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Geometry and Mathematical Physics

Introduction to Topological Field Theories

April, 2021

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Contents

0 Introduction

Overview: a field theory for us is a theory of maps $D \to M$. $\mathrm{Map}(D, M)$ is an infinite dimensional space so the first thing we do is to cut out a finite dimensional locus $\mathcal{M}$, the critical locus of an (action) functional $S[\varphi_x]$. Then we can define and compute invariants (topological, enumerative etc) by using intersection theory on $\mathcal{M}$. There are many caveats: $\mathcal{M}$ is generically singular/non-compact.

Physicists' viewpoint: $\varphi : I \to M$ where $I = [t_a, t_b]$, called a one-diemsnional field theory, is used to study (quantum) mechanics. The trajectory of a particle on M is a map of this kind. The classical trajectory φ_{cl} is a minimum of the action functional $S[\varphi]$. In the quantum world, due to the uncertainty principle, we cannot determine precisely the trajectory of the particle. Rather we define a measure and compute the probability amplitude as the weighted integral

$$Z(x_b, t_b; x_a, t_a; \hbar) = \text{“} \int D\varphi \text{”} e^{iS[\varphi]/\hbar}$$

subject to boundary conditions $\varphi \in \mathrm{Map}(I, M), \varphi(t_a) = x_a, \varphi(t_b) = x_b$. Note as $\hbar \to 0$, Z peaked around S_{cl} which is the minimum of $S[\varphi]$ by *stationary phase semiclassical limit*.

In a topological quantum field theory, the semiclassical limit is *exact*. This makes TQFT a heuristic tool to define and compute topological invariants. It also allows us to ucover unexpected relations via dualities.

Plan of the course:

- 0-dim field theory: $\mathrm{Map}(*, M) = M$, so doing integrals over M. Localisation formula.
- 1-dim field theory: same as quantum mechanics. Relation with Morse theory. Betti numbers, Euler characteristics and their refinements.
- 2-dim string theory: Frobenius manifolds, Gromov-Witten invariants, mirror symmetry.

1 Localisation formulae

1.1 Stationary phase

$$I(s) = \int_{-\infty}^{\infty} \mathrm{d}x g(x) e^{isf(x)}.$$

For large s, $I(s)$ is dominated by the critical points of $f(x)$. Taylor expand at a critical point x_0,

$$f(x) = f(x_0) + \frac{1}{2} f''(x_0)(x - x_0)^2 + \dots$$

so the local contribution is

$$I_0(s) = g(x_0) e^{isf(x_0)} \int \mathrm{d}x \exp[\frac{1}{2} isf''(x_0)(x - x_0)^2].$$

Exercise. Show that

$$I_0(x) = g(x_0) \exp(i(sf(x_0) + \epsilon \frac{\pi}{4})) \left(\frac{2\pi}{s|f''(x_0)|} \right)^{1/2}$$

where ϵ is the sign of $f''(x_0)$.

Generalising to higher dimension, consider

$$I(s) = \int_{\mathbb{R}^n} \mathrm{d}^n x g(x) e^{isf(x)}.$$

At a critical point,

$$f(x) = f(x_0) + \frac{1}{2} \sum_{i,j} f''_{ij}(x_0)(x - x_0)^i (x - x_0)^j + \dots.$$

We have

$$I_0(s) = g(x_0) e^{isf(x_0)} (\frac{2\pi}{s})^{n/2} \frac{e^{i\sigma\pi/4}}{|\det f''(x_0)|^{1/2}}$$

where σ is the signature of the Hessian of f. If f has more than one critical point then the integral has contribution from each of them, so

$$I(s) \simeq_{s \to \infty} (\frac{2\pi}{s})^{n/2} \sum_j g(x_j) e^{isf(x_j)} \frac{e^{i\sigma_j \pi/4}}{|\det f''(x_j)|^{1/2}}.$$

Example. Consider $S^2 = \{x^2 + y^2 + z^2 = 1\}$. Let $g(x, y, z) = 1, f(x, y, z) = z$ and consider

$$I(s) = \int_{S^2} dA e^{isz}.$$

The critical points of f are the north and the south pole. At the north pole

$$z \sim 1 - \frac{1}{2}(x^2 + y^2)$$

and at the south pole

$$z \sim -1 + \frac{1}{2}(x^2 + y^2)$$

so

$$I(s) \simeq \frac{2\pi}{s}(e^{i(-2)\pi/4}e^{is} + e^{i\cdot 2\pi/4}e^{-is}) = \frac{4\pi \sin s}{s}.$$

In this case the integral can also be done exactly by using spherical coordinates:

$$I(s) = \int_{\varphi=0}^{2\pi}\int_{\theta=0}^{\pi} \sin\theta \mathrm{d}\theta \mathrm{d}\varphi e^{is\cos\theta} = 2\pi \int_{-1}^{1} \mathrm{d}\cos\theta e^{is\cos\theta} = \frac{4\pi \sin s}{s}.$$

The secret behind this example is symmetry: there is an S^1-action on S^2 and the fixed points are precisely the poles, and all information is contained in the fixed points.

1.2 Equivariant cohomology

Suppose a compact simply-connected group G acts on a manifold M. *Equivariant cohomology* is a cohomology theory that takes into account the action of G. In the simplest case where G acts freely, we may define

$$H_G(M) = H(M/G).$$

More interestingly if G does not act freely, one has to take into account the fixed points. One approach is use the univeral bundle EG, which is a contractible space with a free G-action. The quotient $BG = EG/G$ is called the *classifying space*. Then one defines

$$H_G^*(M) = H^*(M \times_G EG) = H^*((M \times EG)/G).$$

We will study equivariant cohomology by using the Cartan model. We define a G-action on differential forms. Starting with a function $\phi \in C^\infty(M)$, an element $h \in G$ acts on ϕ via

$$(h \cdot \phi)(x) = \phi(h^{-1}(x)).$$

After differentiating we get an action of Lie algebra which we write in the following way: for a vector field v associated an element $L \in \mathfrak{g}$, the Lie algebra of G, we have

$$(v \cdot \phi)(x) = \frac{d}{d\epsilon}\Big|_{\epsilon=0} \phi(\exp(-\epsilon L)x).$$

In local coordinates

$$v = v^a t_a^i \frac{\partial}{\partial x^i}.$$

Denote by $\mathbb{C}[\mathfrak{g}]$ the algebra of complex valued polynomials on $\mathfrak{g}$. Consider

$$\alpha \in \Omega(M, \mathfrak{g}) := \mathbb{C}[\mathfrak{g}] \otimes \Omega(M),$$

which is the same as a polynomial on $\mathfrak{g}$ valued in $\Omega(M)$. G acts on $\Omega(M, \mathfrak{g})$ by

$$(h \cdot \alpha)(X) = h\alpha(h^{-1}X)$$

where the action on Lie algebra is the adjugate.

An *equivariant differential form* is an α that is invariant under the action of G. In other words, α such that $\alpha(hX) = h\alpha(X)$.

We endow $\Omega(M, \mathfrak{g})$ with a $\mathbb{Z}$-grading

$$\deg(P \otimes \beta) = \deg \beta + 2 \deg P.$$

We define the *equivariant exterior differential* by

$$\mathrm{d}_G \alpha(\xi) = \mathrm{d}\alpha(\xi) + i\iota_V \alpha(\xi)$$

where V is the vector field associated to ξ. Properties:

- $\mathrm{d}_G : \Omega^n(M, \mathfrak{g}) \to \Omega^{n+1}(M, \mathfrak{g})$.
- d_G preserves equivariant forms.
- $\mathrm{d}_G^2(\alpha)(X) = i\mathcal{L}_V \alpha(X)$, which is zero on equivariant differential forms.

It is a theorem of Cartan that

$$H_G^*(M; \mathbb{C}) \cong H((\mathbb{C}[\mathfrak{g}] \otimes \Omega^\bullet(M))^G, \mathrm{d}_G).$$

We will focus on $G = S^1$.

Remark. Note that $\alpha(\xi)$ is a multiform in ordinary de Rham complex. By considering the homogeneous components in de Rham complex, an equivariant form α is closed if and only if

$$\mathrm{d}\alpha_{k-2}(\xi) + \iota_V \alpha_k(\xi) = 0$$

for all k.

Example. Consider the standard action of S^1 on S^2. Let $\omega = \mathrm{d}\cos\theta \mathrm{d}\varphi$ be a symplectic form. $v = \frac{\partial}{\partial \varphi}$ is the vector field generating the action. Then an equivariant closed symplectic form

$$\alpha(\xi) = \omega + \iota_\xi \mu(\theta)$$

such that

$$0 = \mathrm{d}_{S^1}\omega(\xi) = (\mathrm{d} + i\xi\iota_v)(\omega + i\xi\mu)$$

which says

$$\mathrm{d}\omega = 0, i\xi\iota_v\omega + i\xi\mathrm{d}\mu = 0, \iota_v\mu = 0.$$

The only nontrivial condition is the second one, which via

$$\iota_v\omega = -\mathrm{d}\cos\theta$$

gives $\mu = \cos\theta$.

If $\dim_{\mathbb{R}} M = 2m$ then we can define

$$e^{\alpha(\xi)} = \sum_{k=0}^{m} \frac{\omega^k}{k!} e^{i\xi\mu}.$$

1.3 Equivariant integration

Let M be a compact smooth oriented manifold with dimension $2m = n$. Given an equivariant form α, we define

$$\int_M \alpha := \int_M \alpha_n$$

where the second integral is the usual one as de Rham forms. By Stokes' theorem this is independent of cohomology class. Then the equivariant integral can be seen as the pushforward $H_G^*(M) \to H_G^*(*)$.

Recall for $f : F \to M$ a map between compact manifolds. Then we have pullback $f^* : H^*(F) \to H^*(M)$ and pushforward $f_* : H^*(M) \to H^*(F)$ in cohomology. The self-intersection formula says $f^* f_* 1 = e(\nu_F)$.

Theorem 1.1 (Atiyah-Bott localisation formula)**.** *Suppose F is the fixed locus of a G-action on M. Then for an equivariant form α,*

$$\int_M \alpha = \int_F \frac{f^*\alpha}{e(\nu_F)}.$$

We will consider the case of S^1-action on M such that the fixed points are isolated. In this case, for a fixed point x_0, $\nu_{x_0} \cong T_{x_0}M$ is an SO(2)-module of dimension $2m$, which splits into irreducible SO(2)-modules with weights ν_i, $i = 1, \ldots, m$. The vector field generating the S^1-action around x_0 is

$$\sum_{k=1}^m \nu_k (x_k \frac{\partial}{\partial y_k} - y_k \frac{\partial}{\partial x_k}).$$

The Aityah-Bott formula then reads (for a top form)

$$\int_M \alpha = \left(-\frac{2\pi}{i\xi}\right)^m \sum_p \frac{\alpha_0(\xi)(x_p)}{\nu_1^p \cdots \nu_m^p}.$$

Proof via "exact" stationary phase. Introduce a 1-form

$$\psi = \frac{1}{2}(v, \cdot)$$

where $(\cdot, \cdot)$ is an S^1-invariant metric on M. Define an equivariant exact 2-form by

$$\beta(\xi) = \mathrm{d}_{S^1}\psi = \mathrm{d}\psi + i\xi\iota_v\psi = \mathrm{d}\psi + i\xi\frac{\|v\|^2}{2}$$

In a neighbourhood of the fixed point ψ has local expression

$$\psi \sim \frac{1}{2}\sum_{k=1}^m \nu_k (x_k \mathrm{d}y_k - y_k \mathrm{d}x_k)$$

so

$$\beta(\xi) \sim \sum \nu_k \mathrm{d}x_k \wedge \mathrm{d}y_k + \frac{i\xi}{2}\sum \nu_k^2 (x_k^2 + y_k^2).$$

Since $\beta(\xi)$ is equivariant,

$$e^{is\beta(\xi)} = 1 + \sum_{k=1} \frac{(is)^k}{k!}(\mathrm{d}\psi)^k$$

so

$$\int_M \alpha(\xi) = \int_M \alpha(\xi) e^{is\beta(z)}.$$

As LHS is independent of s, we can evaluate RHS by letting $s \to \infty$. Using stationary phase, note that for large positive s, the expression

$$\exp(is\beta(\xi)) = \exp(is\mathrm{d}\psi - s\xi\frac{\|v\|^2}{2})$$

is very small except for the zeros of the vector field. Around a fixed point,

$$\begin{aligned}\int_M \alpha(\xi)e^{is\beta(\xi)} &\sim \alpha_0(\xi)(x_p)\prod_{k=1}^{m}\left(is\nu_k^p \int \mathrm{d}x_k \mathrm{d}y_k e^{-\frac{s\xi}{2}}(\nu_k^p)^2(x_k^2+y_k^2)\right) \\ &= \left(-\frac{2\pi}{i\xi}\right)^m \frac{\alpha_0(\xi)(x_p)}{\nu_1^p\cdots\nu_m^p}\end{aligned}$$

□

alternative proof of localisation formula

> **Lemma 1.2.** *If G is a compact group and α an equivariantly closed form then outside the set of zeroes of the vector field generating the G-action, the top de Rham component of α is exact.*

Proof. Introducce the G-invariant 1-form $\psi = \frac{1}{2}(v, \cdot)$. Then

$$\mathrm{d}_\xi \psi = \mathrm{d}\psi - \frac{1}{2}(v, w).$$

Using α is equivariantly closed and outside the zero locus $\mathrm{d}_\xi\psi$ is invertible, we can write

$$\alpha = \mathrm{d}_\xi \left(\frac{\psi \wedge \alpha}{\mathrm{d}_\xi \psi}\right)$$

so

$$\int_M \alpha = \int_{M \setminus \{B_{x_p}\}} \alpha + \int_{\{B_{x_p}\}} \alpha$$

where B_{x_p}'s are balls around the fixed points. Now apply Stokes' theorem. □

Let's see an example of a symplectif manifold admitting a Hamiltonian S^1-action. Let (M, ω) be a symplectic manifold. A vector field v on M is Hamiltonian if there exists a function H such that

$$\iota_v \omega + \mathrm{d}H = 0.$$

Define the Liouville volume form

$$\mathcal{L} = \frac{\omega^m}{m!} = [e^\omega]_{2m}.$$

Theorem 1.3. *Let (M,ω) be a compact symplectic manifold admitting a Hamiltonian S^1-action. Then*

$$I(\xi) = \int_M \mathcal{L}e^{i\xi H} = \left(-\frac{2\pi}{i\xi}\right)^m \sum_p \frac{e^{i\xi H(x_p)}}{\lambda_1^p \cdots \lambda_m^p}.$$

This is a corollary of localisation formula: consider the equivariant extension of the symplectic form

$$\omega(\xi) = \omega + iH.$$

Then

$$\mathrm{d}_{S^1}\omega(\xi) = (\mathrm{d} + i\xi\iota_v)(\omega + i\xi H) = \mathrm{d}\omega + i\xi(\iota_v\omega + \mathrm{d}H) = 0.$$

Define the equivariant Liouville volume form

$$\mathcal{L}(\xi) = e^{\omega(\xi)} = e^{\omega(\xi)} = e^{i\xi H}\sum_{k=1}^m \frac{\omega^k}{k!}$$

so

$$\int_M \mathcal{L}(\xi) = \int_M \mathcal{L}e^{i\xi H}.$$

Evaluate LHS by localisation formula gives the result.

Let us now revisit some of the examples. S^1 acts on S^2 by rotation. $\omega = \mathrm{d}\cos\theta\mathrm{d}\varphi$. Let $\omega(\xi)$ be the equivariant extension with respect to the S^1-action. Then $\mathrm{d}_{S^1}\omega(\xi) = 0$.

$$\iota_{\partial_\varphi}\mathrm{d}\cos\mathrm{d}\varphi + \mathrm{d}H = 0$$

wher $H = \cos\theta$ is the height function. Then

$$I(\xi) = \int_{S^2} \omega e^{i\xi\cos\theta} = \left(-\frac{2\pi}{i\xi}\right)\left(\frac{e^{i\xi z_N}}{\nu_N} + \frac{e^{i\xi z_S}}{\nu_S}\right) = \left(-\frac{2\pi}{i\xi}\right)(-e^{i\xi} + e^{-i\xi}) = 4\pi\frac{\sin\xi}{\xi}$$

This is the equivariant volume of S^2. Note as $\xi \to 0$ it goes to the regular volume.

Example. ???Compactifying. In $\mathbb{R}^2$, let $\omega = \mathrm{d}x\mathrm{d}y, v = x\partial_y - y\partial_x$. Define

$$\omega(\xi) = \mathrm{d}x\mathrm{d}y + i\xi\frac{x^2+y^2}{2}.$$

Then

$$\int_{\mathbb{R}^2} e^{i\omega(\xi)} = \left(-\frac{2\pi}{i\xi}\right)(e^{-\xi H_{\min}} + e^{-\xi H_{\max}}) = -\frac{2\pi}{i\xi}(1-0).$$

We can also perform the integral directly by

$$i\int_{\mathbb{R}^2} \mathrm{d}x\mathrm{d}y e^{-\xi\frac{x^2+y^2}{2}} = -\frac{2\pi}{i\xi}.$$

Thus we can define $\frac{2\pi}{\xi}$ as the equivariant volume of $\mathbb{R}^2$ with S^1-action. Note that unlike the compact case, as $\xi \to 0$ there is a pole (as the volume of $\mathbb{R}^2$ is infinite).

Exercise. Consider $S^2 \cong \mathbb{CP}^1$. In affine coordinate z the Fubini-Study metric has the form

$$\omega_{\mathrm{FS}} = \frac{i}{2\pi} \frac{\mathrm{d}z \mathrm{d}\overline{z}}{(1+|z|^2)^2}.$$

There is a U(1)-action $z \mapsto e^{i\epsilon} z, \overline{z} \mapsto e^{-i\epsilon} \overline{z}$. Find the corresponding vector field. Compute the equivariant extension of ω_{FS} and compute the equivariant volume. Finally generalise to $\mathbb{CP}^n$, whose Study-Fubini metric in homogeneous coordinates is given by

$$\omega = i \partial \overline{\partial} \log |Z|^2.$$

2 Localisation formula on supermanifolds

Definition (supermanifold)**.** A *supermanifold* is a couple $(M, \mathcal{A})$ where M is a differentiable manifold and $\mathcal{A}$ is a sheaf of $\mathbb{Z}$-graded commutative algebra that extend the structure sheaf on M such that

1. it admits a nilpotent subsheaf $\mathcal{N}$ such that $\mathcal{A}/\mathcal{N} \cong C^\infty(M)$. $\sigma : \mathcal{A} \to C^\infty(M)$ is called the *body map*.
2. locally $\mathcal{A}$ is a sheaf of exterior algebra of smooth functions on M, i.e. locally
$$\mathcal{A} \cong C^\infty(M) \otimes \bigwedge V$$
where V is a n-dimensional vector space.

Superfunctions $M^{(m,n)}$ defined in terms of generators ψ_a of the exterior algebra $\mathcal{A}$. There is a splitting $\mathcal{A} = \mathcal{A}^+ \oplus \mathcal{A}^-$ and there is a commutator defined by
$$[\alpha, \beta] = \alpha\beta - (-1)^{\deg \alpha \deg \beta} \beta\alpha.$$
The even part is called *bosons* and the odd par tis called *fermions*. We then define a superfunction to be
$$f(x, \psi) = f_0(x) + f^a(x)\psi_a + f^{a,b}(x)\psi_a\psi_b + \cdots + f^{1,\ldots,n}(x)\psi_1 \cdots \psi_n.$$
Note $\psi_a^2 = 0$ for all a.

