p_{i\ell} \in D_i$ for each i and fixing all periods determine all other blowup centres $\{p_{ij}\}$. We will introduce the notion of *marked period points*. Finally we will remove the marking and get the weaker Torelli theorem.

Exercise. Study families of Looijenga pairs obtained by moving blowup centres of the toric model $\pi : (Y, D) \to (\overline{Y}, \overline{D})$. By moving the blowup centre p_1 while keeping the other p_i's fixed, we get a family $(\mathfrak{Y}, \mathfrak{D})^\circ \to \Delta^*$. Prove that the family extends over the full disk to $(\overline{Y}, \overline{D}) \to \Delta$ and describe the extension.

$(\mathfrak{Y}_0, \mathfrak{D}_0)$ is the blow up at all p_i for $i > 1$ and an infinitesimally close point to p_2, given by the intersection of the proper transform of D_1 with the exceptional divisor at p_2.



Proof. By taking an analytic neighbourhood suffcies to consider the case $p_1 = (-1,0), p_2 = (0,0) \in \mathbb{C}^2$ and the divisor D is given by $\{y = 0\}$. First blowup the trivial family $\mathrm{pr} : \mathbb{C}^2_{x,y} \times \Delta_t \to \Delta_t$ at p_2 at all time t to obtain

$$\mathrm{Bl}_{\{0,0,t\}}(\mathbb{C}^2 \times \Delta) \xrightarrow{\pi_1} \mathbb{C}^2 \times \Delta \xrightarrow{\mathrm{pr}} \Delta.$$

This is flat over Δ_t becuase when the base of a family $f : \mathfrak{X} \to S$ is smooth and 1-dimension and $\mathfrak{X}$ is reduced, then f is flat if and only if all irreducible components dominate the base.

Next we blowup along the proper transform of $\{(t,0,t)\}$. Again this is flat. Check that

$$(\mathrm{pr} \circ \pi_1 \circ \pi_2)^{-1}(t) = \begin{cases} \mathrm{Bl}_{(0,0),(t,0)} \mathbb{C}^2 & t \neq 0 \\ \mathrm{Bl}_q \mathrm{Bl}_{(0,0)} \mathbb{C}^2 & t = 0 \end{cases}$$

where q is the intersection of the proper transformation of D_1 with the exceptional divisor. $\square$

It follows that $\mathrm{Pic}(Y_t)$ are all isomorphic for $t \in \Delta_t$ (since they are isomorphic to $H^2(Y_t, \mathbb{Z})$ via the first Chern class, which is locally constant). In fact we know

$$\mathrm{Pic}(Y_t) = \cdots \oplus \mathbb{Z}E_1 \oplus \mathbb{Z}E_2$$
$$\mathrm{Pic}(Y_0) = \cdots \oplus \mathbb{Z}E_2 \oplus \mathbb{Z}E_2'$$

where we use the ellipses to indicate direct summands that can be identified. In fact we can choose a basis for $\mathrm{Pic}(Y_0)$ such that the intersection forms are the same: let $C_1 = E_2, C_2 = E_2 + E_2'$ so

$$C_1^2 = -1, C_1 \cdot D_i = 1, C_2^2 = -1, C_2 \cdot D_i = 1.$$

This tells us that under the parallel transport with respect to the Gauss-Manin connection, E_2 is sent to C_1 and E_1 is sent to C_2.

This generalises to more points coming together: suppose we have a sequence of exceptional divisors with self-intersection $-1, -2, -2, \ldots, -2$. Then we can choose the basis

$$C_1 = E_1, C_2 = E_1 + E_2, \ldots, C_m = E_1 + \cdots + E_m$$

which satisfy

$$(C_i)^2 = -1, C_i \cdot C_j = 1 \text{ if } |i-j| = 1, C_i \cdot D = 1.$$

These reducible curves are (confusingly) called *exceptional curves*.

Exercise. Suppose $\pi : (Y', D') \to (Y, D)$ is a toric blowup. Then $\pi^* : \mathrm{Pic}(Y) \to \mathrm{Pic}(Y')$ induces a lattice isomorphism $D^\perp \cong (D')^\perp$.

Markings A *marking* of D is a choice of smooth points $p_i \in D_i$ for all i. Giving a marking, a *marked period point* $\phi_{(Y,D,\{p_i\})} \in \mathrm{Hom}(\mathrm{Pic}(Y), \mathrm{Pic}^0(D))$ given by $L \mapsto L^{-1}|_D \otimes \mathcal{O}_D(\sum (L \cdot D_i) p_i)$.

At this point it is useful to fix isomorphisms

1. $\mathrm{Pic}^0(D) \cong \mathbb{C}^*$,
2. $\mu : \mathrm{Pic}(Y) \cong \mathrm{Pic}(Y_0)$ for some fixed Looijenga pair (Y_0, D_0).

Then we have $\phi_{(Y,D,\mu,\{p_i\})} \in \mathrm{Hom}(\mathrm{Pic}(Y), \mathbb{C}^*) \cong (\mathbb{C}^*)^m$ where m is the rank of $\mathrm{Pic}(Y_0)$.

Lemma 3.5. *Given a cyclic order of the components of D, we get a canonical identification* $\mathrm{Pic}^0(D) \cong \mathbb{C}^*$.

