Scuola Internazionale
Superiore di Studi
Avanzati

Geometry and Mathematical Physics

Infinitesimal Deformation Theory

January, 2021

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0 Introduction

Throughout this course we fix a base field $k = \overline{k}$. All schemes will be k-schemes and morphisms k-morphisms. By algebra we mean k-algebra. For a scheme X by a point we mean a morphism $x : \operatorname{Spec} k \to X$ or equivalently $\kappa(x) = k$ as k-algebra.

Motivation: a scheme X is determined up to canonical isomorphism by its *functor of points* on $\mathbf{Aff}^{\mathrm{op}}$. In fact many moduli schemes are defined this way. For example if X is a projective scheme and $\mathcal{F}$ is a coherent sheaf on X one defines the quotient scheme $\operatorname{Quot}_X(\mathcal{F})$ by its associated functor of points

$$S \mapsto \{\mathcal{G} \subseteq p_X^* \mathcal{F} : \mathcal{F} \in \mathbf{Coh}(X \times S), p_X^* \mathcal{F}/\mathcal{G} \text{ flat over } S\},$$

give a morphism $f : S_1 \to S$, it induces $(f \times \mathrm{id}_X)^* \mathcal{G} \subseteq p_X^* \mathcal{F}$ (we have inclusion because of flatness). Grothendieck proved that $\operatorname{Quot}_X(\mathcal{F})$ is a disjoint union of countably many projective schemes. The special case $\operatorname{Hilb}_X = \operatorname{Quot}_X(\mathcal{O}_X)$ can be defined also by

$$S \mapsto \{Z \subseteq X \times S \text{ closed subscheme} : Z \text{ flat over } S\}$$

with morphisms defined by pullback

$$f^* Z = Z \times_{X \times S} X \times S_1 \subseteq X \times S_1.$$

Infinitesimal deformation means studying the restriction of the functor of points to the subcategories of $\mathbf{Aff}$ of fat points, i.e. schemes S of finite type such that $S_{\mathrm{red}} = \operatorname{Spec} k$. In other words, there is no topological information and everything is about algebra. Since such S has only one nonempty open set, it must be affine, say A. Then $A/\sqrt{0} = k$. Thus the nilradical must be the (necessarily unique) maximal ideal $\mathfrak{m}_A \subseteq A$.

Proposition 0.1. *Let A be a finitely generated local k-algebra. Then TFAE*

1. $\operatorname{Spec} A$ *is a fat point.*
2. *A is finite-dimensional as a k-vector space.*
3. *$A/\mathfrak{m}_A = k$ and elements of $\mathfrak{m}_A$ are nilpotent.*
4. *$A/\mathfrak{m}_A = k$ and $\mathfrak{m}_A$ is nilpotent.*
5. *$A/\mathfrak{m}_A = k$ and A is artinian.*

Recall

Proposition 0.2 (Nakayama's lemma)**.** *Let R be a local ring and M a finitely generated R-module. If $\mathfrak{m}_R M = M$ then $M = 0$.*

Proof. The discussion before the statement of the proposition shows $1 \iff 3$. $3 \implies 4$ since A is noetherian, and $4 \implies 3$ trivially. If A is finite-dmimensional then $A/\mathfrak{m}_A$ is finite-dimensional over k so must be k since k is algebraically closed. Also any descending chain of ideals must stabilise. This shows $2 \implies 5$. $5 \implies 4$ by Nakayama since the chain

$$\mathfrak{m}_A \supseteq \mathfrak{m}_A^2 \supseteq \cdots$$

must stabilise. Finally 4 $\implies$ 2 since

$$\dim A = \sum \dim \mathfrak{m}_A^i/\mathfrak{m}_A^{i+1}.$$

□

Definition. We denote by **Art** or $\mathbf{Art}_k$ the category of algebra satisfying any of the equivalent conditions. Its is opposite to the category of fat points.

Remark. k is both an initial and final object in **Art**. Thus each A in **Art** is canonically isomorphic to $k \oplus \mathfrak{m}_A$.

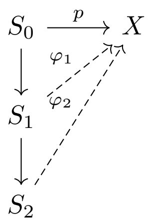
Corollary 0.3. *Let $F : \mathbf{Art} \to \mathbf{Set}$ be a functor. Then $F = \coprod_{x \in F(k)} F_x$ where*

$$F_x(A) = \{a \in F(A) : F(\pi)(a) = x\}$$

where $\pi : A \to k$. In particular if X is a scheme thn $h_X = \coprod_{x \in X} h_{X,x}$ where

$$h_{X,x}(A) = \{\varphi : \operatorname{Spec} A \to X : (\operatorname{im}\varphi)_{red} = x\}.$$

Example. Let $A_n = k[t]/t^{n+1}, S_n = \operatorname{Spec} A_n$.The natural surjections $A_n \to A_{n-1}$ induces closed embeddings $S_{n-1} \hookrightarrow S_n$. We want to study a scheme X near a point p by solving the following question: given $p : S_0 \to X$, does φ_1 exist? If so how many choices are there? What about φ_2 etc?



We will work out the case $X = \operatorname{Spec} R$ where $R = k[x,y]/f$ where $f(0,0) = 0$. Let $p^\# : R \to k$. We will look at the locus of the following four equations:

1. $y^2 - x^3 - x$,
2. $y^2 - x^3 - x^2$,
3. $y^2 - x^3$,
4. $y^2 - x^2$.