Example. Define the *tautological supermanifold*as
$$\mathcal{A} \cong \bigwedge T^*M.$$
$M^{(m,m)}$

2.1 Integration on supermanifold

Berezin rules:
$$\int \psi \mathrm{d}\psi = 1, \int 1 \mathrm{d}\psi = 0$$
?is a derivation. By anticommutativity
$$\int \psi_1\psi_2 \mathrm{d}\psi_1 \mathrm{d}\psi_2 = 1, \int \psi_2\psi_1 \mathrm{d}\psi_1 \mathrm{d}\psi_2 = -1$$
and
$$\int \psi_1 \cdots \psi_n \mathrm{d}\psi_1 \cdots \mathrm{d}\psi_n = 1, \int \psi_1 \cdots \hat{\psi}_a \cdots \psi_n \mathrm{d}\psi_1 \cdots d\psi_n = 0.$$

2.2 Euler class and Pfaffians

Let $G = \mathrm{SO}(2m)$, $g \in S^2V^\vee$ where V is a $\mathbb{R}$-vector space of dimension $2m$. The *Pfaffian* is
$$\mathrm{Pf} : \mathfrak{so}(2m, \mathbb{R}) \to \mathbb{R}$$

where for $x \in \mathfrak{sO}(2m,\mathbb{R})$, ... explicitly in terms of elements of x',

$$\mathrm{Pf}(x') = \frac{1}{2^m m!} \sum_{\sigma \in S_{2m}} \mathrm{sgn}(\sigma) \prod_{i=1}^{m} x_{\sigma(i-1)\sigma(2i)}.$$

It satisfies $\mathrm{Pf}(x)^2 = \det x$.

Let P be a principal $\mathrm{SO}(2m)$-bundle over M with curvature form F. Then the Euler class of P is given by

$$e(P) = \frac{1}{(2\pi)^m} \mathrm{Pf}(F).$$

Example. Let $P = TM$. Then

$$e(TM) = \frac{1}{(2\pi)^m} \mathrm{Pf}(R).$$

2.3 Boson and fermion integration

Boson: let A be an $N \times N$ a symmetric? real matrix.

$$\int_{\mathbb{R}^N} \exp(-\frac{1}{2}x^T Ax]\mathrm{d}vol$$

Diagonalise A by an orthogonal transformation, we get

$$\prod_{i=1}^{N} \int_{\mathbb{R}} \mathrm{d}y_i e^{-\lambda_i y_i^2/2} = \prod_i \sqrt{\frac{2\pi}{\lambda_i}} = (\sqrt{2\pi})^N (\det A)^{-1/2}.$$

If some λ_i is zero then we work with $\det' A$, a measure on $\ker A$.

In the complex case

$$\int_{\mathbb{C}^N} e^{-z^t Hz} \prod \frac{\mathrm{d}z_i \mathrm{d}\overline{z}_i}{2\pi i} = (\det H)^{-1}.$$

Fermions: we consider skew-symmetric matrix ω.

$$\int e^{\frac{1}{2}\psi^t \omega \psi} \mathrm{d}\psi_1 \cdots \mathrm{d}\psi_{2m} = \mathrm{Pf}(\omega)$$

This is because by definition of Pfaffian

$$\mathrm{Pf}(\omega)\psi_1 \cdots \psi_{2m} = \frac{1}{m!} \left(\frac{1}{2}\psi^t \omega \psi\right)^m .$$

Lecture 20/05

Recall that a supermanifold $M^{(m,n)}$ is obtained by enlarging the sheaf of regular functions to $\mathcal{A}$, a $\mathbb{Z}_2$-graded algebra with odd generators $(\psi^1, \ldots, \psi^n)$.

If E is a bundle over M then $\bigwedge^* E$ is one of the easiest example

Tautological supermanifold ΠTM, where Π is the parity reversing functor. $E \cong TM$.

Superfunctions: $f(x,\psi) = f^0(x) + f_a(x)\psi^a + \cdots + f_{1\cdots n}\psi^1 \cdots \psi^n$.

We defined integration on $M^{(m,n)}$. On the odd part this is given by Berezin rule.

Now we add a group action.

A *supergroup* is a Lie algebra containing odd generators with respect to $\mathbb{Z}_2$-grading (correspondingly exists odd vector fields generating the actions

$$\frac{d}{dt}|_{t=0}\exp(tQ)\cdot y$$

). For example $f(x,\psi+t), Q=\frac{\partial}{\partial\psi}$.

Remark. Odd generators are loosely called "supersymmetry".

Witten's fixed point argument: the integral of a a sypersymmetric invariant superfunction on a superspace $\mathcal{E}$ gets contributions only from fixed points of the supersymmetry. Proof: suppose Q acts on E freely. Form the fibre bundle (what is F?)

$$F\hookrightarrow E\to E/F.$$

For an invariant superfunction f, i.e. $Qf=0$ (?),

$$\int_E f=\underbrace{\int_F dt}_{\text{odd}}\int_{E/F}f=0$$

by Berezin rule. This is a theorem for compact supermanifolds with compact odd vector fields (A. Schwarz et al, CMP)

2.4 Duistmaat-Heckmann on supermanifolds

One formulate a version of DH using tautological supermanifold. Let $\mathcal{A}=\bigwedge A^*TM$. $\Pi TM=M^{(2m,2m)}$.

$\mathrm{d}x^1\cdots\mathrm{d}x^{2n}\leftrightarrow\psi^1\cdots\psi^{2n}$. $\eta_{i_1\cdots i_s}\mathrm{d}x^{i_1}\cdots\mathrm{d}x^{i_s}\to\eta_{i_1\cdots i_s}\psi^{i_1}\cdots\psi^{i_s}=\hat{\eta}$. Then

$$\int_M\eta=\int_{\Pi TM}\hat{\mu}\hat{\eta}$$

DH says

$$\int_M\omega^n e^{iH}=i^{-n}\int_M e^{i(\omega+H)}=i^{-n}\int_{\Pi TM}\exp[i(H+\omega_{ab}\psi^a\psi^b)].$$

Supersymmetry: $Q=\psi^a\frac{\partial}{\partial x^a}+v^a\frac{\partial}{\partial\psi^a}$ where v^a are local components of the vector field generating the S^1-action. Then $Q^2=\mathcal{L}_v$. Thus supersymmetric fixed points are in bijection with fixed points of the S^1-action. Moreover the integral is Q-invariant. The measure is also Q-invariant.

$$\int_M\frac{\omega^n}{n!}e^{iH}=\int-\Pi TM\hat{\mu}e^{i(S+sQ\lambda)}$$

which is independent of s, so we can evaluate by taking $s\to\infty$. Remeber $\lambda=v_a\psi^a$, so $Q\lambda\sim\|v\|^2$. The critical points are then the zeroes of v. The rest of the proof is left as an exercise.

2.5 An example

Consider $\mathbb{R}^{(1,2)}$ and the integral

$$Z = \int_{\mathbb{R}^{(1,2)}} \widetilde{\mu} e^{-S}$$

where

$$S(x, \psi_1, \psi_2) = S_0(x) + \psi_1\psi_2 S_1(x)$$

and the measure is

$$\widetilde{\mu} = \mathrm{d}x\mathrm{d}\psi_1\mathrm{d}\psi_2.$$

The supersymmetry is given by

$$\begin{aligned} \delta_\epsilon x &= \epsilon^1\psi_1 + \epsilon^2\psi_2 \\ \delta_\epsilon \psi_1 &= \epsilon^2 h' \\ \delta_\epsilon \psi_2 &= -\epsilon^1 h' \end{aligned}$$

for some $h : \mathbb{R} \to \mathbb{R}$. Claim to make it equivaraint, i.e. $\delta_t S = 0$ we need to choose

$$S_0(x) = \frac{1}{2}(h')^2, S_1(x) = h''$$

Proof:

$$\begin{aligned} \delta_\epsilon S &= h'h''\delta_\epsilon x - h''\delta_\epsilon\psi_1\psi_2 - h''\psi_1\delta_\epsilon\psi_2 - h'''\delta_\epsilon x\psi_1\psi_2 \\ &= h'h''(\epsilon^1\psi_1 + \epsilon^2\psi_2) - h''\epsilon^2 h'\psi_2 + h''\psi_2\epsilon^1 h^1 - \underbrace{h'''(\epsilon^1\psi_1 + \epsilon^2\psi_2)}_{=0} \end{aligned}$$

which is indeed 0 by antisymmetry. As an exercise, show $\mathrm{d}x\mathrm{d}\psi_1\mathrm{d}\psi_2$ is also invariant.

Fixed point argument: $h' \neq 0$ on $\mathbb{R}$, $\epsilon^1 = \epsilon^2 = -\psi_1/h'$. Substitude

$$\hat{x} = x - \frac{\psi_1\psi_2}{h'}, \hat{\psi}_1 = \psi_1 - h'\frac{\psi_1}{h'} = 0, \hat{\psi}_2 = \psi_1 + \psi_2.$$

Then

$$\int \mathrm{d}x\mathrm{d}\psi_1\mathrm{d}\psi_2 e^{-S} = \int \mathrm{d}\hat{x}\mathrm{d}\hat{\psi}_1\mathrm{d}\hat{\psi}_2 e^{-S(\hat{x},0,\hat{\psi}_2)}.$$

Suppose now that h is a polynomial of degree n with isolated critical points (what??). Summing over all fixed points x_c,

$$\begin{aligned} Z &= \sum_c \frac{1}{\sqrt{2\pi}} \int \mathrm{d}x\mathrm{d}\psi_1\mathrm{d}\psi_2 \exp[-\frac{1}{2}h''(x_c)^2(x-x_c)^2 + h''\psi_1\psi_2] \\ &= \sum_c \frac{h''(x_c)}{|h''(x_c)|} \end{aligned}$$

which is the signed count of isolated critical points. This is invariant under local deformations of h provided we do not change asymptotic behaviour at $\pm\infty$. In other words

$$Z = \begin{cases} 0 & n \text{ odd} \\ \pm 1 & n \text{ even} \end{cases}$$

$$\begin{aligned} Z &= \sum_c \frac{1}{\sqrt{2\pi}} \int \mathrm{d}x\mathrm{d}\psi_1\mathrm{d}\psi_2 \exp[-\frac{1}{2}(h')^2 + h''\psi_1\psi_2] \\ &= \frac{1}{\sqrt{2\pi}} \int \mathrm{d}x \exp[-\frac{1}{2}(h')^2 h''] \\ &= \frac{1}{\sqrt{2\pi}} D \int_y \mathrm{d}y e^{-y} \end{aligned}$$

where D is the multiplicity coming from Jacobian. For example if h is odd, hence h' even, we have $D = 0$. If h is even, hence h' odd, $D = \pm 1$.

More on deformation invariance Suppose we have a "local deformation" (in the sense that it does not change behaviour at $\pm\infty$) $h \to h + \rho$, then

$$S(h+\rho) = \frac{1}{2}(h' + \rho')^2 - (h'' + \rho'')\psi_1\psi_2$$

so the variation with respect to ρ is

$$\delta_\rho S = \rho' h' - \rho''\psi_1\psi_2 = \delta_\epsilon(\rho'\psi_1)$$

with $\epsilon_1 = \epsilon_2 = \epsilon$: Indeed

$$\delta_\epsilon(\rho'\psi_1) = \epsilon(\rho''\psi_1\psi_2 - \rho' h').$$

$$\delta Z = \int \mathrm{d}x\mathrm{d}\psi_1\mathrm{d}\psi_2 e^{-\delta_\epsilon g} = 0.$$

3 Supersymmetric quantum mechanics

Recall that a Hilbert space $\mathcal{H}$ is a vector with a Hermitian inner product such that the associated norm makes it a complete metric space. We will use the Dirac bracket notation: $|\alpha\rangle \in \mathcal{H}$ for an element in the Hilbert space, $\langle\beta| \in \mathcal{H}^*$ for an element in the dual space, and use the physicists' convention

$$\langle a\alpha + b\beta|\gamma\rangle = a^* \langle\alpha|\gamma\rangle + b^* \langle\beta|\gamma\rangle .$$

A SUSY quantum mechanics is a $\mathbb{Z}_2$-grading on $\mathcal{H}$. The even bit is called fermions and the odd bit bosons. A SUSY operator Q exchanges (?) the parity

$$\frac{1}{2}\{Q, Q^\dagger\} = H,$$

the Hamiltonian of the generator of translations on I or S^1.

Fermion number operator $(-1)^F$ which is 1 on $\mathcal{H}_B$ and -1 on $\mathcal{H}_F$. Have

$$[(-1)^F, Q] = -Q, [(-1)^F, H] = 0, [Q, H] = 0.$$

Definition of commutator

$$[\alpha, \beta] = \alpha\beta - (-1)^{\deg\alpha \deg\beta}\beta\alpha.$$

Then by super Jacobi identity

$$[Q, \{Q, Q^\dagger\}] + [Q^\dagger, \underbrace{\{Q, Q\}}_{=0}] + [Q, \{Q^\dagger, Q\}] = 0$$

$H = \frac{1}{2}\{Q, Q^\dagger\}$ has the following properties:

1. all energies E (eigenvalues?) are nonnegative.
2. $E = 0$ if and only if $Q|\alpha\rangle = Q^\dagger|\alpha\rangle = 0$. The zero energy states are called *ground states*. Stated in a different way, this says ground states are supersymmetric (i.e. annihilated by both Q and $Q^\dagger$). This follows easily from

   $$\langle\alpha|H|\alpha\rangle = \frac{1}{2}(\langle\alpha|QQ^\dagger|\alpha\rangle + \langle\alpha|Q^\dagger Q|\alpha\rangle)$$

   For nonzero energy, there is an isomorphism $\mathcal{H}_B^{E\neq 0} \cong \mathcal{H}_F^{E\neq 0}$ realised by $Q_1 = Q + Q^\dagger$. This is because

   $$Q_1^2 = \{Q, Q^\dagger\} = 2H$$

   which is invertible on nonzero energy states.

We can define the Witten index to be the difference between number of zero boson states and number of zero fermion states

$$\Omega = \dim \mathcal{H}_B^{E=0} - \dim \mathcal{H}_F^{E=0} = \mathrm{tr}_{\mathcal{H}^{E=0}}(-1)^F = \mathrm{tr}_{\mathcal{H}}(-1)^F e^{-\beta H}$$

where β is the radius of $S^1 \to \mathbb{R}$. The last equality is because for nonzero states Q_1 induces an isomorphism. We write in this way as this will fit into the path integral formalism, which we will see in the future.

Ω is invariant under small deformations of H.

SUSY QM on $\mathbb{R}$ Let $\mathcal{H} = L^2(\mathbb{R}, \mathbb{C}) \oplus \overline{\psi} L^2(\mathbb{R}, \mathbb{C})$. A wavefunction is then a superfunction

$$\Phi(x, \overline{\psi}) = \Phi_B(x) + \Phi_F(x)\overline{\psi} \in \mathcal{H}.$$

Position x and momentum p are hermitian operators on $\mathcal{H}$ which do not commute:

$$[x, p] = i$$

(we take $\hbar = 1$). For the Grassmannian part

$$\{\psi, \overline{\psi}\} = 1.$$

A realisation of these commutation relations on $\mathcal{H}$ is by setting

$$p = -i\partial_x, \psi = \partial_{\overline{\psi}}$$

with supercharge

$$Q = \overline{\psi}(ip + h'(x)), Q^\dagger = \psi(-ip + h'(x)).$$

Then the Hamiltonian is

$$\begin{aligned} 2H &= \{\overline{\psi}ip, \psi(-ip)\} + \{\overline{\psi}h', \psi h'\} + i\{\overline{\psi}p, \psi h'\} - i\{\overline{\psi}h', \psi p\} \\ &= p^2 + (h')^2 + i(\overline{\psi}p\psi h' + \psi h'\overline{\psi}p - \overline{\psi}h'\psi p - \psi p\overline{\psi}h') \\ &= p^2 + (h')^2 + i(\overline{\psi}\psi - \psi\overline{\psi})[p, h'] \\ &= p^2 + (h')^2 + h''(\overline{\psi}\psi - \psi\overline{\psi}) \end{aligned}$$

In matrix form, if we write the wavefunction as $\Phi = (\Phi_B, \overline{\psi}\Phi_F)$, then

$$Q = \begin{pmatrix} 0 & 0 \\ \partial_x + h' & 0 \end{pmatrix}, H = \frac{1}{2}\begin{pmatrix} -\partial_x^2 + h'^2 - h'' & 0 \\ 0 & -p_x^2 + h'^2 + h'' \end{pmatrix}$$

Ground states correspond to $H\Psi = 0$, a 2nd order PDE. However by comments before this is equivalent to two 1st order ODEs

$$Q\Phi = 0, Q^\dagger\Phi = 0$$

which has formal solutions

$$\Phi = A_B e^{-h(x)} + A_F\overline{\psi}e^{h(x)}.$$

We need to check if it is normalisable. Several cases:

1. $\lim_{x\to\infty} h = \infty, \lim_{x\to-\infty} h = -\infty$ or $\lim_{x\to\infty} h = -\infty, \lim_{x\to-\infty} h = -\infty$: all solutions diverge so no ground state. $\Omega = 0$.
2. $\lim_{x\to\infty} h = \lim_{x\to-\infty} h = \infty$: one bosonic SUSY ground state and no fermionic state. $\Omega = 1$.
3. $\lim_{x\to\infty} h = \lim_{x\to-\infty} h = -\infty$: opposite situation of 2. $\Omega = -1$.

Example:super harmonic oscillator Take $h(x) = \frac{\omega}{2}x^2$ so

$$H = p^2 + V(x) + h''(\overline{\psi}\psi - \psi\overline{\psi})$$

where $V(x) = \frac{\omega^2}{2}x^2$. By discussion above for $\omega > 0$ there is a bosonic SUSY ground state

$$\Psi_{\omega>0} = e^{-\frac{1}{2}\omega x^2} |0\rangle$$

and for $\omega < 0$ there is a fermionic SUSY ground state

$$\Psi_{\omega<0} = e^{-\frac{1}{2}|\omega|x^2}\overline{\psi} |0\rangle .$$

We can solve this system and show

$$H_B = \frac{1}{2}p^2 + \frac{\omega^2}{2}x^2$$

with discrete spectrum

$$\{\frac{|\omega|}{2} + \ell|\omega|\}_{\ell\geq 0}$$

The fermionnic part can be expressed as $H_F = \frac{\omega}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ with spectrum

$$\{\frac{|\omega|}{2} + \ell|\omega|\}_{\ell\geq 0}.$$

So for $\omega > 0$ the spectrum of $H = H_B + H_F$ has a pairing *except* the zero Bosonic part. The situation is reversed for $\omega < 0$.

