Scuola Internazionale
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Geometry and Mathematical Physics

Localisation in Enumerative Geometry

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Contents

1 Introduction

Plan of the course:

- classical moduli spaces (Grassmannians, Hilbert scheme and Quot scheme),
- equivariant cohomology and localisation, applications: number of lines on cubic surface, number of lines on quintic threefold etc.
- virtual classes and localisation, applications: DT invariants for local CY threefold.

What is enumerative geometry?
Classically, one had questions, such as

1. how many lines in $\mathbb{P}^2$ pass through 2 points?
2. how many lines in $\mathbb{P}^3$ lie on a smooth cubic?
3. how many lines in $\mathbb{P}^4$ lie on a smooth quintic?

The strategy was

1. to construct a proper moduli space M for the objects to enumerate,
2. make sure the incidence conditions define a 0-cycle $\alpha \in A_0 M$,
3. compute $\deg_M(\alpha)$,
4. make sure this number answers the original question.

Remark. In the examples above we have M some Grassmannian, which is smooth and proper.

What is M is too singular/impure to have a fundamental class? Classically we need the fundamental class to compute for $\alpha \in A^* M$

$$\int_M \alpha := \deg_M(\alpha \frown [M]).$$

This leads to the *virtual fundamental class* $[M]^{\mathrm{vir}} \in A_{\mathrm{vd}}(M)$, where vd stands for virtual dimension. We can then define $\int_{[M]^{\mathrm{vir}}} \alpha \in \mathbb{Z}$. The question now is, what does this count? Usually the answer is unclear. But this is OK, because the actual thrill is to study the behaviour of the generating functions. For example we can consider the generating functions of all classes with a fixed Chern class. They have remarkable properties.

Example. Back to the very first example of number of lines through two pionts $p, q \in \mathbb{P}^2$. If p and q are distinct then there is a unique line. If $p = q$ then the answer is infinite. They correspond to transverse and non-transverse case.

Enumerative geometry is the machinery allowing us to prove that the answer is 1 *always*. Where is the answer one hiding in the non-transverse setup? One may be tempted to say that to compute the self intersection number of Z_p, the class of lines passing through p, we may move Z_p to a rationally equivalent cycle Z_q and then $Z_p^2 = Z_p \cdot Z_q = 1$. But we can't do this in general if

Recall that for $X \hookrightarrow Y$ a closed embedding corresponding to the ideal sheaf $\mathcal{I}$, the conormal sheaf $\mathcal{C}_{X/Y} = \mathcal{I}/\mathcal{I}^2$ is a coherent sheaf (all schemes are locally noetherian over $\mathbb{C}$ in this course), and normal sheaf $\mathcal{N}_{X/Y} = \mathcal{C}_{X/Y}^\vee$. Consider the blowup

$$\begin{array}{ccc} \mathbb{P}^1 = E & \longrightarrow & B = \mathrm{Bl}_p \mathbb{P}^2 \\ \downarrow g & & \downarrow \pi \\ p & \longrightarrow & \mathbb{P}^2 \end{array}$$

and note $\mathbb{P}^1$ is our M in case $p = q$. By intersection theory we have an exact sequence

$$0 \longrightarrow \mathcal{N}_{E/B} \longrightarrow g^*\mathcal{N}_{p/\mathbb{P}^2} \longrightarrow \mathrm{Ob}_{p,p} \longrightarrow 0$$

where $\mathrm{Ob}_{p,p}$ is what is called the *excess bundle*. As $\mathcal{N}_{E/B} = \mathcal{O}_E(-1), g^*\mathcal{N}_{p/\mathbb{P}^2} = \mathcal{O}_E \otimes T_p\mathbb{P}^2 = \mathcal{O}_E \otimes O_{\mathbb{P}^1}^{\oplus 2}$. This is exactly the Euler seuqnece twisted by $\mathcal{O}(-1)$. Thus

$$\mathrm{Ob}_{pp} \cong T_{\mathbb{P}^1}(-1) = \mathcal{O}(1),$$

and this is where we find the number one. We get then a uniform answer

$$\int_{\pi^{-1}(q)} e(\mathrm{Ob}_{pq}) = 1$$

(when $p \neq q$, the fibre $\pi^{-1}(q)$ is a point and the first arrow in the short exact sequence is an isomorphism so Ob_{pq} is trivial.)

In this example the excess sheaf records "why" the moduli psace can be "oversized".

Example. Let Y be a smooth variety of dimension n. Let $E = \mathrm{Spec}\,\mathrm{Sym}\,\mathcal{E}^\vee \to Y$ where $\mathcal{E}$ be is locally free of rank r. Then a section $s \in \Gamma(Y, E) = \mathrm{Hom}(\mathcal{O}_Y, \mathcal{E}^\vee)$ gives a Cartesian square

$$\begin{array}{ccc} M = Z(s)Y & \longrightarrow & Y \\ \downarrow & & \downarrow s \\ Y & \xrightarrow{0} & E \end{array}$$

M has expected dimension (virtual dimension) $n - r$, which can be smaller than the actual dimension $\dim M$. Again this inequality is captured by the excess sheaf. Consider

$$0 \longrightarrow \mathcal{I}_{M/Y} \xrightarrow{i} \mathcal{N}_{Y/E}|_M \longrightarrow \mathrm{Ob} \longrightarrow 0$$

$$\mathcal{E}^\vee$$

$s^\vee : \mathcal{E} \to \mathcal{O}_Y$ has image $\mathcal{I} = \mathcal{I}_{M/Y} \subseteq \mathcal{O}_Y$, so $s^\vee|_M : \mathcal{E}|_M \twoheadrightarrow \mathcal{C}_{M/Y}$, so i corresponds to $\mathrm{Spec}\,\mathrm{Sym}(s^\vee|_M)$.

1.1 Grassmanians

Let S be a noetherian scheme and $\mathcal{F}$ a coherent sheaf on S. Let $d \geq 1$. Let $\mathbf{Sch}_S$ be the category of locally noetherian schemes over S. We define a contravariant functor $\mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set}$, sending $g : U \to S$ to

$$\{p : g^*\mathcal{F} \twoheadrightarrow \mathcal{Q} : \mathcal{Q} \text{ locally free of rank } d\}$$

with the equivalence

$$\begin{array}{ccc} g^*\mathcal{F} & \xrightarrow{p} & \mathcal{Q} \\ \| & & \downarrow \cong \\ g^*\mathcal{F} & \xrightarrow{p'} & \mathcal{Q}' \end{array}$$

i.e. $\ker p = \ker p'$ as subsheaves of $g^*\mathcal{F}$. It is a classical result that this functor is representable by a projective scheme $s : G_d(\mathcal{F}) \to S$ and a universal family $s^*\mathcal{F} \twoheadrightarrow Q$.

Notation: if $\mathcal{F}$ is locally free of rank n, $0 < k \leq n$, we call $G(k, \mathcal{F}) := G_{n-k}(\mathcal{F})$, parameterising rank k subbundles of $\mathcal{F}$. For $S = \operatorname{Spec} \mathbb{C}$, we write

$$\mathbb{G}(k-1, n-1) = G(k, \mathbb{C}^n),$$

in accordance with $\{\mathbb{P}^{k-1} \subseteq \mathbb{P}^{n-1}\}$.

If $\mathcal{F}$ is locally free then $\rho : G(k, F) \to S$ is smooth of relative dimension $k(n-k)$, and the tautalogical exact sequence

$$0 \longrightarrow I \longrightarrow \rho^*F \longrightarrow Q \longrightarrow 0$$

of locally free sheaves of rank k, n and $n-k$ respectively.

It is a fact that $\mathcal{L} = \det \mathcal{Q}$ is ρ-very ample, so it defines a closed immersion $G(k, F) \hookrightarrow \mathbb{P}(\rho_*\mathcal{L}) \hookrightarrow \mathbb{P}(\bigwedge^k \mathcal{F})$ via $\bigwedge^k \mathcal{S} \hookrightarrow \rho^* \bigwedge^k \mathcal{F})$. So $\sigma_1 = c_1(\mathcal{L}) = c_1(\mathcal{S}^\vee) \in A^1 G(k, F)$ defines the embedding.

Later we will use

$$T_{G(k,F)/S} \cong \mathcal{H}om(\mathcal{S}, \mathcal{I}) \cong \mathcal{S}^\vee \otimes \mathcal{Q}.$$

Example. Let $\mathcal{F}$ be locally free and $d = 1$. Then $G_1(F) = G_{n-1}(F) = \mathbb{P}(F)$.

1.2 Quot scheme

Let S be a noetherian scheme and $X \to S$ of finite type. Let $\mathcal{F}$ be a coherent sheaf on X. Define a fnctor $\underline{\mathrm{Quot}}_{X/S}(\mathcal{F})$ to be the functor $\mathbf{Sch}_S^{\mathrm{op}} \to \mathbf{Set}$ sending $g : U \to S$ to

$$\{\mathcal{F}_U \twoheadrightarrow \mathcal{E} : \mathcal{E} \text{ is } U\text{-flat}, \operatorname{supp} \mathcal{E} \text{ proper}\}$$

under the equivalence relation. Recall that for Y locally noetherian and $\mathcal{E}$ a coherent sheaf on Y, define

$$\begin{aligned} \gamma : \mathcal{O}_Y &\to \mathcal{H}om_{\mathcal{O}_Y}(\mathcal{E}, \mathcal{E}) \\ f &\mapsto (\tau \mapsto f \cdot \tau) \end{aligned}$$

Then $\ker \gamma$ is a coherent ideal sheaf, and we let $\operatorname{supp} \mathcal{E}$ be the closed subscheme it defines.