Giving φ_1 is the same as giving

$$\begin{aligned} \varphi_1^\# : k[x,y] &\to k[t]/t^2 \\ x &\mapsto a_1 t \\ y &\mapsto b_1 t \end{aligned}$$

such that $f(at, bt) = 0 \in k[t]/t^2$. For 1, for example, we need

$$(b_1 t)^3 - (a_1 t)^3 - a_1 t = 0.$$

Using $t^2 = 0$ we get $a_1 = 0$ and there is no restriction on b_1. For 2, 3, 4 one can check any choice of a_1, b_1 works. Geometrically, this is because the tangent at the origin is the y-axis for 1, and the entire plane for 2, 3, 4.

φ_2 is given by

$$\phi_2^{\#}(x) = a_1 t + a_1 t^2, \varphi_2^{\#}(y) = b_1 t + b_2 t^2.$$

One find in each case

1. given $a_1 = 0$, any choice b_1 lifts to $\varphi_2^{\#}$ and one requires $a_2 = b_1^2$. Similar to before, there is a family of lifts from φ_1 to φ_2.
2. not solvable unless $a_1^2 = b_1^2$, in which case any a_2, b_2 works.
3. not solvable unless $b_1 = 0$
4. Same as 2.

Geometrically for 2 and 4 there are two distinguished tangent direction. For 3 the tangent direction is the horizontal axis.

As an exercise, work out φ_3.

1 Linearisation

Let $f : X \to Y$ be a morphism of schemes, $x \in Y, y = f(x)$. We want to compute $\Omega_f(x)$ (to be defined soon) in terms of $h_{X,x}$ and $h_{Y,x}$.

Convention: let X be a scheme and $x \in X$, $\mathcal{F}$ a quasicoherent sheaf on X. We denote by $\mathcal{F}_x$ the stalk of $\mathcal{F}$ at x and

$$\mathcal{F}(x) = \mathcal{F}_x \otimes_{\mathcal{O}_{X,x}} \kappa(x) = \mathcal{F}_x/\mathfrak{m}_x\mathcal{F}_x,$$

the *fibre* of $\mathcal{F}$ at x. If $\mathcal{F}$ is coherent then $\mathcal{F}_x$ is a finitely generated $\mathcal{O}_{X,x}$-module so $\mathcal{F}(x)$ is a finite-dimensional k-vector space. In fact by Nakayama, $u_1, \dots, u_r$ generate $\mathcal{F}_x$ as an $\mathcal{O}_{X,x}$-modules if and only if their images generate $\mathcal{F}(x)$ as a k-vector space.

Recall that $\Omega_f = \Delta^*\mathcal{I}_{X/X\times_Y X}$ which is a coherent sheaf on X (assuming locally of finite type). It is local in both X and Y and when $X = \operatorname{Spec} S, Y = \operatorname{Spec} R$ and f is induced by $R \to S$, we have $\Omega_f(X) = \Omega_{S/R}$. If S has presentation $S = R[x_1, \dots, x_n]/(f_1, \dots, f_r)$ then we have an exact sequence

$$\bigoplus_{i=1}^r S \cdot f_i \xrightarrow{\alpha} \bigoplus_{i=1}^n S\mathrm{d}x_i \longrightarrow \Omega_{S/R} \longrightarrow 0$$

where

$$\alpha(f_j) = \sum [\frac{\partial f_j}{\partial x_i}]\mathrm{d}x_i.$$

Thus for $p \in X$ there is a presentation

$$\Omega_f(p) = \bigoplus k\mathrm{d}x_i / \langle \sum \frac{\partial f_j}{\partial x_i}(p)\mathrm{d}x_i \rangle.$$

We denote by **Vect** the category of finite-dimensional k-vector spaces with k-linear maps.

Lemma 1.1. *There exists a fully faithful functor* **Vect** $\to$ **Art** *sending V to $k \oplus V$ with multiplication*

$$(a, v) \cdot (b, w) = (ab, aw + bv).$$

In other words $k \oplus V = \bigoplus_{n\geq 0} \operatorname{Sym}^n V / \bigoplus_{n\geq 2} \operatorname{Sym}^n V$.

Remark. The image of the functor is $\{A : \mathfrak{m}_A^2 = 0\}$. It has a left adjoint $A \mapsto \mathfrak{m}_A/\mathfrak{m}_A^2$.

Theorem 1.2. *Let X be a scheme, $x \in X$, $A \in$* **Art** *such that $\mathfrak{m}_A^2 = 0$. Then there is a natural bijection*

$$\operatorname{Hom}_k(\Omega_X(x), \mathfrak{m}_A) \to h_{X,x}(A).$$

Proof. wlog we may assume $X = \operatorname{Spec} R$ and x corresponds to a maximal ideal $\mathfrak{m}_x$. Then

$$h_{X,x}(A) = \{\varphi : R \to A : \ker \pi \circ \varphi = \mathfrak{m}_x\}.$$

In other words φ makes the following diagram commutes

$$\begin{array}{ccc} R & \xrightarrow{\overline{\pi}} & R/\mathfrak{m}_x \\ \downarrow{\scriptstyle \varphi} & & \downarrow{\scriptstyle \cong} \\ A & \xrightarrow{\pi} & A/\mathfrak{m}_A \end{array}$$

Write $A = k \oplus \mathfrak{m}_A$ and $\varphi(f) = (\overline{\pi}(f), \lambda(f))$. Then one can check that φ is a ring map if and only if $\lambda \in \operatorname{Der}_k(R, \mathfrak{m}_k)$ where the R-module structure on $\mathfrak{m}_k$ is via restriction of scalars along $\overline{\pi}$. Thus we have bijections

$$h_{X,x}(A) \cong \operatorname{Der}_k(R, \mathfrak{m}_A) \cong \operatorname{Hom}_R(\Omega_R, \mathfrak{m}_A).$$

Since $\mathfrak{m}_A$ is a $R/\mathfrak{m}_x = k$-module,

$$\operatorname{Hom}_R(\Omega_R, \mathfrak{m}_A) \cong \operatorname{Hom}_k(\Omega_R/\mathfrak{m}_x\Omega_R, \mathfrak{m}_A).$$

□

Note that the isomorphism is natural in both X and A. The next result says that by transport of structure the tangent space T_xX can be "read off" $h_{X,x}$.