Partition function of Gibbs ensemble: $Z = \operatorname{tr}_{\mathcal{H}} e^{-\beta H}$.

$$\begin{aligned} Z_B &= \sum_{n=0}^{\infty} e^{-\beta(n+\frac{1}{2})|\omega|} = \frac{1}{e^{\beta\frac{|\omega|}{2}} - e^{-\beta\frac{|\omega|}{2}}} \\ Z_F &= e^{-\beta\frac{\omega}{2}} + e^{\beta\frac{\omega}{2}} \\ Z = \operatorname{tr} e^{-\beta H} &= \frac{e^{\beta\frac{\omega}{2}} +}{\mathit{denominator}} \end{aligned}$$

Note the Witten index... is independent of β

23/04/21

We now move to higher dimension and consider $\mathbb{R}^{(N,2N)}$. A wavefunction has the form

$$\Phi(x,\overline{\psi}) = \sum_{b_1,\ldots b_N=0,1} \Phi_{b_1,\ldots,b_N}(\overline{\psi}^1)^{b_1}\cdots(\overline{\psi}^N)^{b_N}$$

which is a vector with 2^N components.

$$Q = \sum I\overline{\psi}^I(ip_I + \partial_I h(x))$$

where $h : \mathbb{R}^N \to \mathbb{R}$. One can check $Q^2 = 0$. The Hamiltonian is

$$H = \frac{1}{2}\{Q,Q^\dagger\} = \frac{1}{2}\sum_I(p_I^2 + (\partial_I h)^2) + \frac{1}{2}\sum_{I,J}[\overline{\Psi}^I,\Psi^J]\partial_I\partial_J h.$$

To solve for ground states we need to solve

$$Q\Phi = Q^\dagger \Phi = 0.$$

This is in general difficult but can be solved in some special cases. For example let us consider N copies of one dimensional harmonic oscillator, i.e.

$$h(x) = \frac{1}{2}\sum_{I=1}^{N} \omega_I (x^I)^2.$$

Then

$$\Phi_0(x) = \exp(-\frac{1}{2}\sum_I |\omega_I|(x^I)^2 \prod_{I:\omega_I<0} \overline{\Psi}^I.$$

The number F is the number of negative eigenvalues of the Hessian of h, which is the Morse index of the critical point (note h is a Morse function). The Witten index is $(-1)^\mu$.

Using this result we can compute the Witten index for any h which is Morse. We use deformation invariance: Ω does not depend on local deformation of h, provided we do not modify its asymptotic behaviour $h(x) \to \lambda h(x)$, $\lambda \to \infty$. Classical fixed points are critical points of $h(x)$. The corresponding wave functions are the Gaussians centered at the critical points. Then

$$\Omega = \sum_{x_c \text{ critical}} (-1)^{\mu(x_c)}.$$

Riemannian manifolds Consider all maps $S^1 \to M$. The Hilbert space in question is isomorphic to differential forms on M. F equals to the de Rham degree.

$$\Phi = \Phi_0 + \Phi_I \mathrm{d}x^I + \Phi_{IJ}\mathrm{d}x^I \wedge \mathrm{d}x^J.$$

Using the tautological supermanifold we identify $\mathrm{d}x^I$ with $\overline{\psi}^I$. The inner product on $\mathcal{H}$ corresponds to the inner product of differential forms:

$$\langle \Psi | \Phi \rangle = \int \overline{\Psi} \wedge \star \Phi$$

where $\star$ is the Hodge star operator. In a local coordinate

$$\langle \Psi | \Phi \rangle = \sum_{\ell=0}^{N} \int \mathrm{d}^N x \sqrt{g} g^{I_1 J_1} \cdots g^{I_\ell J + \ell} \overline{\Psi}_{I_1 \cdots I_\ell} \Phi_{J_1 \cdots J_\ell}.$$

Operators:

$$Q = \mathrm{d} = \mathrm{d}x^I \nabla_I$$

$$H = \frac{1}{2}(QQ^\dagger + Q^\dagger Q) = \Delta,$$

the Laplacian on M. In local coordinates,

$$\overline{\psi}^I = \mathrm{d}x^I \wedge (-), \psi^I = g^{IJ}\iota_{\partial_J}$$

with

$$\{\psi^I, \overline{\psi}^J\} = g^{IJ}$$

and

$$H = -\frac{1}{2}g^{IJ}\nabla_I\nabla_J + R_{IJKL}\psi^I\overline{\psi}^J\psi^K\overline{\psi}^L.$$

The Witten index is $\sum(-1)^F$ summing over all SUSY ground states. The equation for gound state means that Φ is a harmonic differential forms, so

$$\Omega = \sum_{F=0}^{N}(-1)^F b_F(M)$$

which is the Euler characteristic of M.

Adding a potential $h : M \to \mathbb{R}$, the SUSY charge gets deformed. By conjugation

$$Q_h = e^{-h}Q_{h=0}e^h = e^{-h}\mathrm{d}e^h = \mathrm{d} + \mathrm{d}h.$$

Note that adding h does not change Ω. Then the Hilbert space of zero energy states $H_{\text{SUSY}} \cong \ker Q/\operatorname{im} Q$

Proof. For all $|\alpha\rangle$ with positive energy E,

$$|\alpha\rangle = \frac{1}{2E}(Q^\dagger Q + QQ^\dagger)|\alpha\rangle = Q(\frac{1}{2E}Q^\dagger|\alpha\rangle)$$

so all positive energy states are trivial in the Q-cohomology.

On the other hand for zero energy state $|\alpha\rangle$, ??

If $|\alpha\rangle = Q|\beta\rangle$ then

$$\langle\alpha|\alpha\rangle = \langle\beta|Q^\dagger|\alpha\rangle = 0$$

so $|\alpha\rangle = 0$. □

Now adding h does not change the cohomology: $\ker\alpha \to e^{-h}|\alpha\rangle$ sends elements in the cohomology of Q_0 to elements to those of Q_h, which induces an isomorphism.

3.1 Path integral for QM

We are interested in integrals over the space of all maps

$$Z = \int_{q:S^1\to M} Dq\exp(-\underbrace{\int_0^\beta \mathrm{d}t\mathcal{L}(q,\dot{q})}_{S})$$

where t is the coordinate on S^1. For a harmonic oscillator,

$$A = \frac{1}{2}\int_0^\beta \mathrm{d}tq\underbrace{(-\frac{d^2}{dt^2})}_{D}+\omega^2.$$

Z is formally a Gaussian integral so we expect

$$Z(\beta,\omega)\text{“}=\text{''}(\det D)^{-1/2}.$$

This is given by the process of ξ-renormalisation. D has discrete real positive eigenvalues $\lambda_n > 0$. Define

$$\zeta_D(s) = \frac{1}{\Gamma(s)}\int_0^\infty t^{s-1}\operatorname{tr}(\rho e^{-tD})\mathrm{d}t$$

where ρ is the projection on $\lambda_n > 0$. Then

$$\zeta_D(s) \text{``}=\text{''} \sum n\lambda_n^{-s}, s \in \mathbb{C}$$

It is analytic as $s \to 0$. We define

$$\det{}' D = e^{-\xi_D'(0)}.$$

Expand $q(t)$ in terms of Fourier modes

$$q_n(t) = \exp(\frac{2\pi i}{\beta} nt)$$

such that $Dq_n = \lambda_n q_n, \lambda_n = \omega^2 + \omega^2 + (\frac{2\pi n}{\beta})^2$. If $q(t) = \sum_n c_n q_n(t)$ then

$$Dq = \prod_n \frac{dc_n}{\sqrt{2\pi}}$$

so

$$S = \frac{1}{2} \sum_n c_n^2 \lambda_n$$

and

$$Z(\beta, \omega) = \int \prod_n \frac{dc_n}{\sqrt{2\pi}} \exp(-\frac{1}{2} \sum_n \lambda_n c_n^2).$$

Combine with the observation above,

$$(\det{}' D)^{-1/2} \text{``}=\text{''} \prod_n (\lambda_n)^{-1/2}$$

...

Gelfand-Yaglom theorem Let $D = -\frac{d^2}{dt^2} + V(t)$ be a second order differential operator on $[0, \beta]$. We can pose the eigenvalue problem with Dirichlet boundary condition, namely

$$D\psi_n = \lambda_n \psi_n, \psi_n(0) = 0, \psi_n(\beta) = 0$$

where $0 < \lambda_1 < \cdots < \lambda_n$ is the discrete nondegenerate spectrum bounded from below. One can consider the auxillary problem

$$D\phi_\lambda = \lambda \phi_\lambda, \phi_\lambda(0) = 0, \phi_\lambda'(0) = 1.$$

The for all $\lambda = \lambda_n$,

$$\phi_{\lambda_n}(\beta) = 0.$$

Theorem 3.1.

$$\frac{\det(-\frac{d^2}{dt^2} + V_1(t) - \lambda)}{\det(-\frac{d^2}{dt^2} + V_2(t) - \lambda)} = \frac{\phi_\lambda^{(1)}(\beta)}{\phi_\lambda^{(2)}(\beta)}.$$

Proof. LHS and RHS both have zeros at $\lambda = \lambda_n^{(1)}$ and poles at $\lambda = \lambda_n^{(2)}$. Moreover both sides go to 1 as $\lambda \to \infty$. Thus the ratio of the two sides is a bounded entire function, which is constant 1. □

As an application,

$$\frac{\det(-\frac{d^2}{dt^2}+\omega^2)}{\det(-\frac{d^2}{dt^2})}=\frac{\phi_0^\omega(\beta)}{\phi_0(\beta)}.$$

One can solve for $\phi_0^\omega(t)$ to get

$$\phi_0^\omega(t)=\frac{\sinh\omega t}{\omega}, \phi_0(t)=t.$$

How to reconcile the two results?

3.2 Path integral formulation of SQM

Let $\phi : S^1 \to M$, $\psi, \overline{\psi} \in \Gamma(S^1, \phi^*(TM)\otimes\mathbb{C})$.

Witten index in the path integral formulation: the partition function

$$Z=\int_{\phi,\psi,\overline{\psi}} D\phi D\psi D\overline{\psi}\exp(-S(\phi,\psi,\overline{\psi})).$$

By definition ϕ is periodic: $\psi(0)=\phi(\beta)$. This is a boundary condition for bosons. For fermions, there are two possibilities: $\psi(0)=\pm\psi(\beta)$ (since we have seen the action is quadratic in ψ). The correct choice is the periodic one, making

$$Z_{\text{periodic}}=\operatorname{tr}_{\mathcal{H}}(-1)^F e^{-\beta H}.$$

This choices preserves SUSY as

$$\begin{aligned}\frac{\partial Z_{\text{periodic}}}{\partial\beta}&=-\operatorname{tr}_{\mathcal{H}} H(-1)^F e^{-\beta H}\\&=\frac{1}{2}\int D\phi D\psi D\overline{\psi}\{Q,Q^\dagger\}e^{-S}\\&=-\frac{1}{2}\int D\phi D\psi D\overline{\psi}\delta_{\text{SUSY}}(\cdots)\\&=0\end{aligned}$$

Now for our specific case, we define the action to be

$$S=\int_0^\beta \mathrm{d}\tau(\frac{1}{2}g_{IJ}(\phi)\dot{\phi}^I\dot{\phi}^J+g_{IJ}(\phi)\overline{\psi}^I\nabla_\tau\psi^J$$

where

$$\nabla_\tau\psi^I=\partial_\tau\psi^I+\Gamma^I_{JK}\partial_\tau\phi^J\psi^K.$$

Note that $\delta S=0$ under SUSY where

$$\delta\phi^I=\epsilon\overline{\psi}^I, \delta\overline{\psi}^I=0, \delta\psi^I=\epsilon(-\dot{\psi}^I-\Gamma^I_{JK}\overline{\psi}^J\psi^K).$$

To evaluate the partition function we find fixed points of SUSY. $\dot{\phi}^I=0$ gives constant maps in LM. In terms of Fourier modes expansion

$$\begin{aligned}\phi(\tau)&=\sum_{n\in\mathbb{Z}}\phi_n e^{in\tau}\\\psi(\tau)&=\sum_{n\in\mathbb{Z}}\psi_n e^{in\tau}\\\overline{\psi}(\tau)&=\sum_{n\in\mathbb{Z}}\overline{\psi}_n e^{in\tau}\end{aligned}$$

Then

$$\Delta_B = g_{IJ}(\gamma_0)(-\frac{d^2}{d\tau^2})$$
$$\Delta_F = g_{ij}(\phi_0)(i\frac{d}{d\tau})$$

The eigenvalues are

$$\Delta_B\phi_n = n^2\phi_n, \Delta_F\psi_n = in\psi_n.$$

integration over nonconstant modes produce

$$(\det{}'\Delta_B)^{-1/2} \sim (\prod_{n\neq 0} n^2)^{-1/2} = (\prod_{n>0} n^2)^{-1}$$
$$(\det{}'\Delta_F) \sim \prod_{n\neq 0}(in) = \prod_{n\to 0} n^2$$

In this case the bosonic and fermionic parts cancel completely and Z_{SQM} collapses to integration over M:

$$Z_{\text{SQM}} = \int_M \prod_i \mathrm{d}\psi_0^{I_i} \prod \mathrm{d}\overline{\psi}_0^{J_i} \exp(\frac{1}{2}R_{IJKL}\psi_0^I\overline{\psi}_0^J\psi_0^K\overline{\psi}_0^L)$$
$$= \int_M \prod \mathrm{d}\psi_0^i \exp(\frac{1}{2}\psi_0 R\psi_0) \quad \text{identify } \overline{\psi}^I \sim \mathrm{d}x^I$$

which is the integral over Pfaffians of R, which equals to $\chi(M)$. We thus get Gauss-Bonet theorem.

3.3 SQM with potential

Recall that we considered $Q \to Q_h = e^{-h}Qe^h$ where $h : M \to \mathbb{R}$. There is a deformed action

$$S_\lambda = S_0 + \frac{\lambda^2}{2}g_{Ij}\partial^I h\partial^J h + \lambda D_I\partial_J h\overline{\psi}^I\psi^J$$

SUSY is changed by

$$\delta\phi^I = \epsilon\overline{\psi}^I$$
$$\delta\overline{\psi}^I = 0$$
$$\delta\psi^I = \epsilon(-\dot{\psi}^I - \Gamma^I_{JK}\overline{\psi}^J\psi^K + g^{IJ}\partial_J h)$$

The fixed points are given by

$$\dot{\psi}\phi^I = 0, \partial_J h = 0,$$

which are given by constant maps to the critical points of h. The path integral is localisated to the set of critical points of h.

$$\Delta_B = -g_{IJ}(\phi)\frac{d^2}{d\tau^2} + \lambda^2 D_I\partial_K h D_J\partial^K h$$
$$\Delta_F = ig_{IJ}\frac{d}{d\tau} + \lambda D_I\partial_J h$$

For bosons

$$\prod_{n\neq 0}(g_{IJ}n^2+\lambda^2 D_I\partial_K h D_J\partial^K h)^{-1/2}=\prod_{n>0}(g_{IJ}n^2+\lambda^2 D_I\partial_K h D_J\partial^K h)^{-1}$$

and for fermions

$$\prod_{n\neq 0}(ing_{IJ}+\lambda D_I\partial_J h)=\prod_{n>0}(n^2 g_{IJ}+\lambda^2 D_I\partial_K h D_J\partial^K h)$$

which again cancel. For $n=0$ the set of modes not in $\ker\Delta_B$ is empty.

$$\frac{\det\lambda D_I\partial_J h}{(\det\lambda^2 D_I\partial_K h D_J\partial^K h)^{1/2}}=\mathrm{sign}\cdot\det D_I\partial_J h.$$

In summary,

$$Z=\chi(M)=\sum_p \mathrm{sign}\cdot\det\mathrm{Hess}h|_p.$$

Since $\partial_I h$ is a vector field, tis givens the Poincaré-Hopf theorem.

Mathematical formulation Mathai, Quillen: Superconnections, thom classes, and equivariant differential forms

Let E be a real vector bundle of rank $2n$ over X. Let X^μ be coordinates on X and $\mathrm{d}x^\mu=\psi^\mu$. Let h^i be local coordinates of fibres of E. Let χ^i be coordiates on fibres of ΠE (parity reversing). Let g_{ij} be a metric on E, A^i_μ the 1-form connection of E. Define an odd vector field on the supermanifold ΠT ΠE.

$$\delta x^\mu=\psi^\mu,\delta\psi^\mu=0,\delta\chi^i=h^i-A^i_{j\mu}\psi^\mu x^i,\delta h^i=\delta(A^i_j\psi^\mu\chi^j)$$

so $\delta^2=0$. Define $\alpha\in\Omega^\bullet(\Pi E)$ by

$$\alpha=\frac{1}{(2\pi)^n}\exp(-t\delta V)$$

where $t\in\mathbb{R}_{>0}$. Let $V=\frac{1}{2}g_{ij}\chi^i h^j$.

$$\begin{aligned}\delta(\chi,h)&=(h-A\chi,h)-(\chi,\mathrm{d}A\chi-A(h-A\chi))\\&=(h,h)-(\chi,F_A\chi)\end{aligned}$$

where $F_A=\mathrm{d}A+A\wedge A$.

$$\frac{1}{(2\pi)^{2n}}\int_{\Pi E}DhD\chi\exp(-\frac{1}{2}\delta(\chi,h))=\frac{1}{(2\pi)^n}\mathrm{Pf}(F_A)$$

In particular for $E=TM$ and $\chi=\psi$ we recover the previous result.

One can deform this by a section of the vector bundle. Let $s\in\Gamma(E)$ and $V_s=\frac{1}{2}(\chi,h+\sqrt{-1}s)$. Then the integral over (χ,h) produces $\exp(-\frac{1}{2t}s^2)$, a weight concentrated around $s^{-1}(0)\subseteq X$.

$$\alpha_s=\frac{1}{(2\pi)^n}\exp(-t\delta V_s)$$

is a representative of the Thoms class of E. For $s=0$ we get the Euler class.

For infinite-dimensional vector bundles we can use α_s to give the definition of the regularised Euler class. On the other hand for finite-dimensional vector bundle we can define a defomred Euler characteristic $\chi_s(E)$. For example the partition function of SQM can be regraded as the TQFT-Mathai-Quillen representative of Euler class $\chi_s(LM)$. $\chi_s(LM)=\chi(M)$ and does not depend on s (we saw s constant maps and s vector field $\partial_I h$).

3.4 Index theorems and SQM

Some review of characteristic classes: let P be a principal $\mathrm{GL}(n,\mathbb{C})$-bundle over X. The Chern character is an adjoint invariant function

$$\begin{aligned}\mathrm{ch}: \mathfrak{gl}(n,\mathbb{C}) &\to \mathbb{C}\\ X &\mapsto \operatorname{tr} e^X = \sum_{n=0}^{\infty} \frac{1}{n!} \operatorname{tr} x^n\end{aligned}$$

The eigenvalues of the matrix X are called the Chern roots. In terms of the Chern roots

$$\mathrm{ch}(X) = \sum_{i=1}^{n} e^{X_i}.$$

The Chern classes are defined by

$$\det(1+tX) = \sum_{k=0}^{n} t^k c_k.$$

For example

$$c_1 = \operatorname{tr} X, c_n = \det X.$$

The Todd class is defined by

$$\mathrm{td}(X) = \det \frac{X}{1-e^{-X}} = \prod_{i=1}^{n} \frac{X_i}{1-e^{-X_i}}.$$

Note the function has a power series expansion in terms of the Bernoulli numbers

$$\frac{x}{1-e^{-x}} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} B_k x^k.$$

The $\widehat{A}$-class is defined as

$$\widehat{A} = \det \frac{X}{e^{X/2}-e^{-X/2}} = \prod_{i=1}^{n} \frac{X_i}{e^{X_i/2}-e^{-X_i/2}}.$$

The Atiyah-Singer index theorem for Dirac operator states that

$$\mathrm{ind}(\not{D}, E) = \frac{1}{(-2\pi\sqrt{-1})^n} \int_X \widehat{A}(T_X)\, \mathrm{ch}\, E$$

where the Dirac operator is defined on a spin bundle $\not{D} : S^+ \otimes E \to S^- \otimes E$ and its index is defined as the difference between the dimension of kernel and cokernel.