Hilbert polynomial: for Y a variety over $\mathbb{C}$, let L be a very ample line bundle and $\mathcal{E}$ a coherent sheaf on Y with proper support. Then

$$P_L(\mathcal{E}, m) := \chi(\mathcal{E} \otimes \mathcal{L}^{\otimes m})$$

is a polynomial for $m \gg 0$.

In the relative setting, If $Y \to U$ with $\mathcal{L}$ a relatively very ample line bundle, $\mathcal{E}$ is a coherent sheaf on Y that is flat over U. Then $u \mapsto P_{L_u}(\mathcal{E}_u)$ is locally constant. Conversely if $u \mapsto P_{L_u}(\mathcal{E}_u)$ is locally constant and the base U is reduced then $\mathcal{E}$ is U-flat.

Now fix $\mathcal{L}$ a line bundle which is $(X \to S)$-very ample and $P \in \mathbb{Q}[m]$. Then we have subfunctors $\underline{\mathrm{Quot}}^{P,L}_{X/S}(\mathcal{F}) \subseteq \underline{\mathrm{Quot}}_{X/S}(\mathcal{F})$ of those quotients $\mathcal{E}$ such that $P_{L_u}(\mathcal{E}_u) = P$ for all $u \in U$.

Theorem 1.1 (Grothendieck)**.** $\underline{\mathrm{Quot}}^{P,L}_{X/S}(\mathcal{F})$ *can be represented by a projective scheme* $\mathrm{Quot}^{P,L}_{X/S}(\mathcal{F})$*.*

We define another functor

$$\underline{\mathrm{Hilb}}_{X/S} = \underline{\mathrm{Quot}}_{X/S}(\mathcal{O}_X)$$

which is represented by the *Hilbert scheme* $\mathrm{Hilb}_{X/S}$

Example. Let $P = n$ $S = \operatorname{Spec}\mathbb{C}$ and any $\mathcal{L}$. Let $\mathrm{Quot}_X(\mathcal{F}, n) := \mathrm{Quot}^{n,L}_X(\mathcal{F})$, parameterising

$$\{\mathcal{F} \twoheadrightarrow \mathcal{Q} : \dim\operatorname{supp}\mathcal{Q} = 0, \chi(Q) = n\}/\sim .$$

If $\mathcal{F} = \mathcal{O}_X$ then we get the *Hilbert scheme of points*

$$\mathrm{Hilb}^n(X) := \mathrm{Quot}_X(\mathcal{O}_X, n) = \{Z \hookrightarrow X \text{ closed subschem}, \dim Z = 0, h^0(\mathcal{O}_Z) = n\}.$$

Note we do not have to quotient by an equivalence relation in the closed subscheme description.

Theorem 1.2. *Let X be a smooth variety of dimension d. Then* $\mathrm{Hilb}^n(X)$ *is smooth if and only if $d \leq 2$ or $n \leq 3$.*

Exercise. Show $\mathrm{Hilb}^4(\mathbb{P}^3)$ is singular.

Remark. $\mathrm{Quot}_X(\mathcal{F}, n)$ also exists for quasiprojective X.

If $p = [\mathcal{K} \hookrightarrow \mathcal{F} \twoheadrightarrow Q]$ is a point of $Y = \mathrm{Quot}_X(\mathcal{F})$ then

$$T_pY = \operatorname{Hom}(\mathcal{K}, \mathcal{Q})$$

and

$$\dim\operatorname{Hom}(\mathcal{K}, \mathcal{Q}) \geq \dim\mathcal{O}_{Y,p} \geq \dim\operatorname{Hom}(\mathcal{K}, \mathcal{Q}) - \dim\operatorname{Ext}^1(\mathcal{K}, \mathcal{Q})$$

with first equality means (?) Y is smooth at p and second equality means Y is a lci at p.

Example. Let X be a smooth variety of dimension d.

1. $d = 1$. We show $p = [\mathcal{I}_Z \hookrightarrow \mathcal{O}_X \twoheadrightarrow \mathcal{O}_Z]$ satsifies $\mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{O}_Z) = 0$:

$$\begin{aligned}\mathrm{Ext}^i(\mathcal{I}_Z, \mathcal{O}_Z) &= \mathrm{Ext}^i(\mathcal{O}_X(-Z), \mathcal{O}_Z)\\ &= \mathrm{Ext}^i(\mathcal{O}_X, \underbrace{\mathcal{O}_X(Z) \otimes \mathcal{O}_Z}_{\mathcal{O}_Z})\\ &= H^i(\mathcal{O}_Z)\\ &= 0\end{aligned}$$

   for $i \geq 1$.

2. $d = 2$:

   Theorem 1.3 (Fogatry). $\mathrm{Hilb}^n(X)$ *is smooth* $(2n)$*-dimensional.*

   Proof. There exists open $U = \{p_1 \cup \cdots \cup p_n : p_i$ □

Example. Show $\mathrm{Hilb}^1(X) = X$. What is the universal family?

1.3 Examples of Hilbert scheme

Recall that last time when we set $\mathcal{F} = \mathcal{O}_X$ then we get the Hilbert scheme. We fix the base S to be $\mathrm{Spec}\,\mathbb{C}$. We prove that for X smooth curve, $\mathrm{Hilb}^n(X)$ is smooth of dimension n. For X smooth surface, $\mathrm{Hilb}^n(X)$ smooth of dimension $2n$.

We say that the tangent space to Hilb_X at $[Z]$ is

$$\mathrm{Hom}_X(\mathcal{I}_Z, \mathcal{O}_Z) \cong H^0(Z, \mathcal{N}_{Z/X}).$$

If $\mathrm{Ext}^1_X(\mathcal{I}_Z, \mathcal{O}_Z) = 0$ then $[Z] \in \mathrm{Hilb}_X$ is a smooth point.

1.3.1 Symmetric product

Let X be quasiprojective (so quotient is well-defined). Define

$$S^n X = X^n / S_n = \{\sum a_i x_i : a_i \in \mathbb{N}, \sum a_i = n\},$$

parameterising effective 0-cycles of degree n. Note $S^1 X = X$.

Exercise. Check $S^n \mathbb{A}^1 \cong \mathbb{A}^n, S^n \mathbb{P}^1 \cong \mathbb{P}^n$.

Remark. A fact that will be useful later: take $\alpha = (1^{\alpha_1} \cdots i^{\alpha_i} \cdots k^{\alpha_k})$ a partition of n, i.e. $n = \sum i\alpha_i$. It corresponds to a Young diagram with n boxes. We hav

$$S^n X = \coprod_{\alpha \vdash n} S^n_\alpha X$$

where $S^n_\alpha X$ is the cycle whose support is "decided by α". Among those there is a secial one correpsonding to $\alpha = (n^1)$, corresponding to $X \to S^n X, x \mapsto n \cdot x$. "small diagonal.

For X a smooth variety of dimension at least 2, S^nX is singular for $n \geq 2$ due to nontrivial stabilisers. However we have trivial stabilisers aways from all diagonals, i.e. on the *configuration space*

$$F^nX = \{(x_1, \dots, x_n) : x_i \neq x_j \text{ for all } i \neq j\}.$$

Then the quotient F^nX/S_n is smooth.

Definition. $\overline{\{x_1 \cup \cdots \cup x_n : x_i \neq x_j\}} \subseteq \mathrm{Hilb}^n(X)$ is the *smoothable component.*

We proved that it is an equality for $\dim X = 1, 2$.

1.3.2 Quot-to-Chow morphism

There is a *Hilbert/Quot-to-Chow map*

$$\begin{aligned}\sigma_{\mathcal{F},n} : \mathrm{Quot}_X(\mathcal{F}, n) &\to S^nX \\ [F \twoheadrightarrow T] &\mapsto [\mathrm{supp}(T)] = \sum_{T_X \neq 0} \mathrm{length}(T_X) \cdot x\end{aligned}$$

Exercise. $\mathrm{Hilb}^n(X) \to S^nX$ restricts to an isomorphism above F^nX/S_n.

Fact: if X is smooth and $\mathcal{F}$ is locally free, then $\sigma^{-1}_{\mathcal{F},n}(n \cdot x)$ does not depend on X, x or $\mathcal{F}$, and onlly on $d = \dim X, r = \mathrm{rk}\,\mathcal{F}$.

Exercise. Prove this for $\mathcal{F} = \mathcal{O}_X$.

We can then define

Definition. The punctual Quot scheme

$$\mathrm{Quot}_{\mathbb{A}^d}(\mathcal{O}^{\oplus r}, n)_0 = \sigma^{-1}_{\mathcal{O}^{\oplus r},n}(n \cdot 0)$$

Proposition 1.4. $\mathrm{Quot}_{\mathbb{A}^d}(\mathcal{O}^{\oplus r}, n)_0$ *is proper.*

Proof. Compactify $\mathbb{A}^d \subseteq \mathbb{P}^d$ and consider the Cartesian diagram □

1.3.3 Hypersurfaces of degree d in $\mathbb{P}^n$

Let $Y = V(f) \subseteq \mathbb{P}^n$ where

$$f \in \mathbb{C}[x_0, \dots, x_n]_d = H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d)) \cong \mathbb{C}^N$$

where $N = \binom{n+d}{d}$. We calculate the Hilbert polynomial of Y: by the short exact sequence corresponding to $\mathcal{I}_Y$, we have

$$\begin{aligned}P_Y(m) &= \chi(\mathcal{O}_Y(m)) \\ &= \chi(\mathcal{O}_{\mathbb{P}^n}(m)) - \chi(\mathcal{O}_{\mathbb{P}^n}(m-d)) \\ &= \sum_{i \geq 0} (-1)^i h^i(\mathcal{O}_{\mathbb{P}^n}(m)) - \sum_{i \geq 0} (-1)^i h^i..\end{aligned}$$

Let $\mathbb{P}^{N-1} \cong \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))^\vee$. Claim there is a morphism $\alpha : \mathbb{P}^{N-1} \to \mathrm{Hilb}_{\mathbb{P}^n}^{P_d, \mathcal{O}(1)}$, given by the family

$$Z = \{(x, [f]) \in \mathbb{P}^n \times \mathbb{P}^{N-1} : x \in V(f)\}.$$

The morphism $Z \hookrightarrow \mathbb{P}^n \times \mathbb{P}^{N-1} \to \mathbb{P}^{N-1}$ is flat because the Hilbert polynomial is constant and the base is reduced. Thus we have a map $\alpha : [f] \mapsto [V(f)]$. Note α is bijective (i.e. only the hypersurface can have the Hilbert polynomial).