Corollary 1.3. *Let $D = k[\varepsilon]/\varepsilon^2, D_2 = k[\varepsilon_1, \varepsilon_2]/(\varepsilon_1^2, \varepsilon_1\varepsilon_2, \varepsilon_2^2)$. Let X be a scheme and $x \in X$. Then $T_xX := \Omega_X(x)^\vee$ is canonically isomorphic to the k-vector space V defined as*

- *as a set $V = h_{X,x}(D)$.*
- *given $\lambda \in k$, multiplication by λ is induced by $h_{X,x}(\alpha_\lambda)$ where $\alpha_\lambda : D \to D$ is multiplication by λ.*
- *define maps*

$$\begin{aligned} \rho_i, \pi &: D_2 \to D \\ \rho_i &: \varepsilon_j \mapsto \delta_{ij}\varepsilon \\ \pi &: \varepsilon_j \mapsto \varepsilon \end{aligned}$$

Then

$$h_{X,x}(D_2) \xrightarrow{(\rho_1,\rho_2)} h_{X,x}(D) \times h_{X,x}(D) = V \times V$$

is a bijection and the sum on V is the composition

$$V \times V \to h_{X,x}(D_2) \xrightarrow{\pi} h_{X,x}(D) = V.$$

Proof. Exercise. □

Proposition 1.4. *Let $\varphi : X \to Y$ be a morphism, $x \in X, y = \varphi(x)$. Then $T_x\varphi := \Omega_\varphi(x)^\vee$ is naturally isomorphic to the kernel of the map*

$$h_{X,x}(D) \to h_{Y,y}(D).$$

Proof. The exact sequence of modules

$$\varphi^*\Omega_Y \longrightarrow \Omega_X \longrightarrow \Omega_\varphi \longrightarrow 0$$

pulls back to

$$\Omega_Y(y) \xrightarrow{\mathrm{d}F(x)} \Omega_X(x) \longrightarrow \Omega_\varphi(x) \longrightarrow 0$$

Dualise to get

$$\begin{array}{ccccccc} 0 \longrightarrow & T_x\varphi & \longrightarrow & T_xX & \xrightarrow{\mathrm{d}F(x)^\vee} & T_yY \\ & & & \| & & \| \\ & & & h_{X,x}(D) & & h_{Y,y}(D) \end{array}$$

To complete the proof, one check that by the remark on naturality $\mathrm{d}F(x)^\vee$ is equal to $h_{X,x} \to h_{Y,y}$ induced by φ. □

Remark. Note that if we define a tangent sheaf by $\mathcal{T}_X := \mathcal{H}om(\Omega_X, \mathcal{O}_X)$ then in general $\mathcal{T}_X(x) \neq T_xX$. As an exercise, compute both for $k = \mathbb{C}$ (or any field whose characteristic is not 2), $X = \operatorname{Spec} k[u,v,w]/(uv - w^2), \mathfrak{m}_x = (u,v,w)$.

Proof. Let $R = k[u,v,w]/(uv - w^2)$. $\Omega_X(x)$ has a presentation by tensoring with k the exact sequence

$$R \xrightarrow{u,v,-2w} R^3 \longrightarrow \Omega_R \longrightarrow 0$$

so $\dim T_xX = \dim \Omega_X(x) = 0$.

On the other hand $\mathcal{T}_X(x)$ does not have a "nice" presentation since upon taking dual we get a sequence that is in general only exact on the left

$$0 \longrightarrow \operatorname{Hom}(\Omega_R, R) \longrightarrow R^3 \longrightarrow R$$

and tensor product is right exact. Nevertheless we can proceed as below. Let $S = k[u,v]$ and then $R = S \oplus Sw$. Then an element of $\operatorname{Hom}(\Omega_R, R)$ can be expressed as

$$(f_1 + wg_1, f_2 + wg_2, f_3 + wg_3)$$

with f_i, g_i's in S, subjecting to

$$v(f_1 + wg_1) + u(f_2 + wg_2) - 2w(f_3 + wg_3) = 0.$$

Collecting terms, we get

$$\begin{aligned} vf_1 + uf_2 &= 2uvg_3 \\ vg_1 + ug_2 &= 2f_3 \end{aligned}$$

so we deduduce $u \mid f_1, v \mid f_2$. Write $f_1 = u\widetilde{f_1}, f_2 = v\widetilde{f_2}$, we can then express f_3 and g_3 in terms of $\widetilde{f_1}, \widetilde{f_2}, g_1, g_2$:

$$\begin{aligned} f_3 &= \frac{1}{2}(vg_1 + ug_2) \\ g_3 &= \frac{1}{2}(\widetilde{f_1} + \widetilde{f_2}) \end{aligned}$$

In other words, $\mathrm{Hom}(\Omega_R, R)$ is generated as an S-module by

$$(u, 0, \frac{w}{2}), (0, v, \frac{w}{2}), (w, 0, \frac{v}{2}), (0, w, \frac{u}{2}).$$

It is easy to check that there are no relations among them. Hence $\mathcal{T}_X(x)$ has dimension 4. □

Exercise. Let X be a scheme and $x \in X$. Show $T_x X$ is naturally isomorphic to $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$ (hint: use the adjunction below Lemma 1.1).