Formally it looks very similar to the Witten index. We can "prove" various index theorems by identifying Q with the appropriate operator (c.f. Alvarez-Gaumé, CMP SUSY and index theorems).

For example consider

$$S = \int_{S^1} \mathrm{d}\tau (\frac{1}{2} g_{\mu\nu} \dot{x}^\mu \dot{x}^\nu + \dot{x}^\mu A_\mu + \frac{1}{2} g_{\mu\nu} \psi^\mu \nabla_\tau \psi^\nu - \frac{1}{2} \psi^\mu \psi^\nu F_{\mu\nu}$$

The integral is

$$Z = \int_{\substack{x \in LX \\ \psi \in \Gamma(S^1, x^*(T_X))}} DxD\psi e^{-S}.$$

Note ψ is real. Z is independent of the radius of S^1. Taking $\beta \to \infty$ will give the exact result.

$$\begin{aligned} x^\mu(\psi) &= x_0^\mu + \hat{x}^\mu(\tau) \\ \psi^\mu(\tau) &= \psi_0^\mu + \hat{\psi}^\mu(\tau) \end{aligned}$$

Rescale by

$$\begin{aligned} \hat{x}^\mu(\tau) &\to \widehat{x}^\mu(\tau)/\sqrt{\beta} \\ \hat{\psi}^\mu(\tau) &\to \hat{\psi}^\mu(\tau)/\sqrt{\beta} \end{aligned}$$

The measure is invariant (exercise).

$$\beta S \to \int_0^\beta \mathrm{d}\tau(\frac{1}{2}g_{\mu\nu}(x_0)\dot{\hat{x}}^\mu\dot{\hat{x}}^\nu + \frac{1}{2}\hat{\psi}^i\eta_{ij}\partial_\psi\hat{\psi}^j - \frac{1}{2}\psi_0^\mu\psi_0^\nu F_{\mu\nu} + \frac{1}{2}R_{ij\mu\nu}\psi_0^i\psi_0^j\hat{x}^\mu\dot{\hat{x}}^n) + O(\frac{1}{\sqrt{\beta}})$$

so

$$Z = \int \mathrm{d}^{2n}x_0\mathrm{d}^{2n}\psi_0 e^{iF_{\mu\nu}\psi_0^\mu\psi_0^\nu}[\det{}'(\delta_\nu^\mu\partial_\tau - R_\nu^\mu)\partial_\tau]^{-1/2}[\det{}'\partial_\tau]^{1/2}$$

The fermionic part cancels the second factor of the bosonic part and the exponential is $\operatorname{ch} E$ so we are left with $\det{}'(\delta_\nu^\mu\partial_\tau - R_\nu^\mu)$. Expand in Fourier modes,

$$\begin{aligned} \det{}'(\frac{\partial}{\partial\tau} - R^{(i)})^{-1/2} &= \prod_{k\neq 0}(ik - \lambda^{(j)})(ik + \lambda^{(j)}) \\ &= \prod_{kne0}(k^2 + (\lambda^{(j)})^2)^{-1/2} \\ &= \prod_{k=1}^\infty (k^2 + (\lambda^{(j)})^2)^{-1} \end{aligned}$$

where $R = \begin{pmatrix} 0 & \lambda^{(j)} \\ \lambda^{(j)} & 0 \end{pmatrix}$. By ξ-function regularisation this is

$$\frac{\lambda^{(j)}/2}{\sinh \lambda^{(j)}/2}.$$

$\lambda^{(j)}$ are the Chern roots so this is exact the $\widehat{A}(R)$-class:

$$\widehat{A}(R) = \prod_j \frac{\lambda_j}{e^{\lambda_j/2} - e^{-\lambda_j/2}}.$$

Thus

$$Z = \operatorname{ind}(\not{D}, E) = \dim\ker\not{D} - \dim\operatorname{coker}\not{D} = \int_X \operatorname{ch} E\widehat{A}(T_X).$$

Hirzebruch-Riemann-Roch:

$$\operatorname{ind}(\overline{\partial}, E) = \frac{1}{(-2\pi i)^n}\int_X \operatorname{td}(T_X^{1,0})\operatorname{ch} E$$

which can be derived from Atiyah-Singer by noting that on a comples manifold

$$\not{D} = \overline{\partial} \otimes K^{1/2}$$

where K is the canonical class. Thus we use the Todd class instead of $\widehat{A}$-class.

3.5 Morse theory and SQM

Let M be a compact manifold with Morse function f. Let M_p be the number of critical points of f with p negative eigenvalues. The weak Morse inequality says

$$M_p \geq b_p$$

and the strong Morse inequality says that

$$\sum_p M_p t^p - \sum_p b_p t^p = (1+t) \sum_p Q_p t^p$$

where $Q_p \geq 0$. We will see that SQM provides refinements of Morse inequalities via Morse-Witten index.

Recall

$$Z_\lambda = \int D\phi D\psi D\overline{\psi} e^{-S_\lambda(\phi,\psi,\overline{\psi},h)}.$$

For $\lambda = 0$ (zero potential), the Q-fixed points are the constant maps. The integral gives Guass-Bonnet. For $\lambda \neq 0$, Q_h-fixed points are critical points to critical points of h, giving Poincaré-Hopf.

...

$Q_h = e^{-\lambda h} Q e^{\lambda h}, Q_h^\dagger = e^{\lambda h} Q^\dagger e^{-\lambda h}$ so

$$H_\lambda = \frac{1}{2}\{Q_h, Q_h^\dagger\} = \frac{1}{2}\Delta + \frac{1}{2}\lambda \nabla_I \partial_J h[\overline{\psi}^I, \psi^J] + \frac{1}{2}\lambda^2 g^{IJ} \partial_J h \partial_J h.$$

In the limit $\lambda \to \infty$, expand H around the critical point x_i of h,

$$H(x_i) = \frac{1}{2}\sum_I p_I^2 + \lambda^2 c_I^2 (x^I)^2 + \frac{1}{2}\lambda c_I [\overline{\psi}^J, \psi_J] + O(\frac{1}{\lambda}).$$

Remark. This is called pertubation theory with parameter $\frac{1}{\lambda}$. The gradient flow lines actionwhich is proportional to λ is *not* analytic in $\frac{1}{\lambda}$. These corrections $e^{-\lambda}$ are "non-perturbative instanton" corrections.

The ground states are Gaussians centered around the critical points. Let

$$|a_i\rangle = e^{-\lambda \sum_I |c_I(x^I)^2} \prod_{J:c_J<0} \overline{\psi}^J |0\rangle$$

be the ground state asscociated with x_i. Note the number of J such that $c_J < 0$ is exactly the Morse index μ_i. Identify $\overline{\psi}^J \sim \mathrm{d}x^I$, $|a_i\rangle \in \Omega^{\mu_i}(M) \otimes \mathbb{C}$. For λ finite, gradient flow lines produce an overlap among these states

$$\langle a_j | Q_h | a_i \rangle = \int \overline{a}_j \wedge \star(\mathrm{d} + \mathrm{d}h) a_i$$

$\mu_j = \mu_i + 1$. In physics terms, these are ascending flow lines (instantons). Conversely if $\mu_j = \mu_i - 1$ they are descending flow liens (anti-instantons).

Define the *Morse-Witten complex*

$$0 \longrightarrow X^0 \xrightarrow{Q_h} X^1 \xrightarrow{Q_h} X^2 \longrightarrow \cdots \longrightarrow X^n \longrightarrow 0$$

where $X^\mu = \bigoplus_{\mu=\mu_i} a_i$,

$$\begin{aligned} Q_h &: X^\mu \to X^{\mu+1} \\ &|a_i\rangle \mapsto \sum_{a_j} n(a_j, a_i)\,|a_j\rangle \\ Q_h^\dagger &: X^\mu \to X^{\mu-1} \end{aligned}$$

where $n(a_i, a_j) = \pm 1$ according to the orientation. The orientation between two different points can be defined using the Hessian

$$\frac{D^2 h}{\partial\phi^I \partial\phi^J} : T_{x_i}M \to T_{x_j}M.$$

Example. (Example on S^2)

path integral derivation Expand S around the gradient flow line:

$$\frac{d}{dt}\phi^I - \lambda g^{IJ}\partial_J h = 0.$$

$$S = \lambda(h(x_i) - h(x_j)) + \int_{\mathbb{R}} (\frac{1}{2}|D_-\xi|^2 - D_-\overline{\psi}\psi)\mathrm{d}t$$

where $D--$ is first order variation of Levi-Civita connection, plus Hessian:

$$D_-\xi^I = D_\tau \xi^I - \lambda g^{IJ} D_J \partial_K h \xi^K$$

We study zero modes of D_-. Under genericity assumption $\ker D_-^\dagger = 0$ so

$$\mathrm{ind} D_- = \dim\ker D_- = \mu_i - \mu_j$$

(see remark below).

$$\langle a_j|Q_h|a_i\rangle = \frac{1}{h(x_i) - h(x_j) + O(\frac{1}{\lambda})} \lim_{T\to\infty} \langle a_j|e^{-TH}[Q,h]e^{-TH}|a_i\rangle$$

Note $\langle a_j e^{-TH} hQ e^{-TH}|a_i\rangle$ is the projection on zero energy states (for nonzero states it vanishes)

$\phi(-\infty) = x_i, \phi(\infty) = x_j)$

$$\begin{aligned} \langle a_j|e^{-TH}[Q,h]e^{-TH}|a_i\rangle &= \int D\phi D\psi D\psi D\overline{\psi} e^{-S}\overline{\psi}^I \partial_I h \\ &= \int \mathrm{d}t_0 \prod_I \mathrm{d}\overline{\psi}_0^I \prod_{n\neq 0} \mathrm{d}\xi_n \mathrm{d}\psi_n^I \mathrm{d}\overline{\psi}_n^I \partial_J h \overline{\psi}^I \\ &\exp(-\lambda(h(x_j) - h(x_i)) + \frac{1}{2}\int |D_-\xi|^2 - (D_-\overline{\psi}, \psi)) \\ &= \int \mathrm{d}t_0 \prod_I \mathrm{d}\overline{\psi}_0^I \overline{\psi}_0^I \partial_I h \underbrace{\frac{\det{}' D_-}{\sqrt{\det{}' D_-^\dagger D_-}}}_{\pm 1} e^{-\lambda(h_j - h_i)} \\ &= \pm e^{-\lambda(h_j - h_i)}(h_j - h_i) \end{aligned}$$

Thus

$$\langle a_j|Q_h|a_i\rangle = \sum_\gamma n_\gamma e^{-\lambda(h_j-h_i)}$$

where the sum is over all γ from x_i to x_j such that $\mu_i = \mu_j - 1$.

spectral flow and relative Morse index The Hessian is a linear map $H(h) : T_xM \to T_xM$. In an orthonormal local coordinate it is a symmetric matrix so can be diagonalised with real eigenvalues. A gradient flow ϕ gives a family of eigenvectors and eigenvalues:

$$H(h(\phi(\tau))e_I(\tau) = \lambda_I(\tau)e_I(\tau)$$

for $-\infty < \tau < \infty$. λ_I is called the spectral flow. $D_- = D_\tau - H_h(\phi)$ so $\ker D_-$ is given by

$$f_{I,\pm}(\tau) = e_I(\tau)\exp(\pm\int_0^\tau \lambda_I(t)\mathrm{d}t).$$

provided it is normalisable. $f_{I,\pm}$ is normalisable if and only if

$$\lambda_I(-\infty) > 0, \lambda_I(\infty) < 0.$$

The difference

$$\Delta\mu = \dim\ker D_- - \dim\ker D_-^\dagger$$

since $\dim\ker D_-$ is the number of I such that $\lambda_I(-\infty) > 0, \lambda_I(\infty) < 0$.

perfect Morse function A perfect Morse function is one such that Q_h vanishes. A class of perfect Morse functions are the moment maps of S^1-action. Morse index can jump only by an even number (due to S^1-action, the tangent space decompose into plane).

4 String theory

We consider maps $\Sigma \to M$ from closed surfaces. Consider the supermanifold $\mathbb{R}^{(2,4)}$. We split 4 as 2+2, and work with the so called $\mathcal{N} = (2,2)$ supersymmetry. Let x^m be coordinates on $\mathbb{R}^2$. The rotation groups acts on it via

$$\begin{pmatrix} x^1 \\ x^2 \end{pmatrix} = \begin{pmatrix} \cos\gamma & \sin\gamma \\ -\sin\gamma & \cos\gamma \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Odd superspace coordinates $\theta_\alpha, \overline{\theta}^\alpha$ where $\alpha = \pm 1$. $\theta \in S$, a spinor bundle. There is a splitting $S = S^+ \oplus S^-$ where S^+ is Weyl spinor. $\theta_\alpha = (\theta_+, \theta_-), \overline{\theta}_\alpha = (\theta_\alpha)^\dagger$. $\overline{\theta}^\alpha = \epsilon^{\alpha\beta}\overline{\theta}_b$ where

$$\epsilon^{+-} = 1, \epsilon^{-+} = -1, \epsilon^{++} = \epsilon^{--} = 0.$$

Clifford algebra $Cl(2)$

$$\{\gamma^m, \gamma^n\} = \gamma^m\gamma^n + \gamma^n\gamma^m = 2\delta^{mn},$$

$$(\gamma^1)^\beta_\alpha = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} (\gamma^2)^\beta_\alpha = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}$$

Transformation under rotation: $\theta \mapsto e^{i/2\omega^{mn}S_{mn}}\theta$, $S_{mn} = \frac{i}{h}[\gamma_m, \gamma_n]$, $\theta^\pm \to e^{\pm i\omega/2}\theta^\pm, \overline{\theta}^{\pm i\omega/2}\theta$ where ω is antisymmetric with components

$$\omega^{mn} = \begin{pmatrix} 0 & \omega \\ -\omega & 0 \end{pmatrix}$$

$\mathcal{N} = (2,2)$ superfields are maps

$$\Phi : \mathbb{R}^{(2,4)} \to \mathbb{C}$$

with components

$$\Phi(x^m, \theta^\pm, \overline{\theta}^\pm) = f(x^m) + \theta^+ f_+(x^m) + \overline{\theta}^+ g_+ + \cdots + \theta^+\theta^-\overline{\theta}^+\overline{\theta}^- F$$

which has in total has 2^4 components.

supercharges The supersymmetry algebra is defined as follow. They satisfy the relation

$$\{Q_\alpha, \overline{Q}_\beta\} = 2i(\gamma^m)_{\alpha\beta}\partial_m$$

The supercharges are linear realisation as derivative operations:

$$Q_\alpha = \frac{\partial}{\partial\theta^\alpha} - i(\gamma^m)_{\alpha\beta}\overline{\theta}^\beta\partial_m$$
$$\overline{Q}_\alpha = -\frac{\partial}{\partial\overline{\theta}^\alpha} + i(\gamma^m)_{\beta\alpha}\theta^\beta\partial_m$$

Note that compared to the definition of γ^1 above,

$$(\gamma^1)_{\alpha\beta} = \epsilon_{\beta\gamma}(\gamma^1)^\gamma_\alpha = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$$

Similarly

$$(\gamma^2)_{\alpha\beta} = \epsilon_{\beta\gamma}(\gamma^2)^{\gamma}_{\alpha} = \begin{pmatrix} i & 0 \\ 0 & i \end{pmatrix}$$

Therefore we can simplify the notation by writing

$$Q_{\pm} = \frac{\partial}{\partial\theta^{\pm}} + \overline{\theta}^{\pm}\partial_{\pm}$$
$$\overline{Q}_{\pm} = -\frac{\partial}{\partial\overline{\theta}^{\pm}} - \theta^{\pm}\partial_{\pm}$$

where

$$\partial_{\pm} = \frac{1}{2}(\frac{\partial}{\partial x^2} \pm i\frac{\partial}{\partial x^1}),$$

the (anti)holomorphic derivatives on $\mathbb{C} = \mathbb{R}^2$. Then the physical interpretation

$$\{Q_{\pm}, \overline{Q}_{\pm}\} = 2\partial_{\pm} = H \pm P$$

H is the operator generating traslation in "time" x^2.

R-symmetry There is not only rotation in the even part, but also in the odd part. Consider *vector*

$$\Phi(x^m, \theta^{\pm}, \overline{\theta}^{\pm}) \to e^{i\alpha q_v}\Phi(x^m, e^{-i\alpha\theta^{\pm}}, e^{i\alpha}\overline{\theta}^{\pm})$$

Axial R-symmetry

$$\Phi(x^m, \theta^{\pm}1, \overline{\theta}^{\pm}) \to e^{i\beta q_A}\Phi(x^m, e^{\mp i\beta}\theta^{\pm}, e^{\mp i\beta}\overline{\phi}^{\pm})$$

where q_v is the vector R-charge and q_A is the axial R-charge. Call the generator of these two symmetries F_v and F_A.

4.1 Superalgebra

Superalgebra: $\{Q_{\pm}, \overline{Q}_{\pm}\} = 2\partial_{\pm}$,

$$\{Q_+, Q_-\} = Z \text{ central charge}, \{\overline{Q}_+, \overline{Q}_-\} = Z^*$$
$$\{Q_-, \overline{Q}^+\} = \widetilde{Z}, \{Q_+, \overline{Q}_-\} = \widetilde{Z}^*$$
$$Q_+^2 = Q_-^2 = \overline{Q}_+^2 = \overline{Q}_+^2 = \overline{Q}_-^2 = 0$$
$$[iM, Q_{\pm}] = \mp iQ_{\pm}, [iM, \overline{Q}_{\pm}] = \mp i\overline{Q}_{\pm}$$
$$[iF_v, Q_{\pm}] = -iQ_{\pm}, [iF_v, \overline{Q}_{\pm}] = i\overline{Q}_{\pm}$$
$$[iF_A, Q_{\pm}] = \mp iQ_{\pm}, [iF_A, \overline{Q}_{\pm}] = \pm i\overline{Q}_{\pm}$$

irreps of superalgebra Superspace derivative

$$D_{\alpha} = \frac{\partial}{\partial\theta^{\alpha}} + i(\gamma^m)_{\alpha\beta}\overline{\theta}^{\beta}\partial_m$$
$$\overline{D}_{\alpha} = -\frac{\partial}{\partial\overline{\theta}^{\alpha}} - i\theta^{\beta}(\gamma^m)_{\alpha\beta}\partial_m$$

In components

$$D_\pm = \frac{\partial}{\partial\theta^\pm} - \overline{\theta}^\pm \partial_\pm$$
$$\overline{D}_\pm = -\frac{\partial}{\partial\overline{\theta}^\pm} + \theta^\pm \partial_\pm$$

A *chiral superfield* is a superfield such that

$$\overline{D}_\pm \Psi = 0.$$

To solve this, introduce a new variable

$$y^\pm = x^\pm - \theta^\pm \overline{\theta}^\pm$$

which is the superspace analogue of (anti)holomorphic coordinates. Indeed (notation???)

$$\overline{D}_\pm y^\pm = 0, \overline{D}_\pm y^\mp = 0.$$

Then the solution is

$$\Phi(y^\pm, \theta^+, \theta^-) = \phi(y^\pm) + \theta^+ \psi_+(y^\pm) + \theta^- \psi_-(y^\pm) + \theta^+\theta^- F(y^\pm).$$

Similarly we can solve antichiral superfield $D_\pm \overline{\Phi} = 0$ using $\overline{y}^\pm = x^\pm - \overline{\theta}^\pm \theta^\pm$.