Note $\mathrm{Hilb}_{\mathbb{P}^n}^{P_d, \mathcal{O}(1)}$ is smooth:...

Thus α is a bijective morphism of smooth $\mathbb{C}$-varieties, so is an isomorphism by Zariski's main theorem. Thus we have

$$\mathrm{Hilb}_{\mathbb{P}^n}^{P_d, \mathcal{O}(1)} \cong \mathbb{P}H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))^\vee \cong \mathbb{P}^{\binom{n+d}{d}} - 1.$$

Curves in $\mathbb{P}^3$ Let $C \hookrightarrow \mathbb{P}^n$ be a smooth curve of genus g and degree d. Let us compute its Hilbert polynomial

$$\begin{aligned} P_C(m) &= \chi(\mathcal{O}_C(m)) \\ &= \int_C \mathrm{char}(\mathcal{O}_C(m)) \cdot Td(C) \\ &= \int_c \exp(mH) \cdot (1 + \frac{1}{2} c_1(T_c)) \\ &= \int_C (1 + mH)(1 + \frac{1}{2}(2 - 2g)) \\ &= dm + (1 - g) \end{aligned}$$

Example. For $d = 1, g = 0$, i.e. a line, $\mathrm{Hilb}_{\mathbb{P}^n}^{m+1} \cong \mathbb{G}(1, n)$, a Grassmannian smooth of dimension $2n - 2$.

From now on we set $n = 3$. As an example, for a line plus a point, we have $P(m) = m + 2$. We expect

$$\dim \mathrm{Hilb}_{\mathbb{P}^3}^{m+2} = \dim \mathbb{G}(1, 3) + \dim \mathbb{P}^3 = 4 + 3 = 7.$$

We have to be more careful. There is an open locus of $\mathbb{P}^3 \times \mathbb{G}(1, 3)$ or "point not lying on the line". The complement is precisely the universal line $\mathcal{L}$ over $\mathbb{G}(1, 3)$.

Example (twisted cubic)**.** The degree 2 Veronese $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$ up to a change of coordinate. $P(m) = 3m + 1$. Let V be the closure of twisted cubics in $\mathrm{Hilb}_{\mathbb{P}^3}^{3m+1}$.

$$\dim V = 4 \cdot h^0(\mathcal{O}_{\mathbb{P}^1}(3)) - \dim \mathbb{C}^\times - \dim \mathrm{Aut}(\mathbb{P}^1) = 4 \cdot 4 - 1 - 3 = 12.$$

Let V' be the closure of an elliptic curve and a disjoint point

Description of Hilbert scheme of 2 points As a set

Description of $\mathrm{Quot}_{\mathbb{A}^d}(\mathcal{O}^{\oplus r}, n)$ Fix $d, r \geq 1, n \geq 0$. Set $R = \mathbb{C}[x_1, \ldots, x_d] = \Gamma(\mathcal{O}_{\mathbb{A}^d})$. Then a quotient $\mathcal{O}^{\oplus r} \twoheadrightarrow T$ is described by

1. an n-dimension $\mathbb{C}$-vector space T,
2. an R-module structure on T
3. that is induced by a surjection $R^{\oplus r} \twoheadrightarrow T$.

In the linear algebra language,

1. fix $V \cong \mathbb{C}^n$ (we will get rid of the choice later).
2. Ring homomorphism $\varphi : R \to \mathrm{End}_{\mathbb{C}}(V)$, i.e. d elements $A_1, \ldots, A_d \in \mathrm{End}_{\mathbb{C}}(V)$.
3. $\tau : R^{\oplus r} \to V$ is specified by r vectors $v_1, \ldots, v_r \in V$. τ surjective means that the set
$$\{A_1^{a_1} \cdots A_d^{ad} \cdot v_i\}$$
for $a_1, \ldots, a_d \geq 0$ span V. So we must look at the GIT quotient
$$\{(A_1, \ldots, A_d, v_1, \ldots, v_r) : \text{ spanning } V\} / \mathrm{GL}_n$$
where GL_n acts via
$$g \cdot (A_1, \ldots, A_d, v_1, \ldots, v_r) = (gA_1g^{-1}, \ldots, gA_dg^{-1}, gv_1, \ldots, gv_r).$$
It is an exerise that the spanning condition ensures that the GL_n-action is free. Thus we get a smooth quasiprojective of dimension $(d-1)n^2 + rn$. The resulting scheme is called noncommutative Quot scheme.

   We need
$$\mathrm{Quot}_{\mathbb{A}^d}(\mathcal{O}^{\oplus r,n}) = \{[A_1 \ldots, A_d, v_1, \cdots, v_r] \in nc\,\mathrm{Quot} : [A_i, A_j] = 0\}$$
which is a closed subscheme.

   Note that for $n = 1$, the commutativity condition is free so they coincide.

Import special case: $d = 3$.
...

2 Equivariant cohomology

Motivation: to compute stuff via torus localisation. The Atiyah-Bott localisation formula tells us that given a so-called equivariant class, we can compute its integral by considering only the fixed locus.

Let G be a Hausdorff topological group. We can form the functor

$$\begin{aligned} P_G : \mathbf{HTop}^{\mathrm{op}} &\to \mathbf{Set} \\ S &\mapsto \{\text{principal } G\text{-bundles}\}/\text{homotopy} \end{aligned}$$

By Brown representability P_G can be represented by an object $(BG, \eta_G \in P_G(BG)$. Milnor showed that there exists a universal (in the topological sense, i.e. the total space EG is contractible. This implies "universal" in the sense of category theory) principal G-bundle $EG \to BG$ such that G acts freely on EG (we assume on the right). Thus η_G contains one such representative, that we denote $EG \to BG = EG/G$. BG is called the *classifying space* for principal G-bundles. Note that it is well-defined up to homotopy.

Example.

G	$EG \to BG$
$\{e\}$	$* \to *$
$\mathbb{R}$	$\mathbb{R} \to *$
$\mathbb{Z}^n$	$\mathbb{R}^n \to (S^1)^n, (y_1, \ldots, y_n) \mapsto (e^{\pi i y_1}, \ldots, e^{\pi i y_n})$
$\mathbb{Z}/2\mathbb{Z}$	$S^\infty \to \mathbb{RP}^\infty$
$\mathbb{C}^*$	$\mathbb{C}^\infty \setminus 0 \to \mathbb{P}^\infty$, i.e. $\varinjlim(\mathbb{C}^m \setminus 0 \to \mathbb{P}^{m-1})$
S^1	$S^\infty \hookrightarrow \mathbb{C}^\infty \setminus 0 \to \mathbb{P}^\infty$, i.e. $\varinjlim(S^{2m-1} \hookrightarrow \mathbb{C}^m \setminus 0 \to \mathbb{P}^{m-1})$
$(\mathbb{C}^*)^n$	$(\mathbb{C}^\infty \setminus 0)^n \to (\mathbb{P}^\infty)^n$
$\mathrm{GL}_n(\mathbb{C})$	$F_n(\mathbb{C}^\infty) \to G(n, \mathbb{C}^\infty)$, orthonormal n-frame $\mapsto$ its span

Note that $\mathbb{C}^* \simeq S^1$ and $\mathbb{C}^\infty \setminus 0 \simeq S^\infty$. Also $EG \to BG$ is multiplicative.

Remark. Note that this formulation is not yet algebraic geometry-ready: in most cases $\dim EG$ is infinite. But we have to use an infinite dimensional space since for example $\mathbb{C}^m \setminus 0 \to \mathbb{P}^{m-1}$ is not contractible, so we need to take direct limit. We will address this problem in the next section by using *approximation spaces* $E_m \to B_m$.

Definition (Borel space, equivariant cohomology)**.** For a topological space X with a left G-action, we define the *Borel space* to be

$$EG \times^G X = \frac{EG \times X}{(e \cdot g, x) \sim (e, g \cdot x)}.$$

We define the *equivariant cohomology* of X to be the singular cohomology of the Borel space, i.e.

$$H^*_G(X) = H^*(EG \times^G X).$$

In particular we define

$$H_G^* = H_G^*(*) = H^*(BG).$$

Example.

1. For G trivial we recover singular cohomology.

2. Let $X = G = \mathbb{C}^*$ with left multiplication action. Then

$$H_G^1(X) = H^1(\mathbb{C}^\infty \setminus 0 \times^{\mathbb{C}^*} C^*) = H^1(\mathbb{C}^\times \setminus 0) = 0.$$

   This shows that in general we cannot recover H^* from H_G^*.

3. In general $H_G^*(X) \neq H^*(X/G)$ unless the G-action on X is free. To see this do the following exercise: compute the equivariant cohomology of the action of $G = S^1$ on $X = S^2 \subseteq \mathbb{R}^3$ by rotation along z-axis, and compare it with the quotient X/S^1.