Proof. Fix $L \in \mathrm{Pic}^0(D)$ and let $p_{i,i+1} = D_i \cap D_{i+1}$. Choose trivialisations $\sigma_i \in H^0(D_i, L|_{D_i})$. Define

$$\lambda(L) = \prod_i \frac{\sigma_{i+1}(p_{i,i+1})}{\sigma_i(p_{i,i+1})}.$$

One checks that $\lambda(L \otimes N) = \lambda(L)\lambda(N)$ and λ does not depend on the choice of σ_i. □

Consider $\mathrm{Aut}^0(D)$, the connected component of identity in $\mathrm{Aut}(D)$.

Lemma 3.6. *Given a cyclic order, we get a canonical identification* $\mathrm{Aut}^0(D) \cong (\mathbb{C}^*)^n$.

Proof. The element $(1, 1, \ldots, \lambda, 1) \in (\mathbb{C}^*)^n$ with λ at ith place acts trivially on D_j for $j \neq i$ and sends $[x, y] \mapsto [x, \lambda y]$ on D_i. Here the homogeneous coordinates on D_i is such that $[1, 0] = p_{i-1,i}, [0, 1] = p_{i,i+1}$. □

Proposition 3.7 (global Torelli for Looijenga pairs with given toric models of the same type)**.** *Let $(Y, D) \to (\overline{Y}, \overline{D}), (Y', D') \to (\overline{Y}, \overline{D})$ be toric models of the same type, i.e. same number of exceptional curves on corresponding components of $\overline{D}$. Let $\mu : \mathrm{Pic}(Y) \cong \mathrm{Pic}(Y')$ be the unique isomorphism such that $\mu([D_i]) = [D_i]$ on boundary components and $\mu([E_{ij}]) = [E_{ij'}]$ on exceptional curves mapping to D_i. Then*

> 1. *if we fix markings $\{p_i\}, \{p'_i\}$ of D and D' respectively then exists $f : (Y, D) \to (Y', D')$ isomorphism as marked pairs so that $\mu = f^*$ if and only if the marked periods points $\tilde{\phi}, \tilde{\phi}'$ are the same, i.e. $\tilde{\phi}' \circ \mu = \tilde{\phi}$.*
>
> 2. *$\mu = f^*$ for some isomorphism as unmarked pairs if and only if the unmarked period points are the same, i.e. $\phi' \circ \mu = \phi$.*

The main tool in the proof is the following lemma

> **Lemma 3.8.** *There is a long exact sequence*
>
> $$\begin{tikzcd} 1 \arrow[r] & \ker(\operatorname{Aut}(Y,D) \to \operatorname{Aut}\operatorname{Pic}(Y)) \arrow[r] & \operatorname{Aut}^0(D) \arrow[r, "\psi"] & \operatorname{Hom}(\operatorname{Pic} Y, \operatorname{Pic}^0(D)) \arrow[dll] \\ & \operatorname{Hom}(D^\perp, \operatorname{Pic}^0(D)) \arrow[r] & 1 & \end{tikzcd}$$
>
> *Here $\psi(\alpha)(L) = L^{-1}|_D \otimes (\alpha^* L)|_D$.*

Exercise. Try to prove it for toric cases, or just for $\mathbb{P}^2$ and $\mathbb{F}_e$.

Proof. Make canonical identification $\operatorname{Aut}^0(D) \cong (\mathbb{C}^*)^n, \operatorname{Pic}^0(D) \cong \mathbb{C}^*$ so ψ is given by

$$\begin{aligned} \psi : (\mathbb{C}^*)^n &\to \operatorname{Hom}(\operatorname{Pic} Y, \mathbb{C}^*) \\ (\lambda_1, \dots, \lambda_n) &\mapsto (L \mapsto \prod_{i=1}^{n} (\lambda_i)^{\deg L|_{D_i}}) \end{aligned}$$

Given a toric model $\pi : (Y, D) \to (\overline{Y}, \overline{D})$, we have a commutative diagram with exact rows and columns:

$$\begin{tikzcd} & & 0 \arrow[d] & 0 \arrow[d] & & \\ & 0 \arrow[r] \arrow[d] & \operatorname{Pic}(\overline{Y}) \arrow[r, "i"] \arrow[d, "\pi^*"] & \bigoplus \mathbb{Z}\overline{D}_i \arrow[r] \arrow[d, equal] & N \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & D^\perp \arrow[r] \arrow[d] & \operatorname{Pic} Y \arrow[r, "i'"] \arrow[d] & \bigoplus \mathbb{Z}D_i \arrow[r] \arrow[d] & N' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \bigoplus \mathbb{Z}E_{ij} \arrow[r, equal] & \bigoplus \mathbb{Z}E_{ij} \arrow[r] \arrow[d] & 0 \arrow[r] & 0 & \\ & & 0 & & & \end{tikzcd}$$

Basic facts about toric surfaces ($\mathbb{P}^2, \mathbb{P}^1 \times \mathbb{P}^1, \mathbb{F}_2, \operatorname{Bl}\mathbb{F}_e$): because of the torus action on $\overline{Y}$, $\operatorname{Pic}\overline{Y}$ is generated by torus-invariant divisors. Thus we have injection i given by $C \mapsto \sum (C \cdot \overline{D}_i)\overline{D}_i$. N is defined to be the cokernel of i. This gives the first line. Pulling back along π gives the second line and the cokernel of π^* are the classes of exceptional divisors.