Another possible constrant is *twisted chiral fields* where

$$\overline{D}_+ U =, D_- U = 0.$$

Introduce $\widetilde{y}^\pm = x^\pm \mp \theta^\pm \overline{\theta}^\pm$. Then

$$\overline{D}_+ \widetilde{y}^+ = \overline{D}_+ \widetilde{y}^- = D_- \widetilde{y}^+ = \overline{D}_- \widetilde{y}^- = 0.$$

Then $U = U(\widetilde{y}^\pm, \theta^+, \overline{\theta}^-)$.

Supersymmetry transformation

$$\delta\Phi = [\zeta_\alpha Q^\alpha + \overline{\xi}^\alpha \overline{\alpha}]\Phi = (\zeta_+ Q_- - \zeta_- Q_+ - \overline{\zeta}_+ \overline{Q}_- + \overline{\zeta}_- \overline{Q}_+)\Phi$$

relation with geometry In order to preserve SUSY specific geometric structures on M has to be introduced. We recall spinors first. Let $V \cong \mathbb{C}^d$ be a complex vector space equipped with a symmetric bilinear form g. Then $\mathrm{Spin}(V)$ is the extension

$$\mathbb{Z}_2 \longrightarrow \mathrm{Spin}(V) \longrightarrow \mathrm{SO}(V)$$

Let S be the complex Dirac module of $\mathrm{Spin}(V)$ with $\dim_{\mathbb{C}} S{=}2^{\lfloor d/2 \rfloor}$. For d odd this is irreducible and for d even $S = S^+ \oplus S^-$. $V(, g)$ gives the Clifford algebra $CL(V)$, the free tensor algebra over V modulo

$$v \cdot v = g(v, v)1.$$

Suppose $\gamma_m, m = 1, \ldots, d$ is a basis of V, $g_{mn} = ...$ For $d = 3, 4, 6$ the spin gropu actsing on irrep $\mathrm{Spin}(V)$ are the groups $\mathrm{SL}(2, \mathbb{F})$ for $\mathbb{F} = \mathbb{R}, \mathbb{C}, \mathbb{H}$.

Now for a superfield $\mathbb{R}^{(d,s)} \to M$, ...

For $\mathcal{N} = (2, 2)$ supersymmetric sigma models, we nned the target manifold to be Kähler.

Lagrangian

$$S = \underbrace{\int \mathrm{d}z\mathrm{d}\overline{z}\mathrm{d}\theta^+\mathrm{d}\theta^-\mathrm{d}\overline{\theta}^+\mathrm{d}\overline{\theta}^- K(\Phi^i, \overline{\Phi}^i)}_{\text{D-term}} + \underbrace{\int \mathrm{d}^2z\mathrm{d}\theta^+\mathrm{d}\theta^- W(\Phi^i)}_{\text{F-term}} + \text{complex conjugate}$$

where $\Phi : \mathbb{R}^{(2,(2,2))} \to M$ is a superfunction to a Kähler manifold. K is the Kähler potential, a real function and W is the superpotential, a holomorphic function.

In terms of local complex coordinates on M, $\phi^i = \varphi^i + \theta^+ + \dots$ and

$$\mathrm{d}s^2 = g_{i,\overline{j}}\mathrm{d}\varphi^i \otimes \mathrm{d}\varphi^{\overline{j}}$$

a hermitian metric on M,

$$\omega = \frac{i}{2} g_{i\overline{j}}\mathrm{d}\varphi^i \wedge \mathrm{d}\varphi^{\overline{j}},$$

a Kähler form. In particular the closedness implies that

$$\partial_k g_{i\overline{j}} = \partial_i g_{k\overline{j}}, \partial_{\overline{k}} g_{i\overline{j}} = \partial_{\overline{j}} g_{i\overline{k}}.$$

The hermitian metric $g_{i\overline{j}}$ can be derived from a real function, namely the Kähler potential:

$$g_{i\overline{j}} = \frac{\partial^2 K}{\partial\varphi^i \partial\varphi^{\overline{j}}}.$$

The Levi-Civita connection has only holomorphic and antiholomorphic indices $\Gamma^i_{jk}, \Gamma^{\overline{i}}_{\overline{j}\overline{k}}$.

For the D-term, K has an expansion

$$K(\Phi^i, \overline{\Phi}^{\overline{i}}) = K_0 + K_+\theta^+ + \dots + \theta^+\overline{\theta}^+\theta^-\overline{\theta}^- K_{\text{top}}.$$

Exercise. The kinetic part is

$$\mathcal{L}_{\text{kin}} = g_{i\overline{j}}\partial_z\varphi^i\partial_{\overline{z}}\varphi^{\overline{j}} + ig_{i\overline{j}}\overline{\psi}^{\overline{j}}_- D_z\psi^i_- + ig_{i\overline{j}}\overline{\psi}^{\overline{j}}_+ D_z\psi^i_+ + R_{i\overline{j}k\overline{\ell}}\psi^i_+\overline{\psi}^{\overline{j}}_+\psi^k_-\overline{\psi}^{\ell}_-$$

where

$$D_z\psi^i_\pm = \partial_z\psi^i_\pm + \Gamma^i_{jk}\partial_z\phi^k\psi^j_\pm.$$

The system has a symmetry $\delta S = 0$ where

$$\delta\Phi = (\zeta_+ Q_- - \zeta_- Q_+ - \overline{\zeta}_+\overline{Q}_- + \overline{\zeta}_-\overline{Q}_+)\Phi$$

as defined before.

R-symmetries Recall that $\psi \in \Gamma(\Pi S \otimes \phi^*(TM) \otimes R)$. For spin, vector R-symmetry and axial R-symmetry,

$$\mathrm{U}(1)_E : \psi^\pm \to e^{\pm i\gamma}\psi^\pm, \overline{\psi}^\pm \to e^{\pm i\gamma}\overline{\psi}^\pm$$
$$\mathrm{U}(1)_V : \psi^\pm \to e^{-i\alpha}\psi^\pm, \overline{\psi}^\pm \to e^{i\alpha}\overline{\psi}^\pm$$
$$\mathrm{U}(1)_A : \psi^\pm$$

D-term: $d^4\theta$ is invariant under $U(1)_E \times U(1)_V \times U(1)_A$. For K, $q_V = 0, q_A = 0$. If $K = f(\overline{\Phi}^i \Phi^i)$ it is invariant for all q_V and q_A. For the F-term, $d^2\theta$ has $U(1)_V$ charge $q_V = -2$. W must have $q_V = +2$.

$$W(\lambda^{q_i}\phi^i) = \lambda^2 W(\phi^i),$$

quasiholomorphic function of degree 2. $d^2\theta$ has $U(1)_A$ charge $q_A = 0$.

If the measure of the path integral is not invariant but S is then it is called an *anomaly* in physics. This is related to index theorey.

4.2 Simplified SUSY model

Take

$$S = \int_{T^2} d^2z(i\overline{\psi}_+ D_z \psi_+ + i\overline{\psi}_- \overline{D}_{\overline{z}} \psi_-)$$

where $D_z = \partial_z + A_z, \overline{D}_{\overline{z}} = \overline{\partial}_{\overline{z}} + A_{\overline{z}}$. In order for S to be invariant, we take

$$\psi_\pm \in \Gamma(T^2, E \otimes \Pi S_\pm), \overline{\psi}_\pm \in \Gamma(T^2, E^* \otimes \Pi S_\pm)$$

where E is a complex vector bundle. Then

$$K = \mathrm{ind}\overline{D} = \dim\ker \overline{D}_{\overline{z}} - \dim\ker D_z$$

which by index theorem is

$$\int_{T^2} c_1(E)$$

since the connection is flat so the Todd class is trivial. Thus the difference between the number of zero states of ψ^- and $\overline{p}si_-$ is K. Same for $\overline{\psi}_+$ and ψ_+.

The measure is

$$D\psi_\pm D\overline{\psi}_\pm = \underbrace{\prod_{\alpha=1}^{K} d\psi_-^{(0)\alpha} d\overline{\psi}_+^{(0)\alpha}}_{q_A = 2K} \underbrace{\prod_{n=1}^{\infty} d\psi_\mp^{(n)} \overline{\psi}_\pm^{(n)}}_{U(1)-\text{invariant}}.$$

Thus for $\mathcal{N} = (2,2)$ nonlinear sigma model,

$$\int_{T^2} d^2z(-2ig_{i\overline{j}}\overline{\psi}_-^{\overline{j}} \overline{D}_{\overline{z}} \psi_-^i + ...)$$

the kinetic term.

$\psi^i \in \Gamma(\varphi^*(T^{1,0}M))$, $E = \varphi^*(T^{1,0}M)$.

$$K = \int_{T^2} c_1(\varphi^*(T^{1,0}M)) = \langle c_1(TM), \varphi_*(T^2)\rangle.$$

To preserve $U(1)_A$ symmetry, the target manifold M need not only be Kähler but also $c_1(TM) = 0$. These two requirements are equivalently to saying M is Calabi-Yau.

Why is it so important to preserve R-symmetry? They are needed to define *topological twists*. For example for

$$\delta\Phi = \zeta_+ Q_- - \zeta_- Q^+ + \dots$$

SUSY requires existence of trivial sections of the spin bundle, which holds for $\mathbb{R}^2$ and T^2. If we want to formulate a SUSY model of maps from a general surface Σ. For a general Σ, the spin bundles $S_+ \cong K^{1/2}, S_- \cong K^{-1/2}$ do not admit sections. This is where the R-symmetry bundles come to rescue. Topological twists gives redefinition of spin connection with the connection of the U(1) R-symmetry bundle

$$\omega' = \omega + A^R$$

where ω is the spin connection and A^R is the R-symmetry connection. There are two choices:

1. A-model uses vector R-symmetry,
2. B-model uses axial R-symmetry.

From $\mathcal{N} = (2,2)$ NLSM one can define two distinct topological string models. We will study fixed points of A- and B-supersymmetry. They are also called BPS solutions.

Example. Gradient flow lines satisfy $S_{SQM} \geq |h(x_i) - h(x_j)|$ which is saturated by

$$\dot{\varphi}^2 - g^{ij}\partial_j h = 0.$$

Topological twist: A-model

$$\mathrm{U}(1)'_E = \mathrm{diag}(\mathrm{U}(1)_E \times \mathrm{U}(1)_V)$$

B-model

$$\mathrm{U}(1)'_E = \mathrm{diag}(\mathrm{U}(1)_E \times \mathrm{U}(1)_A)$$

so $\nabla = \partial + \omega + A^R$. This changes the representation of SUSY charges and fields...

4.3 Scalar SUSY

A-model: $Q_-, \overline{Q}_+$ are scalars. For B-model, $\overline{Q}_-, \overline{Q}^+$ (one form SUSY: Q_-, Q_+ and Q_-, Q_+ respectively). We can define dcohomologies arising from

$$Q_A = \overline{Q}_+ + Q_-, Q_B = \overline{Q}_+ + \overline{Q}_-.$$

We will strudy

1. fixed points of Q_A, Q_B,
2. cohomology of Q_A, Q_B.

Point 1 identifies the BPS solutions and their moduli spaces, while 2 gives the observables, the intersection theory on $\mathcal{M}_{BPS}$.

Remark.

1. A model is defined for all Kähler manifold and also for symplectic ones. Instead B model requires M to be Calabi-Yau. Indeed U(1) is only preserved on CYs.

2. Topological models on noncompact target spaces have $W(\phi) \neq 0$. This is called the Landau-Ginzburg model.

For $\mathcal{N} = (2, 2)$ SUSY,

$$\delta\Phi = (\zeta_+ Q_- - \zeta_- Q_+ - \overline{\zeta}_+ \overline{Q}_- + \overline{\zeta}_- \overline{Q}_+)\Phi.$$

In components

$$\begin{aligned}
\delta\varphi^i &= \xi_+ \psi^i_- - \xi_- \psi^i_+, \delta\varphi^{\bar{i}} = -\overline{\xi}_+ \overline{\psi}^{\bar{i}}_- + \overline{\xi}_- \overline{\psi}^i_+ \\
\delta\psi^i_+ &= 2\overline{\xi}_- \partial_{\bar{z}} \varphi^i + \xi_+ (\Gamma^i_{jk} \psi^j_+ \psi^k_- - \frac{1}{2} g^{i\bar{\ell}} \partial_{\bar{\ell}} \overline{W}) \\
\delta\psi_- &=
\end{aligned}$$

For A model $W = 0, \xi_+ = 1, \overline{\xi}_- = 1, \xi_- = 0, \overline{\xi}_+ = 0$

Fixed points:

$$\begin{aligned}
Q_A \rho^i_{\bar{z}} = 0 &\implies \partial_{\bar{z}} \varphi^i = 0 \\
Q_A \rho^{\bar{i}}_z = 0 &\implies \partial_z \overline{\varphi}^{\bar{i}} = 0
\end{aligned}$$

So BPS solutions of A model are holomorphic maps.

$$S_A = \int_\Sigma \varphi^*(\omega) = 2\pi n$$

where n is the degree of the holomorphic map ϕ. Then

$$\mathcal{M}_{BPS} = \coprod_n \mathcal{M}_n(\Sigma, M).$$

For the B-model $W = 0, \overline{\zeta}_+ = 1, -\overline{\zeta}_- = 1, \zeta_+ = 0, \zeta_- = 0$. The fixed points are

$$\begin{aligned}
&Q_B \varphi^i = 0, Q_B \overline{\varphi}^{\bar{i}} = \overline{\eta}^{\bar{i}}, \\
&Q_B \overline{\eta}^{\bar{i}} = 0, Q_B \theta_i = 0 \\
&Q_B \rho^i
\end{aligned}$$

which is equivalent to say φ is constant. Note that the B model depends on the complex structure of M. The A and B models are exchanged by $\mathbb{Z}_2$ automorphism of $\mathcal{N} = (2, 2)$ algebra.

4.4 Chiral ring

The chiral ring is the ring of observables of topological A and B modesl. Recall that the observables are in bijection with cohomology of Q_A or Q_B. The cohomologies are different but they are exchanged by the $\mathbb{Z}/2$-automorphism which we call mirror symmetry.

Q-invariant operators from superfields Let us focus on the B model. A chiral multiplet is a field Φ such that

$$\overline{D}_\pm \Phi = 0.$$

The lowest component of Φ, which we call φ, is invariant under $\overline{Q}_\pm$:

$$\overline{Q}_\pm \varphi = \overline{Q}_\pm \Phi|_{\theta^\pm=0}.$$

Note

$$\begin{aligned}\overline{Q}_\pm &= -\frac{\partial}{\partial\overline{\theta}^\pm} - \theta^\pm \partial_\pm \\ \overline{D}_\pm &= -\frac{\partial}{\partial\overline{\theta}^\pm} + \theta^\pm \partial_\pm\end{aligned}$$

so

$$\overline{Q}_\pm = \overline{D}_\pm - 2\theta^\pm \partial_\pm.$$

Substitute in and use the condition that Φ is chiral,

$$\overline{Q}_\pm \varphi = 0.$$

Since $Q_B = \overline{Q}_+ + \overline{Q}_-$, it follows that

$$Q_B \varphi = 0.$$

They form a ring by Leibnitz rule, ergo the name chiral ring.

Exercise. Show that the lowest component of a twisted chiral superfield is annihilated by Q_A.

Topological observables The correlators of the observables $\mathcal{O}$ do not depend on the insertion point on the source Σ of the map:

$$\langle \partial_{\overline{z}} \mathcal{O} \cdots \rangle = 0$$

as a consequence of SUSY algebra.

Proof. Write $\partial_{\overline{z}} = \frac{1}{2}(\partial_2 - i\partial_1)$. Then

$$\begin{aligned}\partial_{\overline{z}} \mathcal{O} &= [H + P, \mathcal{O}] \\ &= [\{Q_+, \overline{Q}_+\}, \mathcal{O}] \\ &= \{[Q_+, \mathcal{O}], \overline{Q}_+\} + \{Q_+, [Q_+, \mathcal{O}]\} \\ &= \{[Q_+, \mathcal{O}], \overline{Q}_+\} - \{Q_+, [\overline{Q}_-, \mathcal{O}]\} \\ &\text{using } [Q_B, \mathcal{O}] = 0, [\overline{Q}_+, \mathcal{O}] = -[\overline{Q}_-, \mathcal{O}] \\ &= \{[Q_+, \mathcal{O}], \overline{Q}_+\} - [\{Q_+, \overline{Q}_-\}, \mathcal{O}] + \{\overline{Q}_-, [Q_+, \mathcal{O}]\} \\ &= \{Q_B, [Q_+, \mathcal{O}]\}\end{aligned}$$

Similarly

$$\partial_z \mathcal{O} = \{Q_B, [Q_-, \mathcal{O}]\}.$$

Recall that upon B twist, Q_+ is antiholomorphic one form. Thus (?) the correlator is independent of insertion point. □

Descent equations From the derivation above we get

$$\mathrm{d}\mathcal{O}^{(0)} = \{Q_B, \mathcal{O}^{(1)}$$

where

$$\mathcal{O}^{(1)} = \mathrm{d}z[Q_-, \mathcal{O}^{(0)}] + \mathrm{d}\overline{z}[Q_+, \mathcal{O}^{(0)}].$$

Continuing,

$$\mathrm{d}\mathcal{O}^{(1)} = \{Q_B, \mathcal{O}^{(2)}$$

where

$$\mathcal{O}^{(2)} = \mathrm{d}z\mathrm{d}\overline{z}\{Q_+, [Q_-, \mathcal{O}^{(0)}]\}.$$

Of course it stops here and $\mathrm{d}\mathcal{O}^{(2)} = 0$.

We have seen that the chiral ring observables do not depend on the metric of the source Σ. What about the target?

A variation of the Kähler potential gives

$$\int \mathrm{d}^4\theta \Delta K \sim \{\overline{Q}_+, [\overline{Q}_-, \int \mathrm{d}^2\theta \Delta K]\}|_{\overline{\theta}=\overline{\theta}^-=0}$$

up to a coboundary because (?) $\overline{Q}_+ = -\frac{\partial}{\partial \overline{\theta}^+} + \theta^+ \partial_+$. Then RHS is

$$\{\overline{Q}_+, [\overline{Q}_-, \int \mathrm{d}^2\theta \Delta K]\}|_{\overline{\theta}=\overline{\theta}^-=0}$$

so chiral ring does not depend on Kähler structure of the target M.

For variation by a twisted chiral superpotential, $\overline{Q}_+ \Delta\widetilde{W} = 0$ so

$$\begin{aligned}\int \mathrm{d}^2 z \sqrt{h} \mathrm{d}\theta^+ \mathrm{d}\overline{\theta}^- \Delta\widetilde{W} &\sim \{Q_+, [\overline{Q}_-, \Delta\widetilde{W}]\} \\ &\sim \{Q_+, [Q_B, \Delta\widetilde{W}]\} \\ &= -\{Q_B, [Q_+, \Delta\widetilde{W}]\} + 0\end{aligned}$$

since $\partial_z = 0$ (no boundary on Σ and $\widetilde{Z}^* = 0$ (since we assume the twisted central charge is 0). This shows that the chiral ring does not depend on twisted chiral deformations.

Independence on antichiral deformations

Dependence on chiral deformation:

$$\int \mathrm{d}^2\theta \Delta W \sim \{Q^+, [Q_-, \Delta W]\} \sim \Delta W^{(2)}$$

so-called marginal deformation.

Exercise. Repeat the computation for the A model.