Lemma 2.1. *$H_G^*(X)$ is well-defined.*

Proof. Suppose $EG \to BG, FG \to BG$ are two universal principal G-bundles. Let $Y = (EG \times FG \times X)/G$. Then we have two fibre bundles

$$EG \hookrightarrow Y \to FG \times^G X$$
$$FG \hookrightarrow Y \to EG \times^G X$$

so by the homotopy long exact sequence

$$\cdots \longrightarrow \underset{\substack{\| \\ 0}}{\pi_n(FG)} \longrightarrow \pi_n(Y) \longrightarrow \pi_n(EG \times^G X) \longrightarrow \underset{\substack{\| \\ 0}}{\pi_{n-1}(FG)} \longrightarrow \cdots$$

so $Y \simeq EG \times^G X$. Same for $FG \times^G X$. □

2.1 H_G^*-module structure

The Borel space can be regarded as a fibration

$$X \hookrightarrow EG \times^G X \xrightarrow{p} BG$$

so $H_G^*(X)$ is endowed with a H_G^*-module structure via p^*.

If G acts on X trivially then $EG \times^G X = BG \times X$ so by Künneth formula (assumptions?)

$$H_G^*(X) \cong H_G^* \otimes H^*(X)$$

As an example if $X = Y^G \subseteq Y$ then

$$H_G^*(Y^G) \cong H_G^* \otimes H^*(Y^G).$$

Definition (equivariantly formal)**.** If G acts on X such that $H_G^*(X) \cong H_G^* \otimes H^*(X)$ then we say the action is *equivariantly formal*.

Remark. If $K \subseteq G$ is a closed subgroup then $EK = EG \to EG/K = BK$ is the classifying space for K. Suppose G acts on X. We can form the orbit space $G \times^K X$, which has a left G-action. Then

$$H_G^*(G \times^K X) = H^*(EG \times^G G \times^K X) = H^*(EG \times^K X) = H_K^*(X).$$

In particular

$$H_G^*(G/K) \cong H_K^*.$$

On the other hand if G/K is contractible then we have a fibration

$$G/K \hookrightarrow EG \times^K X \to EG \times^G X$$

so $H_G * (X) \cong H_K^*(X)$.

Example.

- Let $G = \mathbb{T} = (\mathbb{C}^*)^n$ so

$$H_{\mathbb{T}}^* = H^*((\mathbb{P}^\infty)^n) = \mathbb{Z}[s_1, \dots, s_n]$$

  where s_i has degree 2 and is defined to be the first Chern class of the pullback along the ith projection of $\mathcal{O}_{\mathbb{P}^\infty}(-1)$.

- Let $G = \mathrm{GL}_n(\mathbb{C})$. Then

$$H_{\mathrm{GL}_n(\mathbb{C})}^* = H^*(G(n, \mathbb{C}^\infty) = \mathbb{Z}[e_1, \dots, e_n]$$

  where $e_i = c_i = c_i(\mathcal{S})$, $\mathcal{S}$ the universal rank n bundle on $G(n, \mathbb{C}^\infty)$.

Let G be a Lie ..

$$\int_X \tilde{\alpha} = p_* \tilde{\alpha} \in H_G^*$$

is what we call an *equivariant integral*. So via the relation

$$\in_X \alpha = b^* \int_X \tilde{\alpha}$$

we see that we can compute ordinary integrals using equivariant integrals.

2.2 Approximation spaces

Let X be a complex variety and G an algebraic group. We are mainly interested in $G = \mathbb{C}^*$, for whiich EG infinite dimensional. Now we fix this.

Lemma 2.2. *Let $\{E_m\}_{m \geq 0}$ be connected spaces with free right G-actions.*

Proof. This is another application of double fibration. Let $E = EG$. Let G act diagonally on $E \times E_m$. Then we have

Leray-Hirsch □

Example. $G = \mathbb{T} = (\mathbb{C}^*)^n$. Let $E_m = (\mathbb{C}^m \setminus -)^n \to Bm = (\mathbb{P}^{m-1})^n$. As $\mathbb{C}^m \setminus 0 \simeq S^{2m-1}$, we can pick $k(m) = n(2m-1)$ to achieve $\pi_i(E_m) = 0$ for $0 < i < n(2m-1)$. Thus

$$H^i_{\mathbb{T}} \cong H^i((\mathbb{P}(m-1))^n)$$

for $i < n(2m-1)$.

2.3 Equivariant vector bundle

Lift the action on X to an action on E. Such as choice of lift is called an *equivariant structure* on E.

Example. If X is a point then an equivariant structure is exactly a representation $G \to \mathrm{GL}(E)$.

We can form a new vector bundle

$$V_E = EG \times_G E \to EG \times_G X$$

of rank r. Then we define the *equivariant Chern class* to be

$$c_i^G(E) = c_i(V_E) \in H^{2i}(EG \times_G X) = H^{2i}_G(X).$$

They can also be computed via approximation spaces.
Lecture 4

2.4 Self-intersection formula and pushforward formula

Recall that $f : X \to Y$ is a map between closed connected oriented smooth manifolds of dimension n and m, we can use Poincaré duality to define *pushforward in cohomology*, namely

$$\begin{array}{ccc} H^p(X) & \overset{f_*}{\dashrightarrow} & H^{p+d}(Y) \\ \downarrow \mathrm{pd} & & \downarrow \mathrm{pd} \\ H^{n-p}(X) & \xrightarrow{f_*} & H_{n-p}(Y) \end{array}$$

To get an equivariant version of this, we need to use approximation space (since there is no Poincaré duality for infinite dimensional spaces).

Let $\pi : E \to X$ be a vector bundle of rank r with orientation class $\eta \in H^r(E, E \setminus X)$, corresponding to $1 \in H^0(X)$ under Thom isomorphism. the image of η under

$$H^r(E, E \setminus X) \to H^r(E) \to H^r(X)$$

is the Euler class $e(E)$.

Let $X \to Y$ be a closed embedding of codimenion $d = m - n$. Assume the normal bundle $N = N_{X/Y}$ has an orientation that is compatible with f. Then the cohomological pushforward is the composition

$$h^p(X) \to H^{p+d}(N, N \setminus X) \to H^{p+d}(Y, Y \setminus X) \to H^{p+d}(Y)$$

where the second arrow is by tubular neighbourhood theorem. In particular for $p = 1$ and compose with $f^* : H^d(Y) \to H^d(X)$, we get the *self-intersection formula*

$$f^* f_* 1 = e(N).$$

Informually, the self-intersection $X \cdot X$ means slightly deform X to X', and compute $X \cdot X'$ where the intersection is transerse. To do so we treat X as the zero section of the normal bundle N, and take X' to be a generic section s of N. Then the Euler class $e(N)$ can be interpreted as the Poincaré dual of zeros of s.

Note the formula also works for regular embeddings of varieties.

Now suppose G is a compact Lie group and $f : X \to Y$ a G-equivariant map of closed oriented manifolds. Since G is compact, we can choose the approximation spaces E_i to be compact, and thus the Borel spaces $X^i_G = E_i \times^G X$ and Y^i_G are compact, so we can define pushforward $f^G_* : H^p_G(X) \to H^{p+d}_G(Y)$. They are compatible with i so they define a map $f^G_* : H^*_G(X) \to H^*_G(Y)$.

From now on we let $G = \mathbb{T} = (\mathbb{C}^*)^r$. Denote by

$$\widehat{\mathbb{T}} = \mathrm{Hom}_{\mathbb{Z}}(\mathbb{T}, C^*) \cong \mathbb{Z}^r$$

be the character lattice. Let B be a $\mathbb{T}$-representation, then it can be decomposed as

$$V = \bigoplus_{\chi \in \widehat{\mathbb{T}}} V_\chi.$$

We call $V^{\text{fix}} = V_0$ the *fixed part*, and the rest $V^{\text{mov}} = \bigoplus_{\chi \neq 0} V_\chi$ the *moving part*.

Let X be a smooth variety on which $\mathbb{T}$ acts trivially. (Assume X is equivariantly formal). Let $E \to X$ be a $\mathbb{T}$ equivariant vector bundle. Then E also splits as eigen-subbundles

$$E = \bigoplus_{\chi \in \widehat{\mathbb{T}}} E_\chi.$$

The Borel space $E\mathbb{T} \times^{\mathbb{T}} X = B\mathbb{T} \times X$, and one can show $V_{E_\chi} \cong V_\chi \otimes E_\chi$. Using the formula for the Chern class of a vector bundle and a line bundle, we find

$$c^{\mathbb{T}}_i(E_\chi) =$$

Theorem 2.3 (Iversen, Fogarty)**.** *Let X be a smooth variety, then $X^{\mathbb{T}}$ is also smooth.*

Let $F \subseteq X^{\mathbb{T}}$ be a component and $x \in F$. Then $\mathbb{T}$ acts naturally on the vector space $N_{F/X,x}$. Then the fixed part $(N_{F/X}, x)^{\mathbb{T}} = 0$:

Thus $e^{\mathbb{T}}(N_x) \neq 0$ (since it is a product of nonzero weights). In fact, if $N = \bigoplus_\chi N_\chi$, then $e^{\mathbb{T}}(N)$ becomes invertible in $H^{2d}_{\mathbb{T}}(F)[\chi^{-1}]$.

Now suppose $E \to X$ is a $\mathbb{T}$-equivariant bundle of rank r. Let $F \subseteq X^{\mathbb{T}}$ be a component and suppose F is equivariant formal. $E|_F$ splits into eigen-subbundles as before and we can express the equivariant Chern class...

But $H^{2k} = 0$ for $k > \dim F$, so elements of $H^{>0}(F)$ are nilpotent in $H^*_{\mathbb{T}}(F)$. Thus $c^{\mathbb{T}}_i(E_{F,\chi})$ is a unit if and only if χ is a unit.