Applying $\operatorname{Hom}(-, \operatorname{Pic}^0(D))$, an exact functor since $\operatorname{Pic}^0(D) \cong \mathbb{C}^*$ to the second row, we get a long exact sequence

$$1 \to \operatorname{Hom}(N', \operatorname{Pic}^0(D)) \to \operatorname{Hom}(\bigoplus \mathbb{Z}D_i, \operatorname{Pic}^0(D)) \xrightarrow{\tilde{i}} \operatorname{Hom}(\operatorname{Pic} Y, \operatorname{Pic}^0(D)) \to \operatorname{Hom}(D^\perp, \operatorname{Pic}^0(D)) \to 1$$

Now the task is to identify $\tilde{i}$ with ψ and $\operatorname{Hom}(N', \operatorname{Pic}^0)$ with $\ker(\operatorname{Aut}(Y, D) \to \operatorname{Aut}\operatorname{Pic}(Y))$. The first is easy: $\tilde{i}$, under the canonical identification at the beginning of the proof, has exactly the same formula as ψ.

For the second, let T be the structure torus of $\overline{Y}$:

$$(\mathbb{C}^*)^2 \cong T = \operatorname{Spec}\mathbb{C}[M].$$

Then $N = \operatorname{Hom}(M, \mathbb{Z})$ (either use the classification of toric surfaces or see Fulton). There is also a canonical identification $N \cong M$ because M comes with a nondegenerate skew 2-form (i.e. determinant). The surjection $N \to N'$ thus gives $\operatorname{Hom}(N', \mathbb{C}^*) \subseteq \operatorname{Hom}(M, \mathbb{C}^*)$. The RHS can be identified with $\operatorname{Aut}(\overline{Y}, \overline{D})$.

Claim that the image of $\operatorname{Hom}(N', \mathbb{C}^*) \to \operatorname{Aut}(\overline{Y}, \overline{D})$ consists of elements which fix boundary components supporting nontrivial blowups.

(Aside: In the special case $(Y, D) = (\overline{Y}, \overline{D})$, there is no $D^{\perp}$ so we get a SES

$$1 \longrightarrow T \longrightarrow \operatorname{Aut}^0(\overline{D}) \longrightarrow \operatorname{Hom}(\operatorname{Pic}\overline{Y}, \mathbb{C}^*) \longrightarrow 1$$

so $\ker(\operatorname{Aut}(\overline{Y}, \overline{D}) \to \operatorname{Pic}(\overline{Y})) \cong T$.)

Now let $(\overline{Y}, \overline{D})$ be a toric Looijenga pair and pick $\overline{\phi} \in \operatorname{Hom}(\operatorname{Pic}\overline{Y}, \operatorname{Pic}^0(\overline{D}))$. Then exists a point $p_i \in \overline{D}_i^{\circ}$ such that ϕ is the corresponding marked period points, i.e.

$$\overline{\phi}(L) = L^{-1}|_D \otimes \mathcal{O}_D(\sum (L \cdot D_i) p_i) \in \operatorname{Pic}^0(D).$$

The reason is due to the SES

$$1 \longrightarrow T \longrightarrow \operatorname{Aut}^0(\overline{D}) \longrightarrow \operatorname{Hom}(\operatorname{Pic}\overline{Y}, \mathbb{C}^*) \longrightarrow 1$$

We can pick ant $\{p_i' \in D_i^{\circ}\}$ with period point $\overline{\phi}'$. Then exists ψ such that $\psi \circ \overline{\phi}' = \overline{\phi}$. This also shows that these p_i's are unique up to T-action.

Construction: given $\pi : (Y, D) \to (\overline{Y}, \overline{D})$ a toric model and $\phi \in \operatorname{Hom}(\operatorname{Pic} Y, \operatorname{Pic}^0(D))$, we construct a Looijenga pair (Z, D) over the same base $(\overline{Y}, \overline{D})$ with the same combinatorial type, and $\mu : \operatorname{Pic}(Y) \to \operatorname{Pic}(Z)$, such that $\phi = \phi_{((Z,D),\mu,p_i)}$ (for some marking $p_i \in D_i^{\circ}$).

In the unmarked case, let $\overline{\phi} = \phi \circ \pi^* \in \operatorname{Hom}(\operatorname{Pic}\overline{Y}, \mathbb{C}^*)$ so $\overline{\phi}$ is realised by some marking. Now consider $\phi : \operatorname{Pic}(Y) \to \operatorname{Pic}^0(D)$. Let $E_{ij}, i = 1, \ldots, n$ be the exceptional curves of π. Then there exist unique points $q_{ij} \in D_i^{\circ}$ such that

$$\phi(E_{ij}) = \mathcal{O}_D(-q_{ij}) \otimes \mathcal{O}_D(p_i).$$

Now let Z be the iterated blowup of $\overline{Y}$ at the centres $\{q_{ij}\}$, D the proper transform of $\overline{D}$. Then by definition $\phi = \phi_{((Z,D),\mu,\{p_i\})}$ where $\mu : \operatorname{Pic} Y \to \operatorname{Pic} Z$ is uniquely determined by

$$\mu([D_i]) = [D_i], \mu(E_{ij}) = E_{ij}.$$

Now suppose $\phi = \phi_{(Y,D),\mathrm{id},\{p_i\}}$, i.e. it is the marked period point of (Y, D), Then consider $\pi(r_i) \in \overline{D}_i^{\circ}$. These satisfy the same property as $\{p_i\}$, so $\{p_i\}$ and $\{r_i\}$ are in the same T-orbit. Thus we can change the markings $\{\pi(r_i)\}$ so markings for (Y, D) and (Z, D) are the same.