Proposition 1.5. *Let X be a scheme locally of finite type over k and $x \in X$. Then $\dim_k T_x X$ is finite, and is the smallest n such that exists an open neighbourhood U of x and a closed embedding $U \hookrightarrow M$ with M smooth over k of dimension n.*

The proposition says that if we want to embedding x locally in a smooth scheme M then the dimension of M has to be at least $\dim T_x X$.

Proof. wlog $X = \operatorname{Spec} R$ where $R = k[x_1, \dots, x_N]/(f_1, \dots, f_r)$. Ω_R is finitely generated so $\Omega_X(x)$ is finite-dimensional.

Suppose U is an open neighbourhood of x and $U \hookrightarrow M$ a closed embedding with M smooth of dimension n. Then $\mathcal{O}_{M,x} \twoheadrightarrow \mathcal{O}_{U,x}$ so $\mathfrak{m}_{M,x} \twoheadrightarrow \mathfrak{m}_{U,x}$ so $\Omega_M(x) \twoheadrightarrow \Omega_U(x)$. Since M is smooth of dimension n, $\dim T_x X = \dim T_x U \leq \dim T_x M = n$.

Left to show if $\dim T_x X = n$ then we can find such U and M of dimension n. Recall that $\Omega_X(x) = \mathrm{coker}(\alpha : k^{\oplus r} \to k^{\oplus N})$ where α is the Jacobian at x. Up to linear transformations we can express $\alpha = \begin{pmatrix} I_m & 0 \\ 0 & 0 \end{pmatrix}$. Let $Y = k[x_1, \dots, x_N]/(f_1, \dots, f_r)$ in the new generators and equations. Let $g = \det(\frac{\partial f_i}{\partial x_j})$. Then $g(x) = 1$. We have closed embedding $X \hookrightarrow Y$. $M := Y \cap D(g)$ is smooth of dimension n and $x \in U := X \cap D(g)$. □

2 Formal power series

Let $P = k[x_1, \ldots, x_n]$. We define the power series ring $\widehat{P} = k[[x_1, \ldots, x_n]]$. Formally $\widehat{P} = \prod_{I \subseteq \mathbb{N}^n} k \cdot t^I$ with product $t^I \cdot t^J = t^{I+J}$. For $f \in \widehat{P}$ we write $f(0)$ for its constant term.

Proposition 2.1. *There is a k-algebra homomorphism*

$$\begin{aligned} \widehat{P} &\to k \\ f &\mapsto f(0) \end{aligned}$$

which has kernel $\mathfrak{m}_{\widehat{P}} = (x_1, \ldots, x_n)$. Furthermore $\widehat{P}$ is local.

Proof. The map is obviously a homomorphism. To show the kernel is generated by $x_1, \ldots, x_n$ induct on n. For $n = 1$ if $f = \sum f_i x^i \in \mathfrak{m}_{\widehat{P}}$ then $f_0 = 0$ so $f = x \sum f_i x^{i-1}$. For general n, write $f = g + h$ where $g = \sum_{I : i_n = 0} f_I x^I$. Then $g \in \mathfrak{m}_{k[[x_1, \ldots, x_{n-1}]]}$ and h is a multiple of x_n. Finally to show $\widehat{P}$ is local use the fact that any power series with nonzero constant term has an inverse (informally $(1-x)^{-1} = 1 + x + x^2 + \ldots$). □

Let $P_d \subseteq P$ be the vector space of homogeneous polynomials of degree d. We then have

$$P = \bigoplus_{d \geq 0} P_d, \widehat{P} = \prod_{d \geq 0} P_d$$

and

$$\mathfrak{m}_{\widehat{P}} = \prod_{d \geq 1} P_d, \mathfrak{m}_{\widehat{P}}^a = \prod_{d \geq a} P_d, \bigcap \mathfrak{m}_{\widehat{P}}^a = 0.$$

For every $d \in \mathbb{N}$, $P \hookrightarrow \widehat{P}$ induces an isomorphism $P/\mathfrak{m}_P^{d+1} \to \widehat{P}/\mathfrak{m}_{\widehat{P}}^{d+1}$. It is also worth noting that $\widehat{P} = \varprojlim P/\mathfrak{m}_P^d$.

Lemma 2.2. *For any A in* **Art** *there is a bijection*

$$\begin{aligned} \mathrm{Hom}_{k\text{-alg}}(\widehat{P}, A) &\to h_{\mathbb{A}^n,0}(A) \\ \varphi &\mapsto \varphi|_P \end{aligned}$$

Proof. We first check the map is well-defined. $h_{\mathbb{A}^n,0}(A) = \{\psi : P \to A : \psi^{-1}(\mathfrak{m}_A) = \mathfrak{m}_P\}$. Given $\varphi : \widehat{P} \to A$, we have

$$\widehat{P}/\varphi^{-1}(\mathfrak{m}_A) \hookrightarrow A/\mathfrak{m}_A = k$$

which must be equality so $\varphi^{-1}(\mathfrak{m}_A) = \mathfrak{m}_{\widehat{P}}$. Thus $\varphi|_P^{-1}(\mathfrak{m}_A) = \mathfrak{m}_{\widehat{P}} \cap P = \mathfrak{m}_P$ so defines an element of $h_{\mathbb{A}^n,0}(A)$.