Proposition 2.4. *Let $F \subseteq X^{\mathbb{T}}$*

2.5 Localisation formula

... Let $\Lambda_{\mathbb{T}} = \mathrm{Frac}(H^*_{\mathbb{T}}) = \mathbb{Q}(s_1, \dots, s_n)$.

Theorem 2.5 (Atiyah-Bott locaisation)**.** *$\iota : X^{\mathbb{T}} \to X$ induces an isomorphism*

$$\iota_*^{loc} : H^*_{\mathbb{T}}(X^{\mathbb{T}}) \otimes_{H^*_{\mathbb{T}}} \Lambda_{\mathbb{T}} \to$$

with inverse

$$\sum_F \frac{\iota_F^* \psi}{e^{\mathbb{T}}(N_{F/X})} \hookleftarrow \psi$$

What this means is that a class ψ can be written as

$$\psi = \iota_* \sum_F \frac{\iota_F^* \psi}{e^{\mathbb{T}}(N_{F/X})}$$

Since X is compact, we have

$$\int_X \psi = \sum_F \int_X \frac{\iota_F^* \psi}{e^{\mathbb{T}}(N_{F/X})}$$

Now we prove the theorem in the algebraic setting

Theorem 2.6. *Let X be a msooth variety and suppose $X^{\mathbb{T}} \subseteq X$ is finite. Let $e = \prod_{p \in X^{\mathbb{T}}} e^{\mathbb{T}}(T_p X)$. Let $S \subseteq H^*_{\mathbb{T}} \backslash 0$ be a multiplicative subset containing e. Then*

1. *$S^{-1}\iota^*$ is surjective,*
2. *If $H^*_{\mathbb{T}}(X)$ is a free $H^*_{\mathbb{T}}$-module of rank $r \leq |X^{\mathbb{T}}|$ then $r = |X^{\mathbb{T}}|$ and $S^{-1}\iota^*$ is an isomorphism.*

Proof. 1. The composition

$$S^{-1}(\iota^* \iota_*) :,$$

by self-intersection formula, is diagonal with entries $e^{\mathbb{T}}(T_p X)$, so $\det(S^{-1}(\iota^* \iota_*)) = e$. But e is invertible after inverting S, so the composition is surjective and so $S^{-1}\iota^*$ is surjective.

2. Rank of free module is

□

The assumption that $X^{\mathbb{T}}$ is finite seems to strong

Fact: if X is smooth and projective, $X^{\mathbb{T}}$ is finite then $H^*_{\mathbb{T}}(X)$ is free over $H^*_{\mathbb{T}}$.

Corollary 2.7 (integration formula)**.** *Suppose X is smooth projective and*

$X^{\mathbb{T}}$ is finite then for $\psi \in H^*_{\mathbb{T}}(X) \otimes \Lambda_{\mathbb{T}}$,

$$\int_X \psi = \sum_{p \in X^{\mathbb{T}}}$$

Another application

Corollary 2.8. *Suppose Y is a quasiprojective $\mathbb{C}$-scheme of finite type. Then*

$$\chi(Y) = \chi(Y^{\mathbb{T}}).$$

Proof. We assume Y is smooth projective. Then $F = Y^{\mathbb{T}}$ is also smooth. □

Example. $\chi(G(k,n)) = \binom{n}{k}$.

Lecture 5

3 Application of Atiyah-Bott localisation

3.1 Lines on hypersurfaces

Let $Y \subseteq \mathbb{P}^n$ be a general degree d hypersurface, i.e. $Y = \{f = 0\}$ where $f \in H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(d))$. Let $\ell \subseteq \mathbb{P}^n$ be a line. We say ℓ is contained in Y, written $\ell \subseteq Y$, if under the restriction

$$H^0(\mathcal{O}_{\mathbb{P}^n}(d) \to \mathbb{H}^0(\ell, \mathcal{O}_\ell(d))$$

f is mapped to 0. We regard the set of lines $\{\ell : \ell \subseteq Y\}$ as a subset of the Grassmannian $\mathbb{G} = \mathbb{G}(1, n)$. We would like to define a scheme structure on this subset and show this is 0-dimensional. On $\mathbb{B}$ there is the tautological short exact sequence

$$0 \longrightarrow \mathcal{S} \longrightarrow \mathcal{O}_{\mathbb{G}} \otimes V^* \longrightarrow \mathcal{Q} \longrightarrow 0$$

and $\mathcal{S}|_\ell = ...$

We can make it more explicity:

$$\begin{array}{ccc} \mathcal{L} = \{(p, \ell) \in \mathbb{P}^n \times \mathbb{G} : p \in \ell\} & \xrightarrow{q} & \mathbb{P}^n \\ \downarrow{\scriptstyle \pi} & & \\ \mathbb{G} & & \end{array}$$

We define a sheaf

$$\mathcal{E}_d = \pi_* q^* \mathcal{O}_{\mathbb{P}^n}(d) \cong Sym^d \mathcal{S}^*$$

which is locally free of rank $d + 1$. Let E_d be its total space, which has fibre

$$E_d|_\ell = H^0(\ell, \mathcal{O}_\ell(d)).$$

...

$F(Y)$ the Fano scheme of lines in Y.

Since we assume f is general, τ_f is general so by classical charactertic class theory $F(Y)$ defines a cycle class

$$[F(Y)] = e(E_d) capproduct [\mathbb{G}] \in A_{\dim \mathbb{G} - \mathrm{rk}\, E_d}(\mathbb{G}).$$

$\dim \mathbb{G} = 2n - 2$ and $d + 1$..

- if $d > 2n - 3$ then no lines on Y,
- if $d < 2n - 3$ then infinitely many lines,
- if $d = 2n - 3$ then $\deg[F(Y)]$

If the answer is finite, when is it the actual number of lines? $\in_{\mathbb{G}} e(E_d)$ is the number of lines on Y as long as $F(Y)$ is *reduced*, which happens if $H^0(\ell, \mathcal{N}_{\ell/Y}) = 0$ for all $\ell \in F(Y)$. This holds for qunitic $Y \subseteq \mathbb{P}^4$ since

$$\mathcal{N}_{\ell/Y} =$$

Exercise. Show $H^0(\ell, \mathcal{N}_{\ell/Y}) = 0$ for the cubic surface.

Warming up: Let $\mathbb{T} = \mathbb{C}^*$ act with distinct weights $w_0, w_1, \ldots, w_a$ on

$$F_a = H^0(\mathcal{O}_{\mathbb{P}^1}(1))$$

The fixed points are

$$(\mathbb{P}^a)^{\mathbb{T}} = \{p_0, \ldots, p_a\}$$

It is easy to see the invariant lines are those connecting distinct p_i and p_j

4 Virtual classes and virtual localisation

Goal: apply torus localisation to singular schemes. We will discuss the non-equivariant version first, then the equivariant one.

4.1 Toy model

Suppose Y is a smooth variety of dimension d. Let $E = \operatorname{Spec}\operatorname{Sym}\mathcal{E}^*$ be a vector bundle on Y. Let $s : Y \to E$ be a section. The ideal sheaf of $X = Z(s) \hookrightarrow Y$ is given by

$$\mathcal{I} = \operatorname{im}(s^\vee : \mathcal{E}^* \to \mathcal{O}_Y),$$

giving a surjection $\sigma : \mathcal{E}|_X \twoheadrightarrow \mathcal{I}/\mathcal{I}^2$ which fits into a diagram

$$\begin{array}{ccc} \mathbb{E}_s & & [\mathcal{E}^*|_X \xrightarrow{\mathrm{d}\circ\sigma} \Omega_Y|_X] \\ \downarrow{\scriptstyle \varphi_s} & & \downarrow{\scriptstyle \sigma} \qquad\qquad \| \\ \mathbb{L}_X & & [\mathcal{I}/\mathcal{I}^2 \xrightarrow{\mathrm{d}} \Omega_Y|_X] \end{array}$$

We think of this as a morphism in $\mathbf{D}^{[-1,0]}(\mathbf{QCoh}(X))$, the truncated derived category. We call the top complex $\mathbb{E}_s$ and the bottom complex $\mathbb{L}_X$, which is the *truncated cotangent complex* of X.

Exercise. Show that $\mathbb{L}_X$ is well-defined, i.e. it does not depend on the embedding $X \hookrightarrow Y$ where Y is smooth.

The following definition is due to Behrend-Fantechi.

> **Definition** (perfect obstruction theory)**.** Let X be a scheme. A *perfect obstruction theory* on X is a morphism $\varphi : \mathbb{E} \to \mathbb{L}_X$ in $\mathbf{D}^{[-1,0]}(\mathbf{QCoh}(X))$, where $\mathbb{E}$ is perfect of perfect amplitude $[-1, 0]$, $h^0(\varphi)$ is an isomorphism and $h^{-1}(\varphi)$ is surjective.

Recall that a complex $\mathbb{E}$ is *perfect* of perfect amplitude $[a, b]$ if it is locally isomorphic to a complex

$$[E^a \to \cdots \to E^b]$$

of locally free sheaves of finite rank.

We see immediately that φ_s is a perfect obstruction theory. In fact every perfect obstruction theory looks locally like this.