Now consider $E_{ij} \cap D_i = \tilde{q}_{ij} \in Y$. Then $\pi(\tilde{q}_{ij})$ satisfy the same properties as q_{ij}, so $\pi(\tilde{q}_{ij}) = q_{ij}$ by assumptions. $\square$

We have thus proven Torelli's theorem (both in marked and unmarked case) when two Looijenga pairs are over the same toric base with the same combinatorial type.

Exercise. Construct a pair (Y, D) with infinitely many internal (-2)-curves.

First we construct (Y, D) such that D is a cycle of n (-2)-curves (for example performing toric blowups on $\mathbb{P}^2$). Then bloup smooth points to achieve a cycle of length n such that $D_i^2 = -2$ for all i. $(\mathcal{O}_Y(D)) \in D^\perp$ as

$$(D_1 + \cdots + D_n) \cdot D_i = D_i^2 + D_{i-1} \cdot D_i + D_{i+1} \cdot D_i = 0$$

so $\phi_{(Y,D)}(\mathcal{O}_Y(D)) \in \operatorname{Pic}^0(D) \cong \mathbb{C}^*$.

Remark. $\mathcal{O}_Y(D) = -K_Y$ and $(K_Y)^2 = 0$ so numerically (Y, D) looks like elliptic fibration. Suppose we can achieve $\mathcal{O}_Y(D)|_D \cong \mathcal{O}_D$, then by classification theory of surfaces (Y, D) is a genuine elliptic fibration.

Recall the construction in the proof: given $\pi : (Y, D) \to (\overline{Y}, \overline{D})$, for any $\overline{\phi}$ we can construct $\tilde{\pi} : (Z, D) \to (\overline{Y}, \overline{D})$ such that $\phi_{((Z,D),\mu,\{p_i\})} = \overline{\phi}$. Apply this to $\overline{\phi} = 1$. It is a general fact that (Z, D) is deformation equivalent to (Y, D). Thus up to deforming the complex structure of (Y, D), we can assume it is an eliptic fibration $p : Y \to \mathbb{P}^1$.

Fact: if $n = 7$ then p has infinitely many sections E_k. $\{E_k\}$ are infinitely many (-1)-curves. By pigeonhole principle, exists i such that infinitely many E_k that intersect D_i. In particular $E_k \cap D_i$ must be the same point q (otherwise contradiction to $\phi_{(Y,D)} = 1$!). In fact $q \in D_i^\circ$: define $(Y', D') = (\operatorname{Bl}_q Y, \text{proper transform of } D))$ and E'_k the proper transform of E_k which are internal (-2)-curves.

One would visualise the "usual" (-2)-internal curves of a toric model to have only finitely many (-2)-curves.

concluding proof of global Torelli The idea is to replace the condition of having the same toric base by conditions involving only periods, $\operatorname{Pic}(Y)$ and internal curves. Recall

- $\Delta_Y \subseteq \operatorname{Pic}(Y)$ is the set of classes of internal curves,
- $C^+ = \{x \in \operatorname{Pic}(Y) \otimes \mathbb{R} : x^2 > 0\}$ is the component containing the ample class,
- inside we have $C^{++} = \{x \cdot E \geq 0 \text{ for all } E \in \widetilde{M}\}$.

We will also use the notation $C_D^{++} \subseteq C^{++}$ for the subcone cut out by $\{x \cdot D_i \geq 0\}$.

Recall $\operatorname{Pic}(Y) \cong H^2(Y, \mathbb{Z})$ in our case. We define

- the Mori cone/cone of curve classes $\operatorname{NE}(Y)$ to be the convex hull of the set of all classes in $H^2(X, \mathbb{R})$ represented by curves,
- $\operatorname{Nef}(Y)$ to be the dual of $\operatorname{NE}(Y)$, i.e. all classes in $H^2(Y, \mathbb{R})$ intersecting $\alpha \in \operatorname{NE}(Y)$ nonnegatively.

Remark. $\operatorname{NE}(Y)$ is not finitely generated in general. For example let Y be a rational elliptic surface. In general Y contains infinitely many (-1)-curves. These are extremal rays of $\operatorname{NE}(Y) \subseteq H^2(Y, \mathbb{R})$. If $Y \setminus D$ is affine then $\operatorname{NE}(Y)$ is finitely generated.

If we have $f : (Y, D) \to (Y', D')$ inducing $\mu : \mathrm{Pic}(Y') \to \mathrm{Pic}(Y)$ then

$$\mu(\mathrm{NE}(Y)) = \mathrm{NE}(Y'), \mu(\mathrm{Nef}(Y)) = \mathrm{Nef}(Y').$$

We need numerical conditions that give the same conclusion.

Lemma 3.9. *Fix (Y, D). Then $\mathrm{Nef}(Y)$ is the closure of the subcone of C_D^{++} cut out by $\{x \cdot \alpha \geq 0\}$ for all $\alpha \in \Delta_Y$.*

Proof. We will give a rough "classification" of curve classes in Y. Set $\mathcal{M}$ to be the set of classes of (-1)-curves not contained in D. Claim for $C \subseteq Y$ a curve, either $C^2 \geq 0$ or C is one of $[D_i]$, or C is contained in Δ_Y, or C is contained in $\mathcal{M}$. This follows from the adjunction.