Ring structure Choose a basis $\{\phi_i\}_{i=0}^M$ of the cohomology of Q. We have structure constants

$$\phi_i \phi_j = C_{ij}^k \phi_k$$

up to a coboundary term. Chiral ring is an unital associative commutative algebra over $\mathbb{C}$. In terms of structure constant this is saying

$$C_{i\ell}^m C_{jk}^\ell = ...$$

To compute the structure constants,

$$C_{ijk} = \langle \phi_i \phi_j \phi_k \rangle_0$$

where the subscript 0 denotes integration over maps from genus 0 surface to M.

$$\langle \phi_i \phi_j \phi_k \rangle_0 = \langle \phi_i C^{\ell}_{jk} \phi_\ell \rangle_= = C^{\ell}_{jk} \langle \phi_i \phi_\ell \rangle$$

$\langle \phi_i \phi_\ell \rangle$ is a topological metric $\eta_{i\ell}$.

C_{ijk} depends only holomorphically on the chiral parameters (since antichiral deformation is trivial). More precisely

$$\partial_\ell \langle \phi_i \phi_j \phi_k \rangle_0 = \langle \phi_i \phi_j \phi_k \int_\Sigma \mathcal{O}^{(2)}_\ell \rangle$$

(?)

$$\partial_\ell C_{ijk} = \partial_i C_{\ell jk}$$

leading to WDVV equation. The symmetry of C_{ijk} together with WDVV implies the existence of $\mathcal{F}$, called *prepotential*, such that

$$C_{ijk} = \partial_i \partial_j \partial_k \mathcal{F}.$$

This makes the algebra of chiral ring a *Frobenius algebra*.

For A model with $W = 0$,

$$\begin{aligned} Q_A \varphi^i &= \chi^i, Q_A \varphi^{\bar{i}} = \chi^{\bar{i}} \\ Q_A \chi^i &= 0, Q_A \chi^{\bar{i}} = 0 \\ Q_A \rho^i_{\bar{z}} &= \partial_{\bar{z}} \varphi^i + \Gamma^i_{jk} \rho^j_{\bar{z}} \chi^k \\ Q_A \rho^{\bar{i}}_{\bar{z}} &= \partial_{\bar{z}} \varphi^{\bar{i}} + \Gamma^{\bar{i}}_{\bar{j}\bar{k}} \rho^{\bar{j}}_{\bar{z}} \chi^{\bar{k}} \end{aligned}$$

The Q_A-cohomology in the zero forms. $Q_A \omega = 0$ if and only if ω is closed with respect to de Rham differential. Similar for exact forms. Thus $Q_A = \mathrm{d} = \partial + \overline{\partial}$. Thus the chiral ring for A-model as a vector space is isomorphic to $H^*_{\mathrm{dR}}(M)$.

For B model, the chiral ring is isomorphic to $\bigoplus H^{0,p}(M, \bigwedge^q T^{1,0} M)$.

Landau-Ginzburg B model Observables correspond to holomorphic functions.

$$\{\mathcal{O}_{\mathrm{LG}}\} \cong \mathbb{C}[\varphi^1, \ldots, \varphi^n]/(\partial_j W).$$

4.5 CY moduli spaces

Recall that we define a CY manifold to be a Kähler manifold with $c_1(TM) = 0$. The second condition can be equivalently stated as holonomy $\mathrm{SU}(n)$ or trivial canonical bundle. This means $h^{n,0} = 1$. We also assume the manifold is simply connected so $h^{1,0} = h^{0,1} = 0$. We will focus on $n = 3$ so by Serre duality $h^{2,0} = h^{0,2} = 0$. Thus the Hodge diamond is determined except for $h^{1,1}$, which is associated with Kähler moduli, and $h^{2,1}$, which is associated with the complex structure.

For A model, the Kähler metric gives a form $\omega^{1,1}$. The Kähler cone is the set of $\omega \in H^{1,1}(M)$ such that

$$\int_C \omega \geq 0, \int_D \frac{\omega \wedge \omega}{2} \geq 2, \int_M \frac{\omega^3}{3!} \geq 0.$$

It is a cone of dimension $h^{1,1}$.

For B model,

$$\delta\Omega = \Omega + \omega^a_{\overline{a}} \Omega_{abc} \mathrm{d}\overline{z}^{\overline{a}} \mathrm{d}z^b \mathrm{d}z^c$$

$\omega \in H^1(M, TM)$ infinitesimal deformations. A theorem of Tian-Todorov states that the deformation is unobstructed, so

$$\dim \mathcal{M}_{\mathbb{C}} = h^{2,1} + 1$$

where the number 1 comes from overall scaling of ω. Thus $\mathcal{M}_{\mathbb{C}}$ is a projective space.

4.6 Mirror symmetry

Mirror symmetry is the automorphism

$$Q_- \leftrightarrow \overline{Q}_-, F_V \leftrightarrow F_A$$

thus exchanging A and B model. Using the unique $(n, 0)$-form we can write the B-model chiral ring as $H^{p,q}(M)$. Thus B model chiral ring describes complex structure deformations of M_B. For NLSM with compact CY3 target, mirror symmetry exchanges $H^{1,1}(M_A)$ and $H^{2,1}(M_B)$.

Example. Consider a torus of raius R_1 and R_2. The Kähler form is $\Omega = R_1 R_2 \mathrm{d}\theta_1 \mathrm{d}\theta_2$. The Kähler modulus is $R_1 R_2 = A$.

The complex structure $\tau = i\frac{R_1}{R_2}$. The complex modulus is $\frac{R_1}{R_2}$. Mirror symmetry exchanges A and N, equivalent to invert of radius of one of the circles ($R_2 \mapsto \frac{1}{R_2}$. This is related to T-duality symmetry.

4.7 Topological A-model

$S \geq \int_\Sigma \varphi^*(\omega + iB)$,

Q_A-fixed point:

$$\partial_z \varphi^{\overline{i}} = 0, \overline{\varphi}_{\overline{z}} \varphi^i = 0 \text{ holomorphic maps}$$
$$D_z \psi^{\overline{i}} = 0, D_{\overline{z}} \psi^i = 0 \quad TM$$
$$D_{\overline{z}} \overline{\psi}^{\overline{i}}_z = 0, D_z \psi^i_{\overline{z}} = 0 \text{ obstruction}$$

Then upon localising,

$$\int_D \varphi D\psi D\overline{\Psi} e^{-S} = \int_{M, \ker(D_z, D_{\overline{z}}^\dagger)} dmd\psi^{(0)} \mathrm{d}\overline{\psi}^{(0)} e^{-S}$$

For the bosonic part, there is a stratification

$$\mathcal{M}_g(M, C) = \coprod_d \mathcal{M}_g(M, \beta)$$

where $d \in \mathbb{N}^{b_2(M)}$ and $\beta = \varphi_*(\Sigma_g) \in H_2(M;\mathbb{Z})$. Give an integral basis $\{s_i\}$ of $H_2(M;\mathbb{Z})$ and write $\beta = \sum_i d_i[S_i]$. Then $d = \{d_i\}$ is the degree of the holomorphic map. Then

$$e^{-\int_\Sigma \varphi^*(\omega_\mathbb{C})} = q^\beta := \prod q_i^{d_i}$$

where $q_i = e^{-t_i}$ for t_i the complex Kähler moduli.

For fermions there is an anomaly (which is related to the virtual dimension of the moduli space):

$$\#\psi^{(0)} - \#\overline{\psi}^{(0)} = \operatorname{ind} D_z.$$

By Hirzebruch-Riemann-Roch

$$\operatorname{ind} D_z = \int_{\Sigma_g} \operatorname{ch}(\varphi^*(TM) \operatorname{td}(T_\Sigma) = \dim_\mathbb{C} M(1-g) + \int_{\Sigma_g} \varphi^*(c_1(T_M))$$

Remark. The index is the virtual dimension of the moduli space. If $g = 0$ and M is CY then it is positive so we need to integrate over some forms (in physics language, inserting observables). If $g = 1$ and M is CY then the virtual dimension is 0. For $g > 1$, the virtual dimension is negative, meaning that there is no solution. This suggests we should integrate also over the moduli space of complex structures on Σ_g. In physics language it is topological gravity. The (real) dimension of moduli space of complex structure on genus g surface is $6g - 6$ so after taking it into account the virtual dimension becomes

$$\dim_\mathbb{C} M(1-g) + \int \varphi^*(c_1(T_M)) + 3(g-1) = (\dim_\mathbb{C} M - 3)(1-g) + \int_\Sigma \varphi^*(c_1(T_M))$$

which is zero for M CY3. Non trivial topological string amplitudes at all genera.

What is to come: schematically

$$\langle \prod_k \mathcal{O}(P_k) \rangle_A = \sum_\beta q^\beta N_\beta^g$$

RHS is called the *Gromov-Witten invariants*, informally the “number” of holomorphic maps of degree β from Σ_g to M.

4.8 Evaluation of observables

Generic case: Ind $= K \geq 0, \#\overline{\psi}^{(0)} = 0$. Let $\mathcal{O}_i(x_i)$ be the pullback of $\omega_i \in H^*(M)$ via the evaluation map at $P_i \in \Sigma$

$$\begin{aligned} \operatorname{ev}_i : \mathcal{M}_g(M,\beta) &\to M \\ \varphi &\mapsto \varphi(P_i) \end{aligned}$$

Then

$$\langle \mathcal{O}_1(P_1) \cdots \mathcal{O}_n(P_n) \rangle = \sum_\beta q^\beta \int_{\mathcal{M}_g(M,\beta)} \operatorname{ev}_1^* \omega_1 \wedge \cdots \wedge \operatorname{ev}_1^* \omega_n$$

is called the *Gromov-Witten invariants* and denoted N_g^β. If $[\omega_i]$ is the Poincaré dual of D_i then

$$N_g^\beta(D_1 \cdots D_n) = \#\{\text{holomorphic maps } \varphi \text{ such that } \varphi(P_i) \in D_i, \varphi_*(\Sigma) = \beta\}.$$

$N_g^\beta \in \mathbb{Q}$ in general because of nontrivial automorphism group on the moduli space of stable maps.

The simplest example is $\beta = 0$ so $\varphi_*(\Sigma)$ is a point. Then

$$\mathcal{M}_{g=0}(M, 0) \cong M$$

and $\mathrm{ev}_i = \mathrm{id}_M$ for all P_i. Then

$$\langle \mathcal{O}_1(P_1) \cdots \mathcal{O}_n(P_n) \rangle_0^{\mathrm{const}} = \int_M \omega_1 \wedge \cdots \wedge \omega_n,$$

"recovering" intersection theory on M. For example for M CY3,

$$C_{abc}^{\mathrm{const}} = \langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}_c \rangle_0^{\mathrm{const}} = \int_M \omega_a \wedge \omega_b \wedge \omega_c = \#(D_a \cdot D_b \cdot D_c).$$

Topological metric

$$\eta_{ab} = \langle \widetilde{O}_a \widetilde{O}_b \rangle = \int_M \widetilde{\omega}_a \wedge \widetilde{\omega}_b.$$

4.9 nongeneric case

$H^0(\Sigma, K \otimes \varphi^*(T_M^*)) \neq 0$, which we assume to have constant dimension ℓ. This the rank of the obstruction bundle $\mathcal{O}_M$. Then when integrating there is an extra insertion $e(\mathcal{O}_M)$, an (ℓ, ℓ)-form:

$$\langle \mathcal{O}_1(P_1) \cdots \mathcal{O}_n(P_n) \rangle_\beta = \int_{\mathcal{M}_g(M,\beta)} \mathrm{ev}_1^* \omega_1 \cdots \wedge \mathrm{ev}_n^* \omega_n \wedge e(\mathcal{O}_M).$$

Take derivative with respect to Kähler moduli,

$$\frac{\partial}{\partial t^\ell} \langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \rangle = \langle \mathcal{O}_i \mathcal{O}_j \mathcal{O}_k \int_\Sigma \mathcal{O}_\ell^{(2)} \rangle$$

4.10 Example of topological A model

We consider maps $S^1 \cong \mathbb{P}^1 \to \mathbb{P}^1$. The cohomology of Q_A is the same as the de Rham cohomology, which is $\mathbb{C}$ in degree 0 and 2. The observables of A model are generated by P and Q in H^0 and H^1. From intersection theory we know

$$\int_{\mathbb{P}^1} H = 1$$

for $H \in H^2(\mathbb{P}^1)$ the hyperplane class. This corresponds to the topological metric

$$\eta_{PQ} = \eta_{QP} = 1$$

and 0 otherwise.

Three point correlator: a priori

$$\langle QQQ \rangle = \sum_{n \in \mathbb{N}} \langle QQQ \rangle_n.$$

For φ of degree n, let $\beta = \varphi_*(\mathbb{P}^1) = n[H]$. The expected dimension of $\mathcal{M}_0(\mathbb{P}^1, \beta)$ is

$$\dim_{\mathbb{C}} \mathbb{P}^1(1-g) + \int_{\mathbb{P}^1} \varphi^*(c_1(T^1_{\mathbb{P}})) = 1 + \int_{\mathbb{P}^1} \varphi^*(2H) = 1 + 2n.$$

The axial R-symmetry charge $R = (\#\chi^i, \chi^{\bar{i}})$ is 6 and must equal to twice the expected dimension, so the only contribution is from $n = 1$ maps. Recall $\langle QQQ \rangle_1$ is the "number" of holomorphic maps from the source $\mathbb{P}^1$ to the target $\mathbb{P}^1$ with degree 1 and mapping 3 fixed points of the source $\mathbb{P}^3$ to 3 fixed points of the target $\mathbb{P}^1$. There is only one such map. Thus

$$\langle QQQ \rangle = e^{-t}.$$

The quantum cohomology we get is

$$C_{PPQ} = \eta_{PQ} = 1, C_{PPP} = 0, C_{PQQ} = 0, C_{QQQ} = e^{-t}$$

in which we get a correction by the Kähler moduli. The prepotential is

$$F = -\frac{1}{2} v^2 t + e^{-t}$$

where v is associated to H^0 and t is associated to H^2. One can check $\partial_i \partial_j \partial_k F = C_{ijk}$.

Remark. F can also be obtained as τ-function of estended Toda hierarchy. C.f. Carlet-Dubrovin-Zhang, Okounkov-Pandhariphande

4.11 Local $\mathbb{P}^1$

$\mathbb{P}^1$ is not CY so we want to locally embed it in a CY3 and consider its tubular neighbhourhood. Total space of a rank 2 vector bundle $\mathcal{L}_1 \oplus \mathcal{L}_2 \to \mathbb{P}^1$ with $\mathcal{L}_i = \mathcal{O}(-n_i)$. The condition that this is CY is equivalent to $n_1 + n_2 - 2 = 0$.

Example. The conifold $\mathrm{Tot}(\mathcal{O}(-1) \oplus \mathcal{O}(-1))$ which is the crepant resolution of conifold singularity $ab - cd = 0$.

$\mathbb{P}^1 \to \mathcal{O}(-1) \oplus \mathcal{O}(-1)$ isolated rational curve in the target. Multiple covering maps are known explicitly:

$$\mathbb{P}^1 \to \mathbb{P}^1$$
$$z \mapsto u$$
$$(x, y) \mapsto (s, t)$$

For $d = 1$, $u = \frac{az+b}{cz+d}$ is determined by the three points. For $d > 1$,

$$\frac{s}{t} = \frac{\sum a_i x^i y^{d-i}}{\sum b_i x^i y^{d-i}}$$

so $\overline{\mathcal{M}}_d(\beta) \cong \mathbb{P}^{2d+1}$.

$$C_{abc}(t) = [e_a] \cap [e_b] \cap [e_c] + \sum_k \sum_{d|k} \frac{k^2}{d^3} q^k$$

4.12 Topological Landau-Ginzburg B-model

We assume the critical points $y_1, \ldots, y_N$ are isolated. The chiral ring is $\mathbb{C}[\varphi^1, \ldots, \varphi^n]/\partial_i W$. ...

$$\langle \mathcal{O}_{f_1} \cdots \mathcal{O}_{f_s} \rangle = \sum_{\{y_a\}} f_1(y_a) \cdots f_s(y_a) (\det \partial_i \partial_j W)^{g-1}(y_a)$$

For a sphere ($g = 0$ the topological metric is

$$\eta_{ij} = \sum_{\{y_a\}} = \frac{f_i f_j}{\det \partial_i \partial_j W}(y_a)$$

The three point function is

$$C_{ijk} = \sum_{\{y_a\}} \frac{f_i f_j f_k}{\det \partial_i \partial_j W}(y_a).$$

Example. Sine-Gordon model: $S^2 \to \mathbb{C}^*$, $W = z + e^{-t} z^{-1}$. The chiral ring is generated by $1, z$ subject to $z^2 = e^{-t}$ (since $W' = 1 - \frac{1}{z^2} e^{-t} = 0$. The critical points are $z^* = \pm e^{-t/2}$. The Hessian is

$$z\partial_z(z\partial_z W)|_{z^*} = (z + \frac{e^{-t}}{z})|_{z^*} = \pm 2e^{-t/2}.$$

Correlator

$$\begin{aligned}
\langle 111 \rangle_0 &= \frac{1}{2e^{-t/2}} + \frac{1}{-2e^{-t/2}} = 0 \\
\langle 11z \rangle_0 &= \frac{e^{-t/2}}{2e^{-t/2}} + \frac{-e^{-t/2}}{-2e^{-t/2}} = 1 \\
\langle 1zz \rangle_0 &= \frac{e^{-t}}{2e^{-t}} + \frac{e^{-t}}{-2e^{-t/2}} = 0 \\
\langle zzz \rangle_0 &= \frac{e^{-3t/2}}{2e^{-t/2}} + \frac{-e^{-3t/2}}{-2e^{-t/2}} = e^{-t}
\end{aligned}$$

This is the same as A-model on $\mathbb{P}^1$ target upon identification P with 1 and Q with z. It comes from mirror symmetry. One can prove that the mirror of A-model on $\mathbb{P}^1$ is indeed Landau-Ginzburg B-model on $\mathbb{C}^*$ with Sine-Gordon superpotential.

4.13 B-model on compact CY

Fixed locus are constant maps and the moduli space is M. The observables are generated by the chiral ring

$$\bigoplus_{p,q} H^{0,p}(M, \bigwedge^q TM).$$

The virtual dimension is $\dim_{\mathbb{C}} M(1-g) = n(1-g)$. On a CY3,

$$H^{0,p}(M, \bigwedge^q TM) \cong H^{n-q,p}(M).$$

Vector R-symmetry: $\sum_i p_i = \sum_i q_i$.