Given a perfect obstruction theory, we define its *virtual dimension* to be $\operatorname{rk}\mathbb{E}$. In particular

$$\operatorname{vd}(\varphi_s : \mathbb{E}_s \to \mathbb{L}_X) = \dim Y - \operatorname{rk}\mathcal{E}.$$

Now we consider the special case of the "critical perfect obstruction theory": let $\mathcal{E} = \Omega_Y$. Take an element $f \in \Gamma(Y, \mathcal{O}_Y)$. It has differential $\mathrm{d}f \in H^0(Y, \Omega_Y)$. Let $X = Z(\mathrm{d}f) \subseteq Y$ be the critical locus. Then we have a perfect obstruction theory

$$\begin{array}{ccc} \Omega_Y^*|_X & \xrightarrow{\operatorname{Hess}(f)} & \Omega_Y|_X \\ \downarrow & & \| \\ \mathcal{I}/\mathcal{I}^2 & \xrightarrow{\mathrm{d}} & \Omega_Y|_X \end{array}$$

with virtual dimension 0. This is a key example of

Definition (symmetric perfect obstruction theory)**.** A perfect obstruction theory $(\mathbb{E}, \varphi)$ is called a *symmetric perfect obstruction theory* if there exists an isomorphism $\theta : \mathbb{E} \to \mathbb{E}^\vee[1]$ such that $\theta = \theta^\vee[1]$.

Example. We will use without proof the fact that

$$\mathrm{Quot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n) \subseteq \mathrm{ncQuot}_{\mathbb{A}^3}(\mathcal{O}^{\oplus r}, n)$$

is critical.

The general idea is that whenever there is a perfect obstruction theory, we can define a virtual fundamental class $[X]^{\mathrm{vir}} \in A_{\mathrm{vd}}(X)$. Recall that we have an inclusion of cones over X

$$C_{X/Y} \hookrightarrow N_{X/Y} \hookrightarrow E|_X$$

where $C_{X/Y}$ is pure of dimension $d = \dim Y$, and its pullback along $A_d(E|_X) \to A_{d-\mathrm{rk}\, E}(X)$ gives $[X]^{\mathrm{vir}}$. In fact, in intersection theory it is the same as $0^![Y]$ with respect to the cartesian diagram

$$\begin{array}{ccc} X & \xrightarrow{i} & Y \\ \big\downarrow & & \big\downarrow s \\ Y & \xrightarrow{0} & E \end{array}$$

Note by self-intersection formula we have

$$i_*[X]^{\mathrm{vir}} = i_* 0^![Y] = 0^* s_*[Y] = s^* s_*[Y] = e(E) \frown [Y].$$

If X is proper and has a perfect obstruction theory of virtual dimension 0 then it has a *virtual intersection number*

$$\#^{\mathrm{vir}}(X) := \int_{[X]^{\mathrm{vir}}} 1 = \deg_X [X]^{\mathrm{vir}} \in \mathbb{Z}.$$

We will use this for $X = \mathrm{Hilb}^n(A)$ where A is a smooth projective 3-fold, yielding the Donaldson-Thomas invariant of A.

Example. Suppose Y is smooth and $f = 0$. Then the critical locus is Y itself and $\mathbb{E}_{\mathrm{d}f} = [\Omega_Y^* \xrightarrow{0} \Omega_Y]$, so

$$[Y]^{\mathrm{vir}} = e(\Omega_Y) \frown [Y].$$

If Y is proper then

$$\#^{\mathrm{vir}}(Y) = \int_Y e(\Omega_Y) = (-1)^{\dim Y} \chi(Y).$$

Example. If Y is smooth then $\mathbb{L}_Y$ is isomorphic to the cotangent sheaf so we may take $\mathrm{id} : \mathbb{L}_Y \to \mathbb{L}_Y$ as the perfect obstruction theory. In this case we get

$$[Y]^{\mathrm{vir}} = [Y] \in A_{\dim Y}(Y).$$

4.2 Equivariant sheaves

Let X be a separated noetherian schemes over $\mathbb{C}$. Suppose $\sigma : G \times X \to X$ is an action by an algebraic group G.

Definition (equivariant sheaf)**.** An *equivariant coherent sheaf* is a pair $(\mathcal{F}, \theta)$ where $\mathcal{F}$ is a quasicoherent sheaf and $\theta : p_2^*\mathcal{F} \to \sigma^*\mathcal{F}$ is an isomorphism, plus the compatibility condition along the diagram

$$\begin{array}{ccc} G \times G \times X & \xrightarrow{m \times 1_X} & G \times X \\ \big\downarrow{\scriptstyle 1_G \times \sigma} & & \big\downarrow{\scriptstyle \sigma} \\ G \times X & \xrightarrow{\quad\sigma\quad} & X \end{array}$$

Exercise. Write down the commutative diagram and show that $\mathcal{O}_X$ and Ω_X are naturally G-equivariant.

A morphism of G-equivariant sheaves $(\mathcal{F}, \theta) \to (\mathcal{F}', \theta')$ is a morphism of sheaves $f : \mathcal{F} \to \mathcal{F}'$ such that the following diagram commute

$$\begin{array}{ccc} p_2^*\mathcal{F} & \xrightarrow{p_2^* f} & p_2^*\mathcal{F}' \\ \big\downarrow{\scriptstyle \theta} & & \big\downarrow{\scriptstyle \theta'} \\ \sigma^*\mathcal{F} & \xrightarrow{\sigma^* f} & \sigma^*\mathcal{F}' \end{array}$$

G-equivariant sheaves form an abelian category $\mathbf{QCoh}^G(X)$. In fact we have

$$\mathrm{Hom}_{\mathbf{QCoh}^G(X)}((\mathcal{F}, \theta), (\mathcal{F}', \theta')) = \mathrm{Hom}_{\mathbf{QCoh}(X)}(\mathcal{F}, \mathcal{F}')^G \subseteq \mathrm{Hom}_{\mathbf{QCoh}(X)}(\mathcal{F}, \mathcal{F}')$$

where given a morphism of sheaves $f : \mathcal{F} \to \mathcal{F}'$, we define an action of $g \in G$ as follow: θ restricts to $\theta_g = \theta|_{\{g\} \times X} : \mathcal{F} \to g^*\mathcal{F}$. Then we define $g \cdot f$ vis the following diagram

$$\begin{array}{ccc} \mathcal{F} & \overset{g \cdot f}{\dashrightarrow} & \mathcal{F}' \\ \big\downarrow{\scriptstyle \theta_g} & & \big\uparrow{\scriptstyle (\theta'_g)^{-1}} \\ g^*\mathcal{F} & \longrightarrow & g^*\mathcal{F}' \end{array}$$

There is a forgetful functor $\mathbf{QCoh}^G(X) \to \mathbf{QCoh}(X)$ which induces a functor Φ between the respective derived categories. One way to see Φ is to identify

$$\mathbf{D}(\mathbf{QCoh}^G(X)) = \mathbf{D}(\mathbf{QCoh}[X/G])$$

of the stack quotient $[X/G]$ and pullback via the stack morphism $X \to [X/G]$.

Definition (G-equivariant perfect obstruction theory)**.** A *G-equivariant perfect obstruction theory* is a choice of lift of Φ of a perfect obstruction theory $\mathbb{E} \to \mathbb{L}_X$.

Classical localisation (Atiayh-Bott)	virtual localisation (Graber-Pandharipande)
X smooth with $\mathbb{T}$-action, $\iota : X^{\mathbb{T}} \hookrightarrow X$	$\mathbb{T}$-action on X with POT $\mathbb{E} \to \mathbb{L}_X$
$[X] = \iota_* \sum_{X_i \subseteq X^{\mathbb{T}}} \frac{[X_i]}{e^{\mathbb{T}}(N_{X_i/X})}$ in $H^*_{\mathbb{T}}(X) \otimes \Lambda_{\mathbb{T}}$	$[X]^{\mathrm{vir}} = \iota_* \sum_{X_i \subseteq X^{\mathbb{T}}} \frac{[X_i]^{\mathrm{vir}}}{e^{\mathbb{T}}(N^{\mathrm{vir}}_{X_i/X})}$ in $H^*_{\mathbb{T}}(X) \otimes_{H_{\mathbb{T}}} \Lambda_{\mathbb{T}}$

Why does it live in the ring? In the toy model, suppose Y is smooth with a $\mathbb{T}$-action. Let E be a $\mathbb{T}$-equivariant vector bundle and $s \in H^0(Y, E)^{\mathbb{T}}$. Let $X = Z(s) \hookrightarrow Y$ be the $\mathbb{T}$-equivariant embedding. Taking pullback via a section we get $[X]^{\mathrm{vir}}$ an equivariant class.

We have to check $[X_i]^{\mathrm{vir}}$ exists. As for the equivariant normal bundle, we define T_X^{vir} to be the derived dual $\mathbb{E}^\vee$. For $X_i \subseteq X^{\mathbb{T}}$, we define

$$N^{\mathrm{vir}} = \text{moving part of } T_X^{\mathrm{vir}}|_{X_i}.$$

Why does X_i have a virtual fundamental class? We assume there is an equivariant embedding $X \hookrightarrow Y$ into a smooth Y. $Y^{\mathbb{T}}$ has a stratification into nonsingular Y_i. Then $X^{\mathbb{T}} = X \cap Y^{\mathbb{T}}$ has a stratification into X_i's, which might be singular. We need to show X_i has a perfect obstruction theory.

Step 1: by Fogarty, $\Omega_Y|_{Y_i}^{\mathrm{fix}} \cong \Omega_{Y_i}$. Then via the diagram

$$\begin{array}{ccc} X_i & \longrightarrow & Y_i \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

$\Omega_X|_{X_i}^{\mathrm{fix}} \cong \Omega_{X_i}$.