Next we claim that $\overline{\mathrm{NE}(Y)}$ is the convex hull of $C^+, \Delta_Y, \mathcal{M}, \{[D_i]\}$. This follows from the previous claim if we can show classes in C^+ and $\mathcal{M}$ are effective. We do this for $\mathcal{M}$. Need to show $E^2 = E \cdot K_Y = -1, E \cdot H > 0$ for any ample H then E is effective. This is because we can apply Hirzebruch-Riemann-Roch to $L \in \mathrm{Pic}\, Y$ corresponding to E,

$$h^0(L) + h^0(K_Y - L) - h^1(L) = \frac{1}{12}(K_Y^2 + e(Y)) + \frac{1}{2}(L^2 - L \cdot K_Y)$$

so

$$h^0(L) + h^0(K_Y - L) \geq \frac{1}{12}(K_Y^2 + e(Y)) > 0$$

so one of L and $K_Y - L$ has a section. Now

$$(K_Y - L) \cdot H = K_Y \cdot H - L \cdot H < 0$$

so it cannot have a section. Thus we conclude E is effective. $\square$

Corollary 3.10. *Under the assumption of global Torelli, $\mu(\mathrm{Nef}(Y)) = \mathrm{Nef}(Y')$.*

Now take $\mu : \mathrm{Pic}\, Y_1 \to \mathrm{Pic}\, Y_2$ preserving $\Delta_{Y_1}, C_D^{++}, [D_i]$. Then by the lemma μ preserves the nef cone and the Mori cone. We want to construct $f : Y_2 \to Y_1$ inducing μ. Fix toric models $(Y_1, D) to (\overline{Y}_1, \overline{D}), (Y_2, D) \to (\overline{Y}_2, \overline{D})$ (after toric blowup). μ maps the exceptional curves E_{ij} for $(Y_1, D) \to (\overline{Y}_1, \overline{D})$ to classes $F_{ij} \in \mathrm{Pic}(Y_2)$ with the same intersection properties. Numerically F_{ij} looks like chains and they are effective curve classes so contracting F_{ij} gives a morphism $(Y_2, D) \to (\overline{Y}_2, \overline{D})$. We need to show $(\overline{Y}_2, \overline{D})$ is toric.

Topologically $(\overline{Y}_2, \overline{D})$ is a Looijenga pair, and it is toric if and only if $e(\overline{Y}_2 \setminus \overline{D}) = 0$: for one direction if it is toric then the complement is homeomorphic to $(\mathbb{C}^*)^2$ which has Euler characteristic 0. Conversely, blowing up can only increase Euler characteristic, strictly if at a smooth centre. One can check the topological Euler characteristic is 0, as otherwise the Picard group has the wrong rank.

Finally to check the toric bases are isomorphic, i.e. $(\overline{Y}_1, \overline{D}) \cong (\overline{Y}_2, \overline{D})$, we use the rigidity of toric surfaces: $(\overline{Y}_2, \overline{D})$ is determined, up to isomorphism of toric pairs, by sequence $\{\overline{D}_1^2, \dots, \overline{D}_n^2\}$ (see Fulton).

3.1 Monodromy

Pick a family $f : (\mathfrak{Y}, \mathfrak{D}) \to S$ is a family of Looijenga pair, meaning that $f : \mathfrak{Y} \to S$ is a holomorphic submersion (in complex language) or a flat morphism with smooth fibres (in algebraic language), $\mathfrak{D} \subseteq \mathfrak{Y}$ a divisor restricting to anticononical divisors on fibres. Then $R^2 f_* \mathbb{Z}$ is a local system over S with fibres (stalks) given by $H^2(\mathfrak{Y}_s, \mathbb{Z})$ and $R^2 f_* \mathbb{C}$ is a vector bundle with fibres $H^2(\mathfrak{Y}_s, \mathbb{C})$. We get a flat connection on $R^2 f_* \mathbb{C}$, called the Gauss-Manin connection. We can use the connection to define monodromy. Fix $0 \in S$ and let $(Y, D) = (\mathfrak{Y}_0, \mathfrak{D}_0)$ and let $\pi : [0, 1] \to S$ be a loop based at 0. Consider the ∇^{GM}-parallel transport along π, giving a linear map $\gamma : (R^2 f_* \mathbb{Z})_0 \to (\mathbb{R}^2 f_* \mathbb{Z})_0$.

Definition (monodromy transformation)**.** Such $\gamma \in \mathrm{GL}(H^2(Y, \mathbb{Z}))$ is called a *monodromy transformation.*

Remark. We also regard γ as an automorphism of the lattice $\mathrm{Pic}(Y)$ because the parallel transport preserves the intersection form.

Remark. We will see that all possible monodromy transformations are realised by some single family $(\mathfrak{U}, \mathfrak{D}) \to S$. We call this a *universal family* (which can be confusing since it is not the universal family for Looijenga pairs).

To construct examples of monodromy, we introduce a root of the Picard group. The set of *roots* $\Phi \subseteq \mathrm{Pic}(Y)$ is the set of classes which can be obtained by ∇^{GM}-parallel transport of the (effective) class of an internal (-2)-curve. As the intersection pairing is preserved, any $\alpha \in \Phi$ satisfies $\alpha^2 = -2, \alpha \cdot K_Y = 0$.

Exercise. Consider our previous example of a family where two points p, q on the same component D_i collide at a special point $0 \in \Delta$. Show this gives a root $\alpha \in \Phi$.

Recall $\Delta_Y \subseteq \mathrm{Pic}(Y)$, the set of internal (-2)-curve classes. By definition $\Delta_Y \subseteq \Phi$. Φ also contains as a subset Φ_Y, the subset of roots α such that $\phi_{(Y,D)}(\alpha) = 1$.