We require

$$\sum_{i=1}^{s}(p_i + q_i) = 2n(1-g).$$

For $g = 0$ we need $\sum p_i = \sum q_i = n$.

$$\langle \mathcal{O}_a \mathcal{O}_b \mathcal{O}_c \rangle = \int_M \omega_{\bar{a}}^{i} \omega_{\bar{b}}^{j} \omega_{\bar{c}}^{k} \Omega_{ijk} \wedge \Omega$$

Choose a basis for $H_3(M)$ α_I, β^I for $I = 0, \ldots, h^{2,1}$ such that

$$\alpha_I \cap \beta^J = \delta_I^J.$$

Let $z^I = \int_{\alpha_I} \omega, G_I = \int_{\beta^I} \Omega$.

$$\begin{aligned}
\theta &= \int_M \Omega \frac{\partial}{\partial z^k} \Omega \text{ Hodge-Riemann} \\
&= \int_{\alpha_J} \Omega \int_{\beta^J} \partial_k \Omega - \int_{\alpha_J} \partial_k \Omega \int_{\beta^J} \Omega \\
&= \int_{\alpha_J} (z^I A_I - G_I B^I) \int_{\beta^J} \frac{\partial}{\partial z^k}(z^I A_I - G_I \beta^I) - \int_{\alpha_J} \frac{\partial}{\partial z^k}(z^I A_I - G_I B^I) \int_{\beta^J} (z^I A_I - G_I \beta^I)
\end{aligned}$$

Using

$$\int_{\alpha_J} A_I = \delta_I^J, \int_{\beta^J} B^I = \delta_J^I$$

we get

$$G_k = \frac{\partial}{\partial z^k}(z^J G_J) - G_, G_J = \frac{1}{2}\partial_J(z^I G_I)$$

Define $G = z^I G_I$, we get $2G_J = \partial_J G$. Multiply by z^J and sum over J,

$$2z^J \frac{\partial}{\partial z^J} G = 2G.$$

This shows that G is a homogeneous polynomial of z^I of degree 2. This is the prepotential.

$$C_{IJK} = \partial_I \partial_J \partial_K G.$$

Special geometry:

$$z^I = \int_{\alpha^I} \Omega, \frac{\partial G}{\partial z^I} = \int_{\beta^I} \Omega.$$

5 Linear sigma models

Let ϕ^i, $i = 1, \dots, n$ be scalar fields and consider the Lagrangian

$$\mathcal{L} = -\frac{1}{2}\sum_i (\partial_\mu \phi^i)^2 - U(\phi).$$

For example $U(\phi) = \frac{e^2}{4}(\sum_i (\phi^i)^2 - r)^2$. The lowest energy field configurations are those with ϕ^i constant. If $r \leq 0$ then the unique mimimum is given by $\phi^i = 0$ for all i. If $r > 0$ then the mumimum is given by

$$\sum (\phi^i)^2 = r,$$

which is a sphere. We thus define

$$M_{\text{vac}} = \{(\phi^i) \in \mathbb{R}^n : \sum (\phi^i)^2 = r\} \cong S^{n-1}.$$

At each point of M_{vac} the gradient of U is zero, with Hessian $\partial_i \partial_j U$ symmetric tensor in $\mathbb{R}^n$. It can be diagonalised by an orthogonal transformation with all eigenvalues nonnegative (since it is a minimum).

TM_{vac} corresponds to zero eigenspace, while NM_{vac} are the positive ones. Physically, fields with positive eigenvalues are massive. One can study effective theory of massless modes only. Thus the linear σ-model reduces to nonlinear σ-model of maps to $M_{\text{vac}} \cong S^{n-1}$.

This generalises to the construction of M_{vac} as quotient spaces. Standard example: $\mathbb{CP}^{N-1}$. One can consider $\mathbb{CP}^{N-1}$ as the quotient of $S^{2N-1} \subseteq \mathbb{C}^N$ under the action of $\mathrm{U}(1)$. Physically, we can consider the Langrangian of N complex fields

$$\mathcal{L} = -\sum |D_\mu \phi_i|^2 - U(\phi)$$

where we identify

$$(\phi_1(x), \dots, \phi_N(x)) \sim (e^{i\gamma}\phi_1(x), \dots, e^{i\gamma}\phi_N(x))$$

and the covariant derivative

$$D_\mu \phi_i = \partial_\mu \phi_i + \sqrt{-1} v_\mu \phi_i$$

where $v_\mu \mathrm{d}x^\mu$ is the connection one form of the $\mathrm{U}(1)$-bundle. One checks that under $\phi_i \mapsto e^{i\gamma(x)\phi_i}$, $v_\mu \mapsto v_\mu - \partial_\mu \gamma$ so $D_\mu \phi_i \mapsto e^{i\gamma} D_\mu \phi_i$. and the Lagrangian is invariant under $\mathrm{U}(1)$.

One defines

$$M_{\text{vac}} = \{(\phi_i) : \sum |\phi_i|^2 = r\} / \mathrm{U}(1) \cong \mathbb{CP}^{N-1}.$$

To find the mimimum set $\frac{\partial \mathcal{L}}{\partial v_\mu} = 0$, from which we get

$$\sum_i (\overline{D_\mu \phi_i} \phi_i - \overline{\phi}_i D_\mu \phi_i) = 0$$

which is solved by

$$v_\mu = \frac{i}{2} \frac{\sum_i (\overline{\mu}_i \partial_\mu \phi_i - \partial_\mu \overline{\theta}_i \theta_i)}{\sum_i |\phi_i|^2}.$$

This is the Fubini-Study metric on $\mathbb{CP}^{N-1}$

$$g_{\mathrm{FS}} = \frac{\sum_{i=1}^{N-1} |\mathrm{d}z_i|^2}{1+\sum_{i=1}^{N-1}|z_i|^2} - \frac{\sum_{i=1}^{N-1} |\overline{z}_i \mathrm{d}z_i|^2}{(1+\sum |z_i|^2)^2}.$$

The standard procedure for reducing to NLSM: One takes tangent vectors in $\mathbb{C}^N$, imposes the condition $\partial U = 0$ (i.e. orthogonality to orbits) to obtain tangent space to the quotient target space M_{vac}.

C.f. Hitchin-Karlhede-Lindstrom-Rocek, HyperKähler metrics and SUSY.

5.1 SUSY gauged linear sigma model

Recall that we have met chiral superfield with Lagrangian (in flat space)

$$\mathcal{L} = \int \mathrm{d}^4\theta \overline{\phi}\phi.$$

We want a gauge theory for the transformation $\phi \mapsto e^{iA}\phi$. First note that A itself must be a chiral superfield. To make the Lagrangian invariant, we introduce a new real superfield V which transforms as

$$V \mapsto V + i(\overline{A} - A).$$

The modified Lagrangian is defined to be

$$\mathcal{L} = \int \mathrm{d}^4\theta \overline{\phi} e^V \phi$$

which is now invariant. Expressed in terms of components,

$$\begin{aligned} V &= \theta^-\overline{\theta}^-(v_0 - v_1) + \theta^+\overline{\theta}^+(v_0 + v_1) \\ &= -\theta^-\overline{\theta}^+\sigma - \theta^+\overline{\phi}^-\overline{\phi} \\ &= +i\theta^-\theta^+(\overline{\theta}^- \end{aligned}$$

this is the supersymmetrization of the gauge connection. V is the connection superfield.

The curvature superfield is

$$\Sigma = \overline{D}_+ D_- V$$

which is a twisted chiral superfield. Σ is invariant under $V \mapsto V + i(\overline{A} - A)$ so

$$\overline{D}_+\Sigma = D_-\Sigma = 0.$$

In terms of components,

$$\Sigma = \sigma(\widetilde{y}) + i\theta^+\overline{\theta}_+(\widetilde{y}) - i\overline{\theta}^-\lambda_-(\widetilde{y}) + \theta^+\overline{\theta}^-[D(\widetilde{y}) - iv_{01}](\widetilde{y})$$

where $v_{01} = \partial_0 v_1 - \partial_1 v_0$ is the curvature of v. This is the supersymmetrisation of the curvature.

The SUSY Lagrangian is

$$\begin{aligned}\mathcal{L}_{\text{kin}} &= \int \mathrm{d}^4\theta\overline{\phi}e^V\phi \\ &= -D^\mu\overline{\phi}D_\mu\phi + i\overline{\psi}_-D_{\overline{z}}\psi^- \\ &+ i\overline{\psi}_+D_z\psi_+ + D|\phi|^2 \\ &+ |F|^2 - |\sigma|^2|\phi|^2 \\ &- \overline{\psi}_-\sigma\psi_+ - \overline{\psi}_+\overline{\sigma}\psi_- \\ &- i\overline{\phi}\lambda_-\psi_+\psi^i\overline{\phi}\lambda_+\psi_- \\ &+ i\overline{\psi}_+\overline{\lambda}_-\phi - i\overline{\psi}_-\overline{\lambda}_+\phi\end{aligned}$$

Note $D_{\overline{z}}$ is the U(1)-covariant derivative, not to be confused with D, the auxillary derivative for superfield.

$$\begin{aligned}\mathcal{L}_{\text{gauge}} &= -\frac{1}{2e^2}\int \mathrm{d}^4\theta\overline{\Sigma}\Sigma \\ &= \frac{1}{2e^2}(-\partial^\mu\overline{\sigma}\partial_\mu + i\overline{\lambda}_-\partial_{\overline{z}}\lambda_- + i\overline{\lambda}_+\partial_z\lambda_+ + v_{01}^2 + D^2).\end{aligned}$$

Finally there is the twisted superpotential

$$\widetilde{W}_{\text{FI},\theta} = -t\Sigma$$

where $t = r - i\theta$.

$$\mathcal{L}_{\text{FI},\theta} = \frac{1}{2}(-t\int \mathrm{d}^2\widetilde{\theta}\Sigma + \text{cc}) = rD + \theta v_{01}.$$

The two terms on the far right side are called Fayet-Iliopoules term and θ-angle.

The full Lagrangian is

$$\mathcal{L} = \int \mathrm{d}^4\theta(\overline{\Phi}e^V\Phi - \frac{1}{2e^2}\overline{\Sigma}\Sigma) + \frac{1}{2}(-t\int \mathrm{d}^2\widetilde{\theta}\Sigma + \text{cc}).$$

The potential is

$$U(\phi,\sigma) = |\sigma|^2|\phi|^2 + \frac{e^2}{2}(|\phi|^2 - v)^2.$$

Remark. Recall that we have R-symmetry $\mathrm{U}(1)_V \times \mathrm{U}(1)_A$. $\mathcal{L}$ is invariant if we assign $(0,2)$ to Σ. We can then generalise to multiple bundles: Consider $\prod_{a=1}^k \mathrm{U}(1)$. Φ_i “matter chiral fields” transform as

$$\Phi_i \mapsto e^{iQ_{ai}A_a}\Phi_i.$$

The Lagrangian is then

$$\mathcal{L} = \int \mathrm{d}^4(\theta\sum_i \overline{\Phi}_i e^{Q_{ia}V_a}\Phi_i - \sum_{a,b=1}^k \frac{1}{2e_{a,b}^2}\overline{\Sigma}_a\Sigma_b) + \frac{1}{2}(\int \mathrm{d}^2\widetilde{\theta}\sum_a(-t_a\Sigma_a)).$$

We can add a superpotential term

$$\mathcal{L}_W = \int \mathrm{d}^2\theta W(\Phi_i) + \text{cc}$$

The full potential term is

$$U = \sum_i |Q_{ia}\sigma_a|^2|\phi_i|^2 + \sum_{a,b} \frac{(e^{a,b})^2}{2}(Q_{ia}|\phi_i|^2 - v_a)(Q_{ib}|\phi_i|^2 - v_b) + \sum_i \left|\frac{\partial W}{\partial \phi_i}\right|^2.$$

Example (SUSY $\mathbb{CP}^{N-1}$)**.**

$$U = \sum_i |\sigma|^2|\phi_i|^2 + \frac{e^2}{2}(\sum |\phi_i^2 - r)^2.$$

For $r > 0$, the extrema is given by $\sigma = 0, \sum |\phi_i|^2 = r$.

$$M_{\text{vac}} = \mathbb{CP}^{N-1} = \{(\phi_1, \dots, \phi_N) : \sum_i |\phi_i|^2 = r\}/\,\mathrm{U}(1).$$

The "effective" Lagrangian for TM_{vac} turns out to be the NLSM of $\mathbb{R}^2 \to \mathbb{CP}^{N-1}$.

Equation of motion for fermions:

$$\sum_i \overline{\Phi}_i \psi_{i\pm} = 0, \sum_i \overline{\psi}_{i\pm}\phi_i = 0.$$

In other words, $\psi_\pm = (\psi_{i\pm}, \overline{\psi}_{i\pm})$ is tangent to $\sum_i |\phi_i|^2 = r$ as in NLSM and orthogonal to U(1)-orbit $\delta(\phi_j, \overline{\phi}_j) = (i\phi_j, -i\overline{\theta}_j)$.

Vector multiplet equation:

$$v_\mu = \frac{i}{2}\frac{\sum_i(\overline{\phi}_i\partial_\mu\phi_i - \partial_\mu\overline{\phi}_i\phi_i)}{\sum_i |\phi_i|^2}, \sigma = -\frac{\sum_i \overline{\psi}_{i+}\psi_{i-}}{\sum_i |\phi_j|^2}.$$

Substitute in this solution $v_\mu = v_\mu^*$ we get r times Fubini-Study metric. σ gives four fermion terms.

Recall the complexified Kähler modulus consists of two parts

$$\omega_{\text{FS}} + iB,$$

where B is the pullback via ϕ of curvature 2-form of the U(1)-connection on $\mathbb{CP}^{n-1}$. Indeed v_μ^* is the pullback via ϕ of a U(1) connection A on M_{vac}, $\mathrm{d}A = \frac{\omega_{\text{FS}}}{2\pi}$. Then

$$\frac{\theta}{2\pi}\int \mathrm{d}v = \frac{\theta}{2\pi}\int \mathrm{d}(\phi^* A) = \frac{\theta}{2\pi}\int \phi^*\omega$$

which is the B-field coupling if we set $B = \frac{\theta}{2\pi}\omega_{\text{FS}}$.

$$[\omega] - i[B] = \frac{t}{2\pi}[\omega_{\text{FS}}]$$

where t is the twisted superpotential parameter $t = r - i\theta$. Thus the complexified Kähler class is a twisted chiral parameter.

5.2 Toric manifolds

To work with toric manifolds we work with $\mathrm{U}(1)^k$. Same as before we introduce N fields $\Phi_1, \ldots, \Phi_N$ with charges Q_{ia} where $1 \leq i \leq N, 1 \leq a \leq k$. The coupling constants are

$$\frac{1}{e^2_{a,b}} = \delta_{ab} \frac{1}{e^2_a}.$$

The potential is thus

$$U = \sum_i |Q_{ia}\sigma_a|^2 |\phi_i|^2 + \sum_a \frac{e_a^2}{2} \left(\sum_i Q_{ia} |\phi_i|^2 - r_a \right)^2.$$

The moment maps are indexed by a

$$\mu_a = \sum_i Q_{ia} |\phi_i|^2 - r_a$$

and the potential is minimised at $\mu_a = 0$.

Choose r_a such that $U = 0$ implies $\sigma_a = 0$ for all a. Then

$$X_r = \{(\Phi_1, \ldots, \Phi_N) : \mu_a = 0\} / \, \mathrm{U}(1)^k = \mu^{-1}(0) / \, \mathrm{U}(1)^k$$

is a symplectic quotient.

X_r also has a complex structure, namely that inherited from $\mathbb{C}^N$. Indeed we can regard X_r as the quotient

$$X_r \cong (\mathbb{C}^N - P)/(\mathbb{C}^*)^k$$

where P is the locus of $\mathbb{C}^N$ whose $(\mathbb{C}^*)^k$-orbits do not contain solutions to $\mu_a = 0$. This depends on the choice of r_a's. With respect to this complex structure the symplectic form is Kähler. This whose discussion is related to Marsden-Weinstein theorem relating GIT and symplectic quotient.

The natural torus action on $\mathbb{C}^N$ descends to a $(\mathbb{C}^*)^{N-k}$ action on X_r. This action is free and transitive on an open dense submanifold, making X_r a toric variety.

The specific X_r depends on $\{r_a\}$. Given X_p, the region of $\{r_a\}$ such that $X_r \cong X_p$ is the *Kähler cone*.

There is a geometric interpretation

$$Q_{ia} = c_1(H_i)\alpha_a$$

where $\alpha_a, 1 \leq a \leq k$ generate $H_2(X_p, \mathbb{Z})$ and H_i is the line bundle over X_p admitting ϕ_i as a global section.

$$c_1(H_i) = \sum_a Q_{ia} c_1(\mathcal{L}_a)$$

where $\mathcal{L}_a$ is the line bundle over X_p defined by

$$((\mathbb{C}^N - P) \times \mathbb{C})/(\mathbb{C}^*)^k$$

with action $(\lambda_1, \ldots, \lambda_k) : c \to \lambda_a c$.

$$v_\mu^{*a} = \frac{i}{2} M^{ab} \sum_i Q_{ib}(\overline{\phi}_i \partial_\mu \phi_i - \partial_\mu \overline{\phi}_i \phi_i)$$

is the pullback via ϕ of the connection 1 form of $\mathcal{L}_a$. Here M^{ab} is the inverse of $M_{ab} = \sum_i Q_{ia} Q_{ib}$.

Using the change of basis

$$c_1(X_p) = \sum_i c_1(H_i) = \sum_i \sum_a Q_{ia} c_1(\mathcal{L}_a) = \sum_a b_{1a} c_1(\mathcal{L}_a)$$

(the first equality comes from the SES

$$0 \longrightarrow \mathcal{O}^{\oplus k} \longrightarrow H_1 \oplus \cdots \oplus H_N \longrightarrow T_{X_p} \longrightarrow 0$$

) where $b_{1a} = \sum_i Q_{ia}$. The condition X_p begin CY is equivalently to $b_{1a} = 0$.

Example. Gauge linear sigma model allows us to connect different models. We examine $\mathcal{O}_{\mathbb{P}^{N-1}}(-N)$. Consider U(1) gauge group with N chiral superfields with charge 1 and 1 chiral superfield with charge $-N$. Take $r \gg 0$, the M_{vac} is the total space of $\mathcal{O}(-N)$ over $\mathbb{P}^{N-1}$. For $r \ll 0$,

$$M_{\text{vac}} = \{N|p|^2 = |r| + \sum_i |\phi_i|^2\} / \operatorname{U}(1).$$

Since $|p| \neq 0$, either not all ϕ_i's are zero, $\mathbb{C}^N/\mathbb{Z}^N$.

Example. Consider $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$. There are four fields with charge $(1, 1, -1, -1)$.

$$\mu = |\phi_1|^2 + |\phi_2|^2 - |\phi_3|^2 - |\phi_4|^2 - r.$$

For $r \gg 0$ and $r \ll 0$, the moduli space is the total space of $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(1)$ (note the symmetry). For $r = 0$, we have a conifold singularity $xw = yz$ by letting

$$x = \phi_1\phi_3, y = \phi_1\psi_4, z = \phi_2\phi_3, w = \phi_2\phi_4.$$

The total space of the bundle gives a crepant resolution of the conifold singularity. c.f. Witten, Phases of two-dimension SUSY theories.

5.3 T-duality

Consider maps $x : \mathbb{R} \times S^1 \to S^1_R$. We can think of x as a function $x(t, s)$ where t is the time parameter and x is 2π periodic in s. The action is

$$S = \frac{1}{2\pi} = \int_\Sigma L \mathrm{d}t\mathrm{d}s = \frac{1}{4\pi} \int_\Sigma ((\partial_t x)^2 - (\partial_s x)^2) \mathrm{d}t\mathrm{d}t.$$

The Euler-Lagrange equation asserts that the equation of motion is

$$(\frac{\partial^2}{\partial t^2} - \frac{\partial^2}{\partial x^2})x = 0$$

which has well-known solution

$$x(t, s) = f(t - s) + g(t + s).$$

By periodicity

$$x(t, s + 2\pi) = x(t, s) + 2\pi mR.$$

$m \in \mathbb{Z} \cong \pi_1(S^1)$ is called the *winding modes*.