Step 2: exercise: suppose $\psi : A^\bullet \to B^\bullet$ is a morphism of complexes. Then $h^0(\psi)$ is an isomorphism and $h^{-1}(\psi)$ is onto if and only if the mapping cone is exact on the right:

$$A^{-1} \oplus B^{-1} \longrightarrow A^0 \oplus B^{-1} \longrightarrow B^0 \longrightarrow 0$$

Step 3: $\varphi : \mathbb{E} \to \mathbb{L}_X$ gives rise to

$$\mathbb{E}_i := \mathbb{E}|_{X_i} \xrightarrow{\varphi_i} \mathbb{L}_X|_{X_i} \to \mathbb{L}_{X_i}$$

Step 4: Taking the fixed parts gives a perfect obstruction theory

$$\mathbb{E}_i^{\mathrm{fix}} \to \mathbb{L}_{X_i}.$$

We only need to show the induced map is an isomorphism in degree 0 and surjection in degree 1. We check this for φ_i^{fix} and δ_i. Since $- \otimes_{\mathcal{O}_X} \mathcal{O}_{X_i}$ is right exact, by the characterisation in step 2 ψ_i satisfies the conditions. Since taking fixed part of a torus action is exact, this is done for φ_i^{fix}.

For δ_i^{fix}, we write down the cotangent complex for $X \hookrightarrow Y$ and $X_i \hookrightarrow Y_i$.

$$\begin{array}{ccc} (\mathcal{I}/\mathcal{I}^2)|_{X_i}^{\mathrm{fix}} & \longrightarrow & (\Omega_Y|_X)|_{X_i}^{\mathrm{fix}} \\ \downarrow & & \\ \mathcal{I}_i/\mathcal{I}_i^2 & \longrightarrow & \Omega_{Y_i}|_{X_i} \end{array}$$

Again as taking fixed part is exact,

$$(\Omega_Y|_X)|_{X_i}^{\mathrm{vir}} = \Omega_Y|_{X_i}^{\mathrm{fix}}.$$

Since $\Omega_Y|_{Y_i}^{\mathrm{fix}} \cong \Omega_{Y_i}$, the right arrow is an isomorphism. Similarly the left arrow is a surjection. Thus we have isomorphism in degree 0 and surjection in degree -1.

Missed a lecture 22/04/21

4.3 equivariance of perfect obstruction theory on H

Let X be a toric 3-fold and $H = \mathrm{Hilb}^n(X)$. Let $\sigma_X : \mathbb{T} \times X \to X$ be the toric action. We first lift the action to H. On $X \times H$ there is the universal short exact sequence

$$0 \longrightarrow J \longrightarrow \mathcal{O} \xrightarrow{\xi} \mathcal{O}_Z \longrightarrow 0$$

where $Z \subseteq X \times H$ is the unviersal subscheme. Pullback along $\sigma_X \times 1_H : \mathbb{T} \times X \times H \to X \times H$ to get

$$(\sigma_X \times 1_H)^*(\xi) \in \mathrm{Hilb}^n(X)(\mathbb{T} \times H)$$

which corresponds to an action $\sigma_H : \mathbb{T} \times H \to H$.

Define

$$\varphi : \mathbb{T} \times X \times H \to X \times H$$
$$(t, x, z) \mapsto (\sigma_X(t, x), \sigma_H(t^{-1}, z))$$

then we have a commutative diagram

$$\begin{array}{ccc} \mathbb{T} \times X \times H & \xrightarrow{\varphi} & X \times H \\ \downarrow{p_{12}} & & \downarrow{q} \\ \mathbb{T} \times X & \xrightarrow{\xi_X} & X \end{array}$$

Then

$$\mathcal{O}_{\mathbb{T}\times X \times H} = p_{12}^* p_2^* \mathcal{O}_X \xrightarrow{p_{12}^*\theta} p_{12}^* \sigma_X^* \mathcal{O}_X = \varphi^* q^* \mathcal{O}_X = \varphi^* \mathcal{O}_{X\times H} \xrightarrow{\varphi^*\xi} \varphi^* \mathcal{O}_Z$$

This is a surjection so gives a point in the hilbert scehme ?? $T \times H \to H$. This map is in fact projection to the second factor since

$$\varphi^* \mathcal{O}_Z|_{\{t\}\times X \times \{[Z]\}} = t \cdot \mathcal{O}_{Z\cdot t^{-1}} = \mathcal{O}_Z.$$

This gives an isomrophism of surjections

$$\varphi^* \xi \xrightarrow{\cong} (\mathbb{T} \times X \times H \to X \times H)^* \xi.$$

Thus $J \to \mathcal{O}_Z$ is also $\mathbb{T}$-equivariant. Thus

$$\mathbb{H} = RHom(J, J)_0 \to RHOm(J, J) \xrightarrow{\mathrm{tr}}\!\!\to \mathcal{O}$$

is also equivariant.

To conclude, recall that the perfect obstruction theory we have constructed is the image of Atiyah class $At(J) \in \mathrm{Ext}^1(J, J \otimes \mathbb{L}_{X\times H})$ in $\mathrm{Ext}^1(p^*\mathbb{L}_h) \cong \mathrm{Ext}^{-2}\, Rp_*(\mathbb{H} \otimes \omega_p), \mathbb{L}_H)$. Check

- the Atiyah class in $\mathbb{T}$-equivariant,
- Grothendieck duality preserves equivariance,
- $(-)^{\mathbb{T}}$ corresponds to a morphism in $\mathbf{D}(\mathbf{QCoh}^{\mathbb{T}}(H))$.

Definition (Donaldson-Thomas invariant)**.** Let X be a smooth projective 3-fold, n a nonnegative integer. Then the nth *Donaldson-Thomas invariant* is

$$\mathrm{DT}_n^X = \int_{[\mathrm{Hilb}^n X]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

We define a generating function

$$\mathrm{DT}_X(q) = \sum_n \mathrm{DT}_n^X \cdot q^n$$

.

Remark. If X is toric then DT_n^X can be computed equivariantly.

We want to also define DT invariant for Calabi-Yau 3-folds. For this we need to shift our attention to quasiprojective varieties, whose Hilbert scheme is no longer proper. Nevertheless the torus fixed points are still proper.

From now on X is a smooth quasiprojective 3-fold. Then

$$X = \bigcup_{\alpha \in \Delta(X)} U_\alpha$$

where $U_\alpha \cong \mathbb{A}^3$ with $\mathbb{T} = (\mathbb{C}^*)^3$ acting on $\Gamma(U_\alpha) = \mathbb{C}[x_1^\alpha, x_2^\alpha, x_3^\alpha]$ via

$$(t_1, t_2, t_3) \cdot (x_i^\alpha) = (t_i x_i^\alpha).$$

For a fixed α, we have a unique $\mathbb{T}$-fixed point in U_α, which we call $X^{\mathbb{T}}$.

Exercise. Show

$$\mathrm{Hilb}^n(X)^{\mathbb{T}} = \coprod_{\sum_{\alpha \in \Delta(X)} n_\alpha = n} \prod_{\alpha \in \Delta(X)} \mathrm{Hilb}^{n_\alpha}(U_\alpha)^{\mathbb{T}}$$

But we know $U_\alpha \cong \mathbb{A}^3$ so $\mathrm{Hilb}^{n_\alpha}(U_\alpha)^{\mathbb{T}}$ correpond to monomial ideals or equivalently, plane partitions.

Let $\iota : \mathrm{Hilb}^n(X)^{\mathbb{T}} \hookrightarrow \mathrm{Hilb}^n(X)$ be the fixed locus. By virtual localisation formula,

$$[\mathrm{Hilb}^n(X)]^{\mathrm{vir}} = \iota_* \sum_{\mathcal{I}_Z \in \mathrm{Hilb}^n(X)^{\mathbb{T}}} \frac{[S(\mathcal{I}_Z)]^{\mathrm{vir}}}{e^{\mathbb{T}}(N_Z^{\mathrm{vir}})}$$

where $S(\mathcal{I}_Z) \hookrightarrow \mathrm{Hilb}^n(X)^{\mathbb{T}}$ is the largest subscheme supported at $\{\mathcal{I}_Z\}$.

It is a fact that $\mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{I}_Z)^{\mathbb{T}} = \mathrm{Ext}^2(\mathcal{I}_Z, \mathcal{I}_Z)^{\mathbb{T}} = 0$, so $S(\mathcal{I}_Z)$ is reduced and $[S(\mathcal{I}_Z)]^{\mathrm{vir}} = [*]$, the genuine fundamental class of a point. Thus we have reduced the computation to

$$[\mathrm{Hilb}^n(X)]^{\mathrm{vir}} = \iota_* \sum_{\mathcal{I}_Z} e^{\mathbb{T}}(-N_Z^{\mathrm{vir}}).$$

Recall

$$N_Z^{\mathrm{vir}} = \mathbb{E}^\vee|_{\mathcal{I}_Z}^{\mathrm{mov}} = (\mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{I}_Z)_0 - \mathrm{Ext}^2(\mathcal{I}_Z, \mathcal{I}_Z)_0)^{\mathrm{mov}} = \mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{I}_Z) - \mathrm{Ext}^2(\mathcal{I}_Z, \mathcal{I}_Z)$$

Thus

$$[\mathrm{Hilb}^n(X)]^{\mathrm{vir}} = \iota_* \sum_{\mathcal{I}_Z} \frac{e^{\mathbb{T}}(\mathrm{Ext}^2(\mathcal{I}_Z, \mathcal{I}_Z))}{e^{\mathbb{T}}(\mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{I}_Z))}.$$

Definition. For X quasiprojective, we define the *Donaldson-Thomas invariant* to be

$$\mathrm{DT}_n^X = \int_{[\mathrm{Hilb}^n(X)]^{\mathrm{vir}}} 1 = \sum_{\mathcal{I}_Z} \int_{\{\mathcal{I}_Z\}} \frac{e^{\mathbb{T}}(\mathrm{Ext}^2(\mathcal{I}_Z, \mathcal{I}_Z))}{e^{\mathbb{T}}(\mathrm{Ext}^1(\mathcal{I}_Z, \mathcal{I}_Z))} \in \Lambda_{\mathbb{T}}.$$

interlude on K-theory For Y a toric variety, let $K_0^{\mathbb{T}}(Y) = K_0(\mathbf{Coh}^{\mathbb{T}}(Y))$. For example $K_0^{\mathbb{T}}(pt)$ is generated by three characteres. Have an isomorphism

$$\operatorname{tr} K_0^{\mathbb{T}}(pt) \to \mathbb{Z}[t^\mu : \mu \in \widehat{\mathbb{T}}].$$

Now for the standard action of $\mathbb{T} = (\mathbb{C}^*)^3$ on $\mathbb{A}^3$,

$$\operatorname{tr}(S_g) = \sum_{k_1,\ldots,k_g \geq 0} \prod_{i=1}^{g} t_i^{k_i} = \prod_{i=1}^{g} \frac{1}{1-t_i} \in \mathbb{Z}[[t_1, \ldots, t_g]] \subseteq \mathbb{Q}((t_1, \ldots, t_g)).$$

For $F, G \in K_0^{\mathbb{T}}(\mathbb{A}^g)$, define

$$\chi(F, G) = \sum_{i \geq 0} (-1)^i \operatorname{Ext}^i(F, G) \in K_0^{\mathbb{T}}(pt)$$

and let $\chi(F) = \chi(\mathcal{O}, F)$.