Define the Weil group $W \subseteq \mathrm{Aut}(\mathrm{Pic}(Y))$ to be the subgroup generated by reflections $S_\alpha : \beta \mapsto \langle \alpha, \beta \rangle \alpha$. Let $W_Y \subseteq W$ be the subgroup generated by $S_\alpha, \alpha \in \Delta_Y$.

Theorem 3.11 (GHK)**.** $\Phi_Y = W_Y \cdot \Delta_Y$.

Corollary 3.12. *(Y, D) is generic if and only if $\Phi_Y = \emptyset$, i.e. all roots have nontrivial periods.*

Combining this with the local Torelli theorem, we get

Corollary 3.13. *Every Looijenga pair is deformation equivalent to a generic pair.*

Proof. Consider the period mapping

$$\phi : \mathrm{Def}(Y, D) \to T_Y = \mathrm{Hom}(D^\perp, \mathbb{C}^*).$$

which is an isomorphism onto its image. For each root α of (Y, D), consider the hypertorus (a codimension 1 affine torus)

$$\{\phi_{(Y',D')}(\alpha) = 1\} \subseteq T_Y.$$

Claim this is a proper inclusion because it is locally finite in T_Y (this is a nontrivial fact and we will visit it later). $\square$

roots inducing monodromy Claim that given $\alpha \in \Phi$, $S_\alpha \in \mathrm{GL}(\mathrm{Pic}(Y))$ is a monodromy transformation:

1. by the definition of root, there exists a family $f : (\mathfrak{Y}, \mathfrak{D}) \to S$ which takes $\alpha \in \mathrm{Pic}(Y_0)$ to $[C] \in \mathrm{Pic}(Y_1)$ where C is an internal (-2)-curve.
2. There exists a family $f' : (\mathfrak{Y}', \mathfrak{D}') \to S'$ such that (Y_1, D_1) appears as a fibre, and which is a smoothing for the surface $\widetilde{Y}$ obtained by contracting C. One can show $\widetilde{Y}$ is singular with a single ordinary double point. Picard-Lefschetz theory says that the monodromy "around the ordinary double point" is given by the reflection $S_{[C]} \in \mathrm{GL}(\mathrm{Pic}(Y_1))$.
3. Composing the two paths (exercise: show the composition of monodromies is again a monodromy. hint: the family f' is a versal smoothing family).

Remark. Let us illustrate Picard-Lefschetz theory in low dimension. Consider a family of elliptic curves with a single nodal fibre. Then the monodromy $T \in \mathrm{GL}(H_2) = \mathrm{GL}(\mathbb{Z}^2)$ induced by the loop π along the "singular point" is given by reflection across the class of the vanishing cycle. For more see Voisin Hodge theory.

characterisation of the monodromy group

Definition. Let (Y, D) be a Looijenga pair. Define the group of *admissible monodromy transformations* to be

$\mathrm{Adm}_Y = \{$linear self-maps of $\mathrm{Pic}(Y)$ preserving $[D_i] \in \mathrm{Pic}(Y)$ and $C^{++} \subseteq \mathrm{Pic}(Y)\}$.

Theorem 3.14.

1. *There exists a family $f : (\mathfrak{Y}, \mathfrak{D}) \to S$ such that all monodromy transformations are realised by f.*
2. *The set of monodromy transformations is precisely the set of admissible ones.*

We do not give the proof but it might be instructive to show monodromy transformations are admissible. By the definition of a family of Looijenga pairs, the boundary components are preserved. Thus it is enough to show if $\gamma \in \mathrm{GL}(\mathrm{Pic}(Y))$ is a monodromy transformation then it preserves C^{++}, i.e. given $f : (\mathfrak{Y}, \mathfrak{D}) \to S$ then the cones $C_s^{++} = C^{++}(\mathfrak{Y}_s, \mathfrak{D}_s)$ are preserved by the parallel transports. We can choose a divisor $\mathfrak{H} \subseteq \mathfrak{Y}$ such that the restriction to each fibre is ample. C^{++} *does not* depend on the choice of the ample divisor (recall the computation by Hirzebruch-Riemann-Roch). Clearly the family of cones

C_s^{++} is ∇^{GM}-flat (because it is defined by a condition on intersection numbers, which are themselves preserved by the Gauss-Manin connection). Combining these two points give the desired result.

A Deformation theory

We will have an interlude on deformations of complex manifolds. For now think of a holomorphic submersion $f : \mathfrak{X} \to S$ of complex manifolds. As f is a surjection all fibres are complex manifolds. The following theorem allows us to identify fibres of f as the space underlying space with different complex structures.

Theorem A.1 (Ehresmann)**.** *All fibres of f are diffeomorphic.*

Then we say the deformation of a given manifold $X = \mathfrak{X}_0$ is the germ of a holomorphic submersion as above. Such a germ induces an element of $H^1(X, T_X)$. We call such an element a first order deformation.

We say f is *complete* if all sufficiently small deformations of $X = X_s$ for $s \in S$ appear as fibres, and *versal* if all sufficiently small deformation family $f' : \mathfrak{Y} \to T$, is isomorphic to $\psi^* f$ for some $\psi : T \to S$, and $d\psi$ is uniquely determined.