We can expand x in the Fourier modes of the target space S^1_R, which have momentum $p = \frac{\ell}{R}$. After quantisation there are two quantum numbers, resulting in a direct sum decomposition

$$\mathcal{H} = \bigoplus_{(\ell,m)\in\mathbb{Z}^2} \mathcal{H}_{(\ell,m)}$$

and we label the eigenstates by $|\ell, m\rangle$. The solution of the Euler-Lagrange equation can then be expressed as

$$x_R(t-s) = \frac{x_0 - \hat{x}_0}{2} + \frac{1}{\sqrt{2}}(t-s)P_R + \text{oscillations}_R$$
$$x_L(t-s) = \frac{x_0 - \hat{x}_0}{2} + \frac{1}{\sqrt{2}}(t-s)P_L + \text{oscillations}_L$$

where x_0 is the zero mode (?) and $[x_0, p_0] = i$

$$p_0 |\ell, m\rangle = \frac{\ell}{R} |\ell, m\rangle, w_0 |\ell, m\rangle = mR |\ell, m\rangle$$

and

$$p_L = \frac{1}{\sqrt{2}}(p_0 - w_0), p_L = \frac{1}{\sqrt{2}}(p_0 + w_0).$$

The oscillation part is

$$\text{osc}_R = \frac{i}{\sqrt{2}} \sum_{n\neq 0} \frac{1}{n} \alpha_n e^{-in(t-s)}$$
$$\text{osc}_L = \frac{i}{\sqrt{2}} \sum_{n\neq 0} \frac{1}{n} \widetilde{\alpha}_n e^{-in(t-s)}$$

The associated Hamiltonian is

$$H_R = \frac{1}{2}(H - P) = \frac{1}{2}P_R^2 + \sum_{n=1}^{\infty} \alpha_{-n}\alpha_n + \frac{1}{2}\sum_{n=1}^{\infty} n$$
$$H_L = \frac{1}{2}(H + P) = \frac{1}{2}P_L^2 + \sum_{n=1}^{\infty} \widetilde{\alpha}_{-n}\widetilde{\alpha}_n + \frac{1}{2}\sum_{n=1}^{\infty} n$$

Using ζ-regularisation, we identify

$$\sum n = \zeta(-1) = -\frac{1}{12}.$$

Now we compute the partition function

$$\mathrm{Tr}_{\mathcal{H}}\, e^{-\beta H}.$$

Instead of considering the source as a torus with sides β and 2π, we consider the more general situation $\tau = \tau_1 + i\tau_2$ and the torus has lengths $2\pi\tau_1, 2\pi\tau_2$. We consider

$$Z(\tau, \overline{\tau}) = \mathrm{Tr}_{\mathcal{H}}\, q^{H_R}\overline{q}^{H_L}$$

where $q = e^{2\pi i\tau}$. In our case

$$Z(\tau,\overline{\tau}) = (q\overline{q})^{-1/24}\prod_{n=1}^{\infty}\mathrm{tr}_{\mathcal{H}_n^R}\, q^{\alpha_{-n}\alpha_n}\,\mathrm{tr}_{\mathcal{H}_n^L}\,\overline{q}^{\widetilde{\alpha}_{-n}\widetilde{\alpha}_n}\sum_{(\ell,m)\in\mathbb{Z}^2} q^{\frac{1}{4}(\frac{\ell}{R}-mR)^2}\overline{q}^{\frac{1}{4}(\frac{\ell}{R}+mR)^2}$$

The first two lines are

$$\begin{aligned}\mathrm{tr}_{\mathcal{H}_n^R}\, q^{\alpha_{-n}\alpha_n} &= \sum_{k=0}^{\infty}(q^n)^k = \frac{1}{1-q^n}\\ \mathrm{tr}_{\mathcal{H}_n^L}\, q^{\widetilde{\alpha}_{-n}\widetilde{\alpha}_n} &= \sum_{k=0}^{\infty}(\overline{q}^n)^k = \frac{1}{1-\overline{q}^n}\end{aligned}$$

$$(q\overline{q})^{-1/24}\prod_{n=1}^{\infty}\Big|\frac{1}{1-q^n}\Big|^2 = \Big|\frac{1}{\eta(\tau)}\Big|^2$$

where η is the Dedekind η-function, a modular function. Under the transformation $\tau\mapsto\frac{a\beta+b}{c\tau+d}$,

$$\eta(\tau+1) = e^{\pi i/12}\eta(\tau), \eta(-\frac{1}{\tau}) = (-i\tau)^{1/2}\eta(\tau).$$

$Z(\tau,\overline{\tau})$ is invariant under exchanging momentum ℓ and winding modes m. In other words, we can exhange $R\leftrightarrow\frac{1}{R}, \ell\leftrightarrow m$. This is called *T-duality*.

NLSM on T^2 Consider $T^2 = S^1_{R_1}\times S^1_{R_2}$. It has Kähler moduli

$$A = \frac{\text{area}}{(2\pi)^2} = R_1R_2$$

and complex moduli

$$\sigma = i\frac{R_1}{R_2}.$$

Suppose we perform T-duality on the second cirle, namely $R_2\mapsto\frac{1}{R_2}$, then we exchange the two parameters.

Exercise. Compute the partition function with target space T^2 with B-field ($\rho = \frac{B}{2\pi}+iA$) and show that it is invariant under $\rho\leftrightarrow\sigma$.

T-duality in the path integral formalism Consider maps $\varphi:\Sigma\to S^1_R$ and action

$$S_\varphi = \frac{1}{4\pi}\int_\Sigma R^2h^{\mu\nu}\partial_\mu\varphi\partial_\nu\varphi\sqrt{h}\mathrm{d}^2\sigma$$

where h is a metric on Σ. Intoduce a one-form B on Σ and consider

$$S' = \frac{1}{2\pi}\int_\Sigma\frac{1}{2R^2}h^{\mu\nu}B_\mu B_\nu\sqrt{h}\mathrm{d}^2\sigma + \frac{i}{2\pi}\int_\Sigma B\wedge\mathrm{d}\varphi.$$

One can do two things with S^1

1. either take derivative with respect to B to get S_φ with target space S^1_R,

2. or take derivative with respect to φ to get another action $\widetilde{S}_\theta$ with target $S^1_{1/R}$.

For 1, the equation of motion is $B = iR^2 *_\Sigma \mathrm{d}\varphi$ and

$$S'|_{B=iR^2*_\Sigma \mathrm{d}\varphi} = S_\varphi.$$

For 2, the equation of motion for φ implies B is a closed one form on Σ so we write

$$B = \mathrm{d}\theta_0 + \sum_{i=1}^{2g} a_i \omega_i$$

where ω_i is a basis of harmonic forms of $H^1(\Sigma;\mathbb{R})$. Choose $\gamma_i \in H_1(\Sigma < \mathbb{Z})$ such that

$$\int_{\gamma_i} \omega_j = \delta_{ij}.$$

Note

$$\int_\Sigma \omega^i \wedge \omega_j = J^{ij},$$

a unimodular matrix.

The key observation is that the equation of motion of φ also imposes constraints on a_i. By periodicity

$$\mathrm{d}\varphi = \mathrm{d}\varphi_0 + \sum_i 2\pi n_i \omega^i.$$

Then

$$\int B \wedge \mathrm{d}\varphi = 2\pi \sum_{i,j} a_i J^{ij} n_j$$

so e^{iS} will contain $e^{i\sum_{i,j} a_i J^{ij} n_j}$. For the action to be single-valued, summing over j imposes the condition

$$a_i = 2\pi m_i.$$

This shows $B = \mathrm{d}\theta$ where θ is a periodic variable of period 2π. Plug back into S', one sees

$$S'|_{B=\mathrm{d}\theta} = \widetilde{S}_\theta = \frac{1}{4\pi} \int_\Sigma \frac{1}{R^2} h^{\mu\nu} \partial_\mu \theta \partial_\nu \theta \sqrt{h} \mathrm{d}^2\sigma,$$

a NLSM with target space $S^1_{1/R}$. In summary, upton the substitution $R\mathrm{d}\varphi = \frac{i}{R} * \mathrm{d}\theta$, S_φ becomes $\widetilde{S}_\theta$.

5.4 Strategy for constructive proof of mirror symmetry

We will prove mirror symmetry for toric varieties. Toric varieties as target spaces can be realised from GLSM $\mathcal{N} = (2,2)$ with suitable complex superfields Φ_i and supercharges (using U(1) representation). The phase of Φ_i is S^1, to which we will apply T-duality. This produces a LG model W, the mirror to the toric variety. If it is CY then we can solve it for a geometric model.

For each chiral superfield Φ_i we get Y_i, a neutral (U(1)-invariant) superfield with twisted chiral superpotential.

GLSM

NLSM with toric target ← GLSM → (T-duality) LG model

X $\qquad\qquad$ $\check{X}$

T-duality for SUSY sigma model Consider the case where the target is $\mathbb{C}^* = \mathbb{R} \times S^1_R$. Let B be a real superfield, Θ twisted chiral superfield.

$$\mathcal{L}_1 = \int \mathrm{d}^4\theta(\frac{R^2}{4}B^2 - \frac{1}{2}(\Theta + \overline{\Theta})B).$$

As before we can apply either equations of motion for B or for Θ, getting mirrors of each other. The equation for $\Theta, \overline{\Theta}$ implies that

$$\overline{D}_+ D_- B = 0, D_+ \overline{D}_- B = 0$$

which is solved by

$$B = \Phi + \overline{\Phi}.$$

Substituting back one gets

$$\mathcal{L}_2 = \int \mathrm{d}^4\theta \frac{R^2}{4}(\Phi + \overline{\Phi})^2 = \int \mathrm{d}^2\theta \frac{R^2}{2}\Phi\overline{\Phi}.$$

This is the Lagrangian for a σ-model of maps into a cylinder $\mathbb{R} \times S^1_R$.

On the other hand the eom for B implies

$$B = \frac{1}{R^2}(\Theta + \overline{\Theta}).$$

Substituting back into $\mathcal{L}_1$ gives

$$\mathcal{L}_3 = \int \mathrm{d}^4\theta(-\frac{1}{2R^2}\Theta\overline{\Theta}),$$

a σ-model of maps onto cylinder $\mathbb{R} \times S^1_{1/R}$.

One thus gets T-dual models, which satisfy

$$R^2(\Phi + \overline{\Phi}) = \Theta + \overline{\Theta}.$$

Since we are interested in gauge σ-models, consider

$$\mathcal{L} = \int \mathrm{d}^4\theta(e^{2QV+B} - \frac{1}{2}(Y + \overline{Y})B)$$

where V (superconnection) and B are real superfields, $Y, \overline{Y}$ are twisted chiral superfields with $\mathrm{Im}(Y)$ periodic with period 2π.

Same as before the eom of $Y, \overline{Y}$ demands

$$\overline{D}_+ D_- B = 0, D_+ \overline{D}_- B = 0$$

which is solved by

$$B = \Psi + \overline{\Psi}$$

where Ψ is a chiral superfield with $\operatorname{Im}\Psi$ also 2π-periodic. Substituting back we get

$$\mathcal{L}_1 = \int \mathrm{d}^4\theta e^{2QV+\Psi+\overline{\Psi}} = \int \mathrm{d}^4\theta\overline{\Phi}e^{2QV}\Phi$$

where $\Phi = e^{\Psi}$.

The eom for B gives

$$B = -2QV + \log(\frac{Y+\overline{Y}}{2}).$$

Substituting back gives

$$\mathcal{L}_2 = \int \mathrm{d}^4\theta[QV(Y+\overline{Y}) - \frac{1}{2}(Y+\overline{Y})\log(Y+\overline{Y})].$$

For $\overline{D}_+Y = 0, D_-Y = 0$,

$$\int \mathrm{d}^4\theta VY = \int \mathrm{d}\theta^+\mathrm{d}\overline{\theta}^-(\overline{D}_+D_-V)Y = \int \mathrm{d}^2\widetilde{\theta}\Sigma Y$$

where $\Sigma = \overline{D}_+D_-V$ is the supercurvature. Thus the Lagrangian includes a kinetic term for Σ:

$$\widetilde{\mathcal{L}} = \int \mathrm{d}^4\theta(-\frac{1}{2e^2}\overline{\Sigma}\Sigma - \frac{1}{2}(Y+\overline{Y})\log(Y+\overline{Y})) + \int \mathrm{d}^2\widetilde{\theta}\Sigma Q(Y-t) + \mathrm{cc}$$

This is T-dual to

$$\mathcal{L} = \int \mathrm{d}^4\theta\overline{\Phi}e^{2QV}\Phi + \frac{1}{2e^2}\overline{\Sigma}\Sigma - t\int \mathrm{d}^2\widetilde{\theta}Q\Sigma.$$

They satisfy

$$2\overline{\Phi}e^{2QV}\Phi = Y + \overline{Y}.$$

$\widetilde{W}(Y) = \Sigma(QY - t)$. So far everything is on classical level. Quantum effects add a further term to $\widetilde{W}$ proportional to e^{-Y}.

$$S|_{\mathrm{bos}} = \frac{1}{2\pi}\int \mathrm{d}^2x\mathcal{L}|_{\mathrm{bos}} = \frac{1}{2R}\int \mathrm{d}^2x|D_\mu\phi|^2 + |\sigma\phi|^2 + \frac{1}{2e^2}|\partial_\mu\sigma|^2 + \frac{1}{2e^2}(F_{12}+D^2) + i\theta F_{12}$$

where

$$D = e^2(|\phi|^2 - r_0)$$

where $t = r_0 + i\theta$.

The BPS solutions are

$$F_{12} = e^2(|\phi|^2 - r_0), D_{\overline{z}}\phi = 0, \sigma = 0.$$

These are known as *vortices*. Why do they saturate the bounds? Recall that in SQM we used squaring argument to find the mimimum. Substitute the BPS solutions, we get

$$\frac{1}{2\pi}\int \mathrm{d}^2x|2D_{\overline{z}}\phi|^2 - F_{12}|\phi|^2 + \frac{1}{2e^2}(F_{12}+D)^2 - \frac{1}{e^2}DF_{12} + i\theta F_{12}.$$

Recall that $D = -e^2(|\phi|^2 - r_0)$ so we can rewrite this as

$$\frac{1}{2\pi}\int \mathrm{d}^2x(|2D_{\overline{z}}\phi|^2 + \frac{1}{2e^2}(F_{12}+D)^2) - \frac{t}{2\pi}\int \mathrm{d}^2xF_{12}.$$

Thus $S_{\text{bos}} \geq tK$ where K is the first Chern class of the U(1)-bundle. One then sees that the purported BPS solutions achieves the bound.

Vortices Work with complex coordinates $z = x_1 + ix_2$. As $z \to \infty$, $|\phi|^2 \to r_0$. Let $\phi = \sqrt{r_0}\hat{\phi}$ so $|\widehat{\phi}| \to 1$, thus mapping $S^1_\infty \to S^1$. They are classified by $k \in \mathbb{Z} \cong \pi_1 S^1$, called the charge of the vortex. For $k = 1$, $\phi \to \sqrt{r_0}\frac{z}{|z|}$. Vanishing of covariant derivative imposes as $z \to \infty$

$$A_\mu \to \partial_\mu \arg(z).$$

One can then reparameterise the eom with $f(w), w = |z|^2$:

$$iA = -\frac{1-f}{2}(\frac{\mathrm{d}z}{z} - \frac{\mathrm{d}\overline{z}}{\overline{z}})\phi = \sqrt{r_0}\exp(-\int_{|z|^2}^\infty \frac{\mathrm{d}w}{2w}f(w))\frac{z}{|z|}.$$

Plugging back to the BPS solution, we get an ODE for f

$$wf'' = \frac{e^2r_0}{2}f + ff', f(0) = 1, f(\infty) = 0.$$

There are approximate solutions in terms of modified Bessel functions of the second kind (but we don't need them).

For $|z| \gg \frac{1}{e\sqrt{r_0}}$, $f \sim \sqrt{m(z)}e^{-m|z|}$ where $m = e\sqrt{2r_0}$.

Fermionic part

$$-i(\overline{\Psi}_-, \lambda_+)\begin{pmatrix} 2D_{\overline{z}} & -\phi \\ \phi^\dagger & e^{-1/2}\partial_z \end{pmatrix}\begin{pmatrix} \psi_- \\ \overline{\lambda}_+ \end{pmatrix} + i(\overline{\psi}_+, \lambda_-)\begin{pmatrix} 2D_z & -\phi \\ \phi^\dagger & -\frac{1}{e^2}\partial_{\overline{z}} \end{pmatrix}\begin{pmatrix} \psi_+ \\ \overline{\lambda}_- \end{pmatrix}$$

For $D_{\overline{z}}, D_z$, apply index theorem to get

$$\text{Ind}D_{\overline{z}} = \frac{1}{2\pi}\int F_{12} = K = 1.$$

Are these $(\overline{\psi}_-, \lambda_+), (\psi_+, \overline{\lambda}_-)$ zero modes? The answer is no. We have vanishing theorem

$$\begin{aligned} 0 &= \int \mathrm{d}^2z(|2D_z\psi_+ - \phi\overline{\lambda}_-|^2 + 2e^2|\phi^\dagger\psi_+ - \frac{1}{e^2}\partial_{\overline{z}}\overline{\lambda}_-|^2) \\ &= \int \mathrm{d}^2z(|2D_{\overline{z}}\psi_+|^2 + 2e^2|\phi|^2|\psi_+|^2 + \frac{2}{e^2}|\partial_{\overline{z}}\overline{\lambda}_-|^2 + |\phi|^2|\overline{\lambda}_-|^2) \end{aligned}$$

Remark. ψ, λ are commuting variables:

$$\psi^{(0)} = \sum_n \psi_n^{(0)}\psi_n$$

where $\psi_n^{(0)}$ is Grassmann odd.

Thus the only zero modes in the $K = 1$ vortex background are $\overline{\psi}_+, \psi_-$.

$$\langle \overline{\psi}_+^{(0)} \phi(\phi^\dagger \psi_-^{(0)} \rangle = \int \mathrm{d}^2 z e^{-t_0} \overline{\Psi}_+^{(0)} \phi(z) \phi^\dagger(z) \psi_-^{(0)}.$$

Rewriting in twisted chiral variables

$$\overline{\Phi} e^{2QV} \Phi = Y + \overline{Y}$$

and expanding in components (Y has components $(y, \chi_+, \overline{\chi}_-)$, Φ has components $(\phi, \overline{\psi}_+, \psi_-)$),

$$\chi_+ = 2\overline{\psi}_+ \psi, \overline{\chi}_- = -2\phi^\dagger \psi_-.$$

Since

$$\overline{\psi}_+ \phi \phi^\dagger \psi_- e^{-t} \sim e^{-y} \chi_+ \overline{\chi}_-,$$

the vortex effects generate a potential $\widetilde{W}(Y) = e^{-Y}$.

In summary, we started with GLSM with Φ_i, Q_i wrt U(1). This gives a NLSM with toric variety Q_i, which is T-dual to a LG model with Y_i and superpotential

$$widetildeW(Y_i) = -\Sigma(t - \sum_{i=1}^N Q_i Y_i) + \sum_{i=1}^N e^{-Y_i}.$$

Example ($\mathbb{CP}^{N-1}$)**.** The GLSM has Φ_i for $i = 1, \ldots, N$ and $Q_i = 1$. The mirror has $\widetilde{W} = \Sigma(Y_1 + \cdots + Y_N) + \sum_i e^{-Y_i}$. The eom for Σ is

$$Y_1 + \ldots Y_N = 0$$

so one gets $\widetilde{W} = e^{-Y_1} + \cdots + e^{-Y_{N-1}} + e^{\sum Y_i}$. For $N = 2$, one gets $\mathbb{CP}^1$ in A-model has mirror with Sinh Gordon potential.

Index