To show it is a bijection, note that $\mathfrak{m}_A^{d+1} = 0$ for some d, so $\mathfrak{m}_{\widehat{P}}^{d+1} \subseteq \ker \varphi$ (resp. $\mathfrak{m}_P^{d+1} \subseteq \ker \psi$). Thus φ induces $\widehat{P}/\mathfrak{m}_{\widehat{P}}^{d+1} \to A$ (resp. $P/\mathfrak{m}_P^{d+1} \to A$). But $P/\mathfrak{m}_P^{d+1} \cong \widehat{P}/\mathfrak{m}_{\widehat{P}}^{d+1}$. □

Proposition 2.3. *Let $I \subseteq \mathfrak{m}_P$ be an ideal in P and $\widehat{I} = \widehat{P}I$ the ideal it generates in $\widehat{P}$. Let $R = P/I, \widehat{R} = \widehat{P}/\widehat{I}$, $X = \operatorname{Spec} R \subseteq \mathbb{A}^n$. Then the bijection in the previous lemma induces a bijection*

$$\operatorname{Hom}_{k\text{-alg}}(\widehat{R}, A) \to h_{X,0}(A).$$

For $f \in \widehat{P} = k[[x_1, \dots, x_n]]$, we denote by f_d its component in P_d and $f_{\leq d}$ its image in $\widehat{P}/\mathfrak{m}_{\widehat{P}}^{d+1} = P_{\leq d}$.

Lemma 2.4. *Let $N \geq 0$, $u_1, \dots, u_N \in \mathfrak{m}_{\widehat{P}}, f \in k[[y_1, \dots, y_N]]$. Then there exists a unique $g \in \widehat{P}$ such that $g_{\leq d} = (f_{\leq d}(u_1, \dots, u_N))_{\leq d} \in P_{\leq d}$. We denote this g by $f(u_1, \dots, u_N)$.*

Proof. The condition $u_i \in \mathfrak{m}_{\widehat{P}}$ ensures that $f_d(u_1, \dots, u_N) \in \widehat{P}$ (well-defined as f_d is a polynomial) is in $\mathfrak{m}_{\widehat{P}}^d$. Thus

$$f_{\leq d}(u_1, \dots, u_N) = f_{\leq d-1}(u_1, \dots, u_N) \pmod{\mathfrak{m}_{\widehat{P}}^{d-1}}$$

i.e. defines the same element in $P_{\leq d-1}$. As $\widehat{P} = \varprojlim P_{\leq d}$ they determine a unique element in $\widehat{P}$. □

Remark. $\frac{\partial}{\partial x_i} : P_{\leq d} \to P_{\leq d}$ determines a unique map $\frac{\partial}{\partial x_i} : \widehat{P} \to \widehat{P}$. If we identify $P_{\leq d}$'s with subalgebras of $\widehat{P}$ then it satisfies

$$\pi_{d-1} \circ \frac{\partial}{\partial x_i} = \frac{\partial}{\partial x_i} \circ \pi_d.$$

Concretely

$$\frac{\partial}{\partial x_i}(\sum_I f_I x^I) = \sum_J f_{J+e_i}(J_i + 1)x^J.$$

In particular $\frac{\partial f}{\partial x_i}(0) = f_{e_i}$.

Theorem 2.5 (implicit function theorem)**.** *Let $n \geq 1, \widehat{P} = k[[x_1, \dots, x_n]], \widehat{Q} = k[[x_1, \dots, x_n, y]], f \in \mathfrak{m}_{\widehat{Q}}$ such that $\frac{\partial f}{\partial y}(0) \neq 0$. Then exists a unique $g \in \mathfrak{m}_{\widehat{P}}$ and $u \in \widehat{Q} \setminus \mathfrak{m}_{\widehat{Q}}$ such that $f = u(y - g)$.*

Proof. wlog assume $\frac{\partial f}{\partial y}(0) = 1$. Write $f = y + y\tilde{f} + h$ with $\tilde{f} \in \mathfrak{m}_{\widehat{Q}}$ and $h \in \mathfrak{m}_{\widehat{P}}$. Claim we can find unique $u_i \in Q_i, g_i \in P_i$ such that for all d,

$$\pi_{\leq d}(f) = \pi_{\leq d}(u_0 + \dots + u_{d-1})(y + g_0 + \dots g_d).$$

Induction on d. For $d = 0$ have $f_0 = 0$ so $g_0 = 0$. For $d = 1$

$$f_1 = u_0(y + g_1)$$

is uniquely solved by $u_0 = 1$ and $g_1 = f_1 - y$. Suppose the result holds for $d - 1$. Then we need to solve

$$\begin{aligned} f_d &= [(u_0 + \dots + u_{d-1})(y + g_1 + \dots + g_d)]_d \\ &= [(u_0 + \dots + u_{d-2})(y + g_1 + \dots + g_{d-1})]_d + u_{d-1}(y + g_1) + g_d \end{aligned}$$

which can be solved uniquely. □

Corollary 2.6 (inverse function theorem)**.** *There exists a unique* $g \in \mathfrak{m}_{\widehat{P}}$ *such that* $f(x_1, \dots, x_n, g) = 0$.

Proof. Let $h \in \mathfrak{m}_{\widehat{P}}$. Then $f(x_i, h) = u(x_i, h) \cdot (h - g)$. Note

$$u(x_i, h)(0) = u(0, \dots, 0, h(0)) = u(0, \dots, 0) \neq 0$$

so $v = u(x_i, h)$ is a unit and $f(x_i, h) = 0$ if and only if $h = g$. □

3 Small extensions

Let $\pi : A \to B$ be a surjection of artinian algebras. We say it is a *small extension* if $I = \ker \pi$ satisfies $\mathfrak{m}_A \cdot I = 0$, i.e. if the A-module structure on I is induced by the k-vector space structure via $A \to A/\mathfrak{m}_A \cong k$.

Example.

1. If V is a finite-dimensional vector space then $k \oplus V \to k$ is a small extension.
2. For any $n \geq 1$, $k[t]/t^{n+1} \to k[t]/t^n$ is a small extension.
3. For any scheme X locally of finite type, $x \in X$, $n \geq 1$, then $\mathcal{O}_{X,x}/\mathfrak{m}_x^{n+1} \to \mathcal{O}_{X,x}/\mathfrak{m}_x^n$ is a small extension.