Theorem A.2 (Kuranish)**.** *If we allow S to be a complex analytic space then versal deformations exist.*

The number of moduli or the number of deformation parameters at $s \in S$ is $\dim_s S$. We usually use this terminology in the case when X is *unobstructed*, i.e. when S is actually a manifold.

de Rham/Dolbeault approach to deformations Let X be a compact complex manifold. Let $\mathcal{A}^k(X, \mathbb{C})$ be the sheaf of differential k-forms with complex coefficient. There is a splitting

$$\mathcal{A}^k(X, \mathbb{C}) = \bigoplus_{p+q=k} \mathcal{A}^{p,q}(X).$$

This extends to forms with coefficients in a holomorphic vector bundle, $\mathcal{A}^{p,q}(X, E)$. In particular we can take $E = T_X = T^{1,0}X$. We'll see

- $\mathcal{A}^{0,0}(T_X)$ defines an equivalence relation on deformations given by infinitesimal diffeomorphisms.
- $\mathcal{A}^{0,1}(T_X)$ gives infinitesimal deformations of X as *almost complex manifold*.
- $\phi(t)$, with values in $\mathcal{A}^{0,1}(T_X)$, preserves complex structure property (not just almost complex) if and only if the Maurer-Cartan equation holds (see below for the Lie algebra structure):

$$\overline{\partial}\phi(t) + [\phi(t), \phi(t)] = 0.$$

$(\mathcal{A}^{0,*}(T_X), \overline{\partial})$ has the structure of a dg Lie algebra. The Lie bracket is defined as

$$[\sum_I \mathrm{d}\overline{z}_I \otimes v_I, \sum_J \mathrm{d}\overline{z}_J \otimes w_J] = \sum_{I,J} \mathrm{d}\overline{z}_I \wedge \mathrm{d}\overline{z}_J [v_I, w_J]$$

where v_I, w_J are (complex vector fields) and the bracket on RHS is the usual Lie bracket extended by complex linearity. The differential $\overline{\partial}$ is compatible with the Lie braket in the sense that

$$\overline{\partial}[\alpha, \beta] = [\overline{\partial}\alpha, \beta] \pm [\alpha, \overline{\partial}\beta].$$

Thus the bracket descends to Dolbeault cohomology $H^{0,*}_{\overline{\partial}}(X, T_X) \cong H^*(X, T_X)$.

Why is this relevant? Consider the space of all almost complex structures on X

$$\mathcal{J} = \{J \in \Gamma(X, \mathrm{End}(TX)) : J^2 = -\mathrm{id}\} = \{J \in \Gamma(X, T^*X \otimes TX) : J^2 = -\mathrm{id}\}.$$

A complex structure induces an element $J \in \mathcal{J}$: in local holomorphic coordinates $z_i = x_i + \sqrt{-1}y_i$, J is defined by

$$J(\frac{\partial}{\partial x_i}) = -\frac{\partial}{\partial y_i}, J(\frac{\partial}{\partial y_i}) = \frac{\partial}{\partial x_i}.$$

This is well-defined by the fact that transition functions are holomorphic (i.e. Cauchy-Riemann relation holds).

The converse is false:

Theorem A.3 (Newlander-Nirenberg)**.** *Let J be an almost complex structure. Define $T^{0,1}_J X$ to be the part of $T_{\mathbb{C}}X$ on which $J_{\mathbb{C}}$ acts as $-\sqrt{-1}\,\mathrm{id}$. Then J is induced by a complex structure if and only if $[T^{0,1}_J X, T^{0,1}_J X] \subseteq T^{0,1}_J X$.*

Given this, the strategy to deform a complex manifold (X, J) is to first deform J aas almost complex structure and then impose integrability. Thus we would like to deform the decomposition $T_{\mathbb{C}}X = T^{1,0}_J X \oplus T^{0,1}_J X$. SInce they are complex conjugate to each other, it suffices to form $T^{0,1}_J X \subseteq T_{\mathbb{C}}X$.

Thus given $\phi(t)$ with values in $\mathcal{A}^{0,1}(T_X)$, we define

$$T^{0,1}_t = T^{0,1}_J(X) + \phi(t)(T^{0,1}_J X).$$

This defined an almost complex structure J_t determined by

$$T^{0,1}_{J_t} X = (\mathrm{id} + \phi(t)) T^{0,1}_J X.$$

Now enforcing J_t to be integrable turns out to be equivalent to the Maurer-Cartan equation

$$\overline{\partial}\phi(t) + [\phi(t), \phi(t)] = 0.$$

By the term *infinitesimal deformation* we mean power series solutions

$$\phi(t) = \phi_1(t) + \phi_2 t^2 + \dots$$

to the problem.

We also want to introduce an equivalence relation on the solutions by keeping track of when complex structures are equivalent. Recall that two complex structures (X, J) and (X, J') are equivalent if there exists a diffeomorphism F such that $J' = F^*J = dF \circ J \circ dF^{-1}$. It is often the case in the orbit of formal power series solutions under this action, there is one that converges

first order deformation By a *first order deformation* we mean ϕ_1 that solves Maurer-Cartan to first order. More precisely write Maurer-Cartan equation order by order, we get

$$\overline{\partial}\phi_1 = 0, \overline{\partial}\phi_2 = -[\phi_1, \phi_1], \ldots, \overline{\partial}\phi_k = -\sum_i [\phi_i, \phi_{k-i}].$$

Thus we get an element $[\phi_1] \in H^{0,1}_{\overline{\partial}}(X, T_X) = H^1(X, T_X)$. It is called the *Kodaira-Spencer class* of the first order deformation.

Moreover if $\overline{\partial}\phi_2 = -[\phi_1, \phi_1]$ is solvable then $[\phi_1, \phi_1]$ is $\overline{\partial}$-exact. Since we have a dgla this means

$$[[\phi_1], [\phi_1]] = 0 \in H^{0,2}_{\overline{\partial}}(X, T_X).$$

This is saying that given a first order deformation ϕ_1, the first obstruction to lift ϕ_1 to an infinitesimal deformation is the Lie bracket of the Kodaira-Spencer class $[\phi_1]$ with itself.