Some manipulation

$$(t_1 \cdots t_g) \operatorname{tr}(S_g) = \prod \frac{t_i}{1-t_i} = \prod \frac{1}{t_i^{-1} - 1} = (-1)^g \frac{1}{1 - t_i^{-1}} = (-1)^g \overline{\operatorname{tr}(S_g)}$$

where

$$t_i \mapsto t_i^{-1}$$

Lemma 4.1.

$$\chi(F, G) = \frac{\overline{\chi(F)} \chi(G)}{\overline{\chi}(\mathcal{O}_{\mathbb{A}^g})}.$$

Proof. Ingredients:

1. $\chi(F) = \chi(F|_0)\chi(\mathcal{O})$,
2. $\chi(F \otimes F'|_0) = \chi(F|_0 \otimes F'|_0) = \chi(F|_0)\chi(F'|_0)$,
3. Serre duality: $(-1)^g \overline{\chi(F', F)} = \chi(F, F' \otimes K_{\mathbb{A}^g})$. In particular setting $F' = \mathcal{O}$ we get

$$(-1)^g \overline{\chi(F)} = \chi(F^*, K_{\mathbb{A}^g}) = \chi(F^*) t_1 \cdots t_g.$$

Thus □

Back to perfect obstruction theory...

20/05/21

Recap: if X is a toric CY3 then

$$\sum \mathrm{DT}_n^X \cdot q^n = M(-q)^{\chi(X)}$$

where

$$M(q) = \prod_{m \geq 1} (1 - q^m)^{-m} = 1 + q + 3q^2 + 6q^3 + \ldots$$

4.4 DT/PT correspondence

Let Y be a smooth projective CY3 over $\mathbb{C}$. Fix a homology class $\beta \in H_2(Y, \mathbb{Z})$ and an interger m. We construct two moduli spaces:

1. the DT moduli space

$$I_m(Y, \beta) = \{Z \hookrightarrow Y : [Z] = \beta, \chi(\mathcal{O}_Z) = m\}.$$

Note when $\beta = 0$ we get the Hilbert scheme of points $\mathrm{Hilb}^m(Y)$.

2. the PT moduli space of stable pairs

$$P_m(Y, \beta) = \{[\mathcal{O}_Y \xrightarrow{s} \mathcal{F}] : \mathcal{F} \text{ pure 1 dim}, \operatorname{coker} s \text{ 0 dim}, [\operatorname{supp} s] = \beta, \chi(\mathcal{F}) = m\}.$$

Every stable pair $J^\bullet[\mathcal{O}_Y \xrightarrow{s} \mathcal{F}]$ gives rise to an exact sequence

$$0 \longrightarrow \mathcal{I}_C \longrightarrow \mathcal{O}_Y \xrightarrow{\ s\ } \mathcal{F} \longrightarrow \mathcal{Q} \longrightarrow 0$$

By purity of $\mathcal{F}$, C is Cohen-Macaulay. Since $\dim \mathcal{Q} = 0$, $\operatorname{supp} Q$ is a collection of points on C.

For example let $i : C \hookrightarrow Y$ be a smooth curve. Let $D \subseteq C$ be an effective divisor. We can form

$$[\mathcal{O}_Y \twoheadrightarrow i_*\mathcal{O}_C \xrightarrow{i_* s_D} i_*\mathcal{O}_C(D)].$$

If $C \hookrightarrow Y$ is a smooth curve and is the only curve in class $\beta = [C]$ then all stable pairs are of this form. Thus $P_m(Y, \beta) \cong \mathrm{Sym}^{\#\mathrm{pt}}(C)$, where

$$\#\mathrm{pt} = \mathrm{length}(\mathcal{Q}) = \chi(\mathcal{I}_C) - \chi(\mathcal{O}_Y) + \chi(\mathcal{F}) = \chi(\mathcal{I}_C) - 0 + m = m - \chi(\mathcal{O}_C) = m + g - 1.$$

$P_m(Y, \beta)$ has symmetric POt and $[P_m(Y, \beta)]^{\mathrm{vir}} \in A_0(P_m(Y, \beta))$. Then we can form the generating function

$$\mathrm{PT}_\beta(q) = \sum_{m \in \mathbb{Z}} \mathrm{PT}_{m,\beta} \cdot q^m$$

where

$$\mathrm{PT}_{m,\beta} = \int_{[P_m(Y,\beta)]^{\mathrm{vir}}} 1 \in \mathbb{Z}.$$

Theorem 4.2 (DT/PT correspondence, Bridgeland, Toda)**.**

$$\mathrm{DT}_\beta(q) = M(-q)^{\chi(Y)} \cdot \mathrm{PT}_\beta(q) = \mathrm{DT}_0(q) \cdot \mathrm{PT}_\beta(q).$$

We will dedicate the rest of the course to computing somes terms in the generating function. From now on let Y be the total space $\pi : \mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1) \to \mathbb{P}^1$, the resolved conifold. Note Y is CY3 but not projective. Nevertheless it is toric and we can apply virtual techniques.

Let $C_0 \hookrightarrow Y$ be the 0 section of π. It is the only proper curve in Y. Call β its homology class. We can form $I_m(Y, d\beta), P_m(Y, d\beta)$ for $d \geq 1$. We will focus on $d = 1$ today.

By discussion above

$$P_m(Y,\beta) \cong \mathrm{Sym}^{m-1}(C_0) \cong \mathbb{P}^{m-1}$$

for all $m \geq 1$. This moduli space is nonsingular of dimension $m-1$. Since we know

$$[P_m(Y,\beta)]^{\mathrm{vir}} = e(\mathrm{Ob}) \frown [P_m(Y,\beta)] = e(\Omega_{P_m(Y,\beta)}) \frown [P_m(Y,\beta)]$$

by definition of virtual fundamental class and symmetry,

$$\mathrm{PT}_{m,\beta} = \int_{P_m(Y,\beta)} e(\Omega) = (-1)^{m-1} \int e(T_{\mathbb{P}_m(Y,\beta)}) = (-1)^{m-1} \cdot m$$

so

$$\mathrm{PT}_\beta(q) = \sum_{m\geq 1} (-1)^{m-1} m \cdot q^m = q \sum_{m \geq 0} (-1)^m (m+1) q^m = q(1+q)^{-2}$$

where for the last equality we used $\chi(\sum^m \mathbb{P}^1) = m+1$.

Our goal is to verify DT/PT formula for Y in the first few terms. We do RHS first. $\chi(Y) = \chi(\mathbb{P}^1) = 2$, so the first few terms of the first term is

$$M(-q)^2 = (1 - q + 3q - 6q^3 + \dots)^2 = 1 - 2a + 7q^2 - 18q^3 + \dots$$

so

$$M(-q)^2 \cdot \mathrm{PT}_\beta(q) = q - 4q^2 + 14q^3 + \dots.$$

Now for the LHS, we can generalise the construction of DT in the case of Hilbert scheme of points to $I_m(Y,\beta)$. It has a symmetric POT with finite $\mathbb{T}$ (and $\mathbb{T}_0$) fixed locus given as follows. We cover Y by two copies of $\mathbb{A}^3$, trivialising it over $\mathbb{P}^1 \setminus 0$ and $\mathbb{P}^1 \setminus \infty$. Then

$$I_m(Y,\beta)^{\mathbb{T}} = \{\mathcal{I}_Z : \mathcal{I}_Z|_{\mathbb{A}^3_0}, \mathcal{I}_Z|_{\mathbb{A}^3_\infty} \text{ monomial}\}$$

... $\mathrm{len}(Q) = -1 + m$.

Then the fixed locus is in bijection with the number of ways to stack $m-1$ boxes along (insert diagram).

Lemma 4.3 (Behrend-Bryan)**.**

$$(-1)^{m-d} = (-1)^{\dim T_p(T_m(Y,d\beta))}$$

for all $p \in I_m(Y,d\beta)^{\mathbb{T}}$*.*

We then define

$$\mathrm{DT}(d\beta, m) = (-1)^{m-d} \cdot \chi(I_m(Y,d\beta)).$$

Then by the box interpretation,

$$\mathrm{DT}_{\beta,\leq 0} = 0, \mathrm{DT}_{\beta,1} = 1, \mathrm{DT}_{\beta,2} = -4, \mathrm{DT}_{\beta,3} + 14.$$

$$\mathrm{DT}_\beta(q) = q - 4q^2 + 14q^3 + \dots$$

which indeed agrees.

Index