It is easy to see that any surjection of artinian algebras can be written as the composition of small extensions.

Lemma 3.1. *Let $\widehat{P} = k[[t_1, \ldots, t_n]]$, A artinian. Then there is a bijection*

$$\begin{aligned} \mathrm{Hom}_{k\text{-alg}}(\widehat{P}, A) &\to \mathfrak{m}_A^{\oplus n} \\ \varphi &\mapsto (\varphi(x_1), \ldots, \varphi(x_n)) \end{aligned}$$

Proof. Follows from Lemma 2.2. □

Corollary 3.2. *Let $\pi : A \to B$ be a surjection of Artinian rings. Then for any homomorphism $\varphi : \widehat{P} \to B$ there exists $\psi : \widehat{P} \to A$ such that $\varphi = \pi \circ \psi$. We call ψ a* lifting *of φ.*

Theorem 3.3. *Let $\pi : A \to B$ be a small extension with kernel I. Let $\widehat{P} = k[[x_1, \ldots, x_n]]$, $J \subseteq \mathfrak{m}_{\widehat{P}}^2$ an ideal, $\varphi : \widehat{P} \to B$ a homomorphism with $J \subseteq \ker \varphi$. Let $\psi : \widehat{P} \to A$ be a lifting of φ. Let*

$$\omega = \psi|_J \in \mathrm{Hom}_{\widehat{P}}(J, I) = \mathrm{Hom}_k(J/\mathfrak{m}_{\widehat{P}} J, I).$$

Then ω is independent of the choice of ψ and only depends on π and φ.

Proof. Note that as ψ lifts φ, $\psi(J) \subseteq I$. The Hom sets are identified because $\mathfrak{m}_{\widehat{P}} I = \mathfrak{m}_A I = 0$.

Suppose $\eta : \widehat{P} \to A$ is a different lifting. Then for all i, $\psi(x_i) - \eta(x_i) \in I$. Hence for all $f \in \mathfrak{m}_{\widehat{P}}$,

$$\psi(fx_i) - \eta(fx_i) = f(\psi(x_i) - \eta(x_i)) = 0.$$

As $J \subseteq \mathfrak{m}_{\widehat{P}}^2$ and x_i's generate $\mathfrak{m}_{\widehat{P}}$, any $f \in J$ can be written as linear combinations of fx_i where $f \in \mathfrak{m}_{\widehat{P}}$. □

Definition (obstruction)**.** Let $\pi : A \to B$ be a small extension with kernel I and $\varphi : \widehat{P}/J \to B$ with $J \subseteq \mathfrak{m}_{\widehat{P}}^2$. We call $\omega \in \mathrm{Hom}_k(J/\mathfrak{m}_{\widehat{P}}J, I)$ the *obstruction* to lifting φ to a homomorphism $\psi : \widehat{P}/J \to A$ and denote it by $\mathrm{ob}_\pi(\varphi)$.

Remark. $\mathrm{ob}_\pi(\varphi)$ is the obstruction to lift to $\varphi : \widehat{P} \to B$ in the sense that given a lift $\psi : \widehat{P} \to A$, $J \subseteq \ker \psi$ if and only if $\psi|_J : J \to I$ is zero.

Remark. The obstruction is functorial in small extensions. That is given a commutative diagram of artinian algebras

$$\begin{array}{ccc} A & \xrightarrow{\pi} & B \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ A' & \xrightarrow{\pi'} & B' \end{array}$$

where the rows are small extensions with kernels I and I', and $\varphi : \widehat{P}/J \to A$ with $J \subseteq \mathfrak{m}_{\widehat{P}}^2$ then

$$\alpha \circ \mathrm{ob}_\pi(\varphi) = \mathrm{ob}_{\pi'}(\beta \circ \varphi).$$

Proposition 3.4. *Let X be a scheme, $\pi : A \to B$ a small extension with kernel I. Let $F : \mathrm{Spec}\, B \to X$ be a morphism of scheme, $x = F|_{\mathrm{Spec}\, k} \in X$. Then the set*

$$\{\widetilde{F} : \mathrm{Spec}\, A \to X : \widetilde{F}|_{\mathrm{Spec}\, B} = F\},$$

if nonempty, is a principal homogeneous space for $\mathrm{Hom}_k(\Omega_X(x), I)$.

Smoetimes we say an action of G on S is *simply transitive* if S is a principal homogeneous space for G.

Proof. wlog $X = \mathrm{Spec}\, R$ and F is induced by $\varphi : R \to B$. Suppose $\psi_0 : R \to A$ is a lifting of A. Then there is a bijection

$$\{\text{liftings of } \varphi\} \longleftrightarrow \mathrm{Der}_k(R, I)$$
$$\psi \mapsto \psi - \psi_0$$

Note that all we used here is $I^2 = 0$ (instead of the stronger condition of small extension) so I is a B-module and φ makes I into an R-module. ☐

3.1 Exact sequences of groups and sets

Let A_1, A_2 be abelian groups, S_1, S_2 be sets. We say

$$0 \longrightarrow A_1 \xrightarrow{a} S_1 \xrightarrow{f} S_2 \xrightarrow{\mathrm{ob}} A_2$$

is a *sequence of groups and sets* if a is an action of A_1 on S_1, f and ob are maps such that for all $s \in S_1$, for all $g \in A_1$ we have $f(s) = f(g \cdot s)$ and $\mathrm{ob} \circ f$ is the zero map.