The equivalence relation on first order deformation can be easily verified to be $\phi_1 \sim \phi_1 + \overline{\partial}V$ where $V \in \mathcal{A}^0(T_X)$, i.e. a smooth vector field of type $(1, 0)$. Thus

$$\frac{\{\text{first order deformations}\}}{\{\text{infinitesimal diffeomorphism}\}} = H^{0,1}_{\overline{\partial}}(X, T_X) = H^1(X, T_X).$$

holomorphic approacth to deformation Now we make a new definition: a deformation of a compact complex manifold X is a proper holomorphic submersion $f : \mathfrak{X} \to S$ with $\mathfrak{X}_0 \cong X$ for $0 \in S$. Throughout we only care about the germ of f. By Ehresmann, if S is a connected complex manifold then all fibres of f are diffeomorphic and we have a short exact sequence of sheaves on X

$$0 \longrightarrow T_X \longrightarrow T\mathfrak{X}|_{\mathfrak{X}_0} \longrightarrow T_0S \otimes \mathcal{O}_X \longrightarrow 0.$$

Definition. The *Kodaira-Spencer map* of f is the connecting homolorphism in cohomology $ks : T_0S \to H^1(X, T_X)$.

Proposition A.4. *The Kodaira-Spender map is compatible with the Kodaira-Spencer class*

$$\{\textit{first order deformation}\} \to H^1(X, T_X).$$

Thus the two approaches to deformation theory are equivalent, if we identify the sheaf cohomology group $H^1(X, T_X)$ and Dolbeault cohomology $H^{0,1}_{\overline{\partial}}(X, T_X)$.

We collect here some key results of deformation theory:

1. a deformation of a Kähler manifold is Kähler.
2. There is a notion of *completeness*, that is $f : \mathfrak{X} \to S$ such that all other deformations are obtained from f as base change. This is weaker than *versality*, that is $f : \mathfrak{X} \to S$ complete and such that the differential at 0 of pullbacks are unique: if $g : S' \to S$ is the pullback map, $g(0') = 0$ then $dg_{0'}$ is uniquely determined by the pullback family. Fianlly we have the strongest notion of *universality*, where in addition we require g to be unique.

3. Kuranishi's theorem: A versal deformation of a compact complex manifold exists, but the base is *not* a complex manifold in general (it is a complex analytic space).

4. Universal deformations do not exist in general, but we have the following sufficient condition: if $H^0(X, T_X) = 0$ then a versal family is also universal.

5. If $H^2(X, T_X) = 0$ then X is *unobstructed*: a versal deformation is (the germ of) a complex manifold.

6. Suppose $f : \mathfrak{X} \to S$ with S a complex analytic space is a proper flat morphism of complex analytic spaces such that all fibres are complex manifolds. Then there is a generalisation of the Kodaira-Spencer class

$$ks : T_0 S := (\mathfrak{m}_0/\mathfrak{m}_0^2)^\vee \to H^1(X, T_X).$$

7. In this case, f is complete if and only if ks is onto.

unobstructedness for noncompact CY Kaledin-Verbitsky says that if U is holomorphic symplectic (i.e. has a nondegenerate holomorphic 2-form Ω) and Ω is algebraic and $H^i(U, \mathcal{O}_U) = 0$ for $i > 0$, then U is unobstructed.

Note that for a general pair (Y, D) where $D \in |-K_Y|$ (in any dimension), it is not true that the pair is unobstructed. However luckily for us, the result is true for Looijenga pairs. This follows from a general result of Iacono that gives sufficient condition on D for it to be unobstructed. This result is also know to Looijenga himself. In any case we have a smooth versal deformation germ. Let $S = \mathrm{Defo}(Y, D)$ be the deformation space with a distinguished point $0 \in S$.

Theorem A.5 (Looijenga). *The period map*

$$\begin{aligned} \pi : \mathit{Defo}(Y, D) &\to \mathrm{Hom}(D^\perp, \mathbb{C}^*) \\ (Y', D') &\to \textit{period point} \end{aligned}$$

is a biholomorphism near 0.

In general suppose we have a family of complex manifolds (i.e. proper holomorphic submersion) $f : \mathfrak{X} \to S$. Consider the direct images $Rf_*\mathbb{Z} \subseteq Rf_*\mathbb{C}$ as local systems on S, which is a complex vector bundle W equipped with a flat connection ∇, the *Gauss-Manin connection.* It is characterised by two properties:

1. flatness: $\nabla^2 = 0$.

2. elements of $H^*(\mathfrak{X}_s, \mathbb{Z})$ as s varies give flat section.

In the special case of Looijenga pairs $f : \mathfrak{Y} \to S$, we may look instead at $\mathrm{Pic}(Y_s)$ for $s \in S$. Claim that the first Chern class is an isomorphism $c_1 : \mathrm{Pic}(Y_s) \to H^2(Y_s, \mathbb{Z})$. Indeed up to a blowup Y_s has a toric model $\pi : (Y_s, D_s) \to (\overline{Y}_s, \overline{D}_s)$. The statement is true for $(\overline{Y}_s, \overline{D}_s)$ and lifts to the blowup. Thus we get a vector bundle with fibres $H^2(Y_s, \mathbb{C})$ with a local system $H^2(Y_s, \mathbb{Z})$.

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