Example. If

$$0 \longrightarrow A_1 \xrightarrow{\varphi} S_1 \longrightarrow S_2 \longrightarrow A_2$$

is a complex of ableian groups then we can make it into a sequence of groups and sets by defining the action of A_1 on S_1 to be $a \cdot s = s + \varphi(a)$.

Definition. A sequence of groups and sets

$$0 \longrightarrow A_1 \xrightarrow{a} S_1 \xrightarrow{f} S_2 \xrightarrow{\text{ob}} A_2$$

is called

- *exact at* A_1 if the action of A_1 on S_1 has no fixed points.
- *exact at* S_1 if for all $s, s' \in S_1$ such that $f(s) = f(s')$ then exists $g \in A_1$ with $s' = g \cdot s$. This is equivalent to A_1 acts transitively on nomempty fibres of f.
- *exact at* S_2 if for all $s_2 \in S_2$, $f^{-1}(s_2) \neq \emptyset$ if and only if $\text{ob}(s_2) = 0$.

Exercise. Show that in the previous example exactness of as a sequence of abelian groups is the same as that as a sequence of groups and sets.

Theorem 3.5. *Let* $\widehat{P} = k[[x_1, \dots, x_n]]$, $J \subseteq \mathfrak{m}_{\widehat{P}}^2$ *an ideal,* $\widehat{R} = \widehat{P}/J$. *Let* $\widehat{X} = \operatorname{Spec} \widehat{R}$. *Let* $0 \in \widehat{X}$ *be the point corresponding to* $\mathfrak{m}_{\widehat{R}}$, *the image of* $\mathfrak{m}_{\widehat{P}}$. *Then for every small extension* $\pi : A \to B$ *with kernel* I *we have an exact sequence of groups and sets functorial in* π, *i.e. given a commutative diagram of artinian algebras*

$$\begin{array}{ccc} A & \xrightarrow{\pi} & B \\ \downarrow{\scriptstyle\alpha} & & \downarrow{\scriptstyle\beta} \\ A' & \xrightarrow{\pi'} & B' \end{array}$$

where the rows are small extensions with kernels I *and* I', *we have a commutative diagram of sequence of groups and sets with exact rows*

$$\begin{array}{ccccccccc} 0 & \longrightarrow & T_0\widehat{X} \otimes_k I & \longrightarrow & h_{\widehat{X},0}(A) & \longrightarrow & h_{\widehat{X},0}(B) & \xrightarrow{\text{ob}_\pi} & \operatorname{Hom}_k(J/\mathfrak{m}_{\widehat{P}}J, I) \\ & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ 0 & \longrightarrow & T_0\widehat{X} \otimes_k I' & \longrightarrow & h_{\widehat{X},0}(A') & \longrightarrow & h_{\widehat{X},0}(B') & \xrightarrow{\text{ob}_{\pi'}} & \operatorname{Hom}_k(J/\mathfrak{m}_{\widehat{P}}J, I') \end{array}$$

Proof. We have done most of the work and are merely left to expound the notations. The action in each sequence is defined the same way as in Proposition 3.4 and the other two maps in the sequence are the obvious ones. Then exactness at $T_0\widehat{X} \otimes I$ and $h_{\widehat{X},0}(A)$ is just Proposition 3.4, and exactness at $h_{\widehat{X},0}(B)$ follows from the remark after the definition of obstruction.

The commmutativity of the diagram means that

- $h_{\widehat{X},0}(A) \to h_{\widehat{X},0}$ intertwines the actions of $T_0\widehat{X} \otimes I$ where it acts on the second set via the first vertical map,
- commutativity as sets for the second and third square.

which follows from functoriality of each map. □

Theorem 3.6. *Let $F : \mathbf{Art} \to \mathbf{Set}$ be $A \mapsto \operatorname{Hom}(\widehat{R}, A)$. Then for every small extension $A \to B$ with kernel I there is an exact sequence*

$$0 \longrightarrow \bigoplus I \frac{\partial}{\partial x_i} \longrightarrow F(A) \longrightarrow F(B) \longrightarrow \operatorname{Hom}_k(J/\mathfrak{m}_{\widehat{P}} J, I)$$

Theorem 3.7. *Let $P = k[x_1, \dots, x_n]$, $J = (f_1, \dots, f_r) \subseteq \mathfrak{m}_P = (x_1, \dots, x_n)$. Let $R = \operatorname{Spec} P/J$, $X = \operatorname{Spec} R$. For every small extension $\pi : A \to B$ with kernel I, there exists an exact sequence of groups and sets*

$$0 \longrightarrow T_0X \otimes I \longrightarrow h_{X,0}(A) \longrightarrow h_{X,0}(B) \xrightarrow{\text{ob}} T_0X^1 \otimes I$$

functorial with respect to π, where T_0X and T_0^1X are defined by the exact sequence

$$0 \longrightarrow T_0X \longrightarrow T_0\mathbb{A}^n_k \xrightarrow{\alpha} (J/\mathfrak{m}_P J)^\vee \longrightarrow T_0^1X \longrightarrow 0$$

where α is the dual of

$$\begin{aligned} J/\mathfrak{m}_P J &\to \Omega_{\mathbb{A}^n}(0) \\ f &\mapsto \sum \frac{\partial f}{\partial x_i}(0)\mathrm{d}x_i \end{aligned}$$

Note that if $J \subseteq \mathfrak{m}_P^2$ then all derivatives at the origin vanish and $\alpha = 0$ so we get the previous theorem.

Proof. The only thing new here is the construction of the obstruction map. Given $\varphi : R \to B$, i.e. $\varphi : P \to B$ such that $J \subseteq \ker \varphi$, we can lift it to $\psi : P \to A$, thus defining an element $\psi|_J \in \operatorname{Hom}_P(J, I) = \operatorname{Hom}_k(J/\mathfrak{m}_P J, I)$. The element depends on the choice of ψ and the dependency is described by Proposition 3.4. In other words we get a well-defined element

$$\operatorname{ob}(\varphi) \in \operatorname{Hom}_k(J/\mathfrak{m}_P J, I)/\operatorname{Hom}_k(\Omega_{\mathbb{A}^n}(0), I).$$

Since I is finite-dimensional so flat we get the desired map. □

Exercise. Given P, J, R as in the previous theorem, show exists $\widehat{R}$ in theorem 1 and an equivalence of functors $h_{X,0} \to h_{\widetilde{X},0}$ inducing isomorphism $T_0X \otimes k^{\oplus n}, T_0^1X \cong (\widehat{J}/\mathfrak{m}\widehat{J})^\vee$. Hint: use $h_{X,0} \cong h_{\widehat{X}}$ where $\widehat{S} = k[[x_1, \dots, x_n]]/J\widehat{S}$ and apply inplicit function theorem for formal power series (induction on $\dim \widehat{J}/(\mathfrak{m}\widehat{J})$ (? this step is to verify if $\widehat{J} \subseteq \mathfrak{m}^2$).

Definition (pro-representable functor)**.** Let $F : \mathbf{Art} \to \mathbf{Set}$ be a functor such that $F(k)$ is a singleton. We say F is *pro-representable* if exists n, $\widehat{J} \subseteq \mathfrak{m}_{\widehat{P}}^2$ and an equivalence of functors $\operatorname{Hom}(\widehat{P}/\widehat{J}, -) \to F$.

Exercise. Show F is pro-representable if and only if for all $N > 0$ the restriction of F to $\mathbf{Art}_N$ is representable, where $\mathbf{Art}_N$ is the full subcategory of artinian rings A such that $\mathfrak{m}_A^N = 0$.

Definition (tangent space and obstruction space to a functor)**.** Let $F : \mathbf{Art} \to \mathbf{Set}$ be a functor with $F(k)$ a singleton. Let TF, T^1F be k-vector spaces. We say that TF is the *tangent space* to F and T^1F is an *obstruction space* to F if, for every small extension $\pi : A \to B$ with kernel I, we are given an exact sequence

$$TF \otimes I \longrightarrow F(A) \longrightarrow F(B) \longrightarrow T^1F \otimes I$$

functorial in the small extensions, and such that $0 \to TF \otimes I \to FA$ is also exact if $B = k$.

Theorem 3.8. *F is pro-representable if and only if exists finite-dimensional TF, T^1F such that*

$$0 \longrightarrow TF \otimes I \longrightarrow F(A) \longrightarrow F(B) \longrightarrow T^1F \otimes I$$

is exact for all small extensions $A \to B$.

The only if direction is same as theorem. We will not prove the other direction.

Theorem 3.9. *Assume F has finite-dimensional tangent and obstruction space. Then the tangent space is unique up to a canonical isomorphism, and exists a minimal obstruction space $T^1F_{\min}$ such that every obstruction space is induced by $T^1F_{\min} \hookrightarrow T^1F$.*

Proof. Uniqueness of TF: we say how to recover TF from $TF|_{\mathbf{Art}_2}$. $\mathbf{Art}_2$ is equivalent to finite-dimensional vector space. We can show $TF \to F(k[\varepsilon]/\varepsilon^2)$ is a bijection.

T^1F is not unique: suppose T^1F is an obstruction space. Choose a linear embedding $T^1F \hookrightarrow \widetilde{T}^1F$, then $T^1F \otimes I \hookrightarrow \widetilde{T}^1F \otimes I$. Define $\widetilde{\mathrm{ob}}$ to be the composition of this inclusion with ob. Then $\widetilde{T}^1F$ is also an obstruction space.

Note that this *does not* prove the existence of $T^1F_{\min}$. □

Why didn't we define obstruction space to be $T^1F_{\min}$? This is because in practice, one can compute obstruction spaces but it is unknown if they are minimal, or we do not known how to compute the minimal obstruction space sitting inside a

As a corollary, if we have equalities in both inequalities, i.e. 0 is an obstruction space they we know X is smooth (in fact the converse is also true, which we will prove if we have time)

If instead we have equality on the first inequality then we know the particular TF is minimal.

> **Proposition 3.10.** *Let* $X = \operatorname{Spec} k[x_1, \ldots, x_n]/J$ *where* $J \subseteq \mathfrak{m}_P = (x_1, \ldots, x_n)$. *Then*
> $$(T^1 h_{X,p})_{\min} = \operatorname{coker}((J/\mathfrak{m}J)^\vee \to \Omega_{\mathbb{A}^n}(p)^\vee).$$

Proof. Let $\widetilde{P} = P_{\mathfrak{m}_P} = \mathcal{O}_{\mathbb{A}^n,0}$. Then we have natural maps $P \to \widetilde{P} \to \widehat{P}$. Let $\widetilde{J} - J\widetilde{P}, \widehat{J} = J\widehat{P}, \widetilde{R} =, \widehat{R} =$. Then we have $R \to \widetilde{R} \to \widehat{R}$. We have

$$\begin{array}{ccc}
P & \longrightarrow & \Omega_P = \bigoplus P\mathrm{d}x_i \\
\downarrow & & \downarrow \\
\widetilde{P} & \longrightarrow & \bigoplus \widetilde{P}\mathrm{d}x_i \\
\downarrow & & \\
\widehat{P} & \longrightarrow & \bigoplus \widehat{P}\mathrm{d}x_i \\
 & & \downarrow{\scriptstyle f \mapsto f(0)} \\
 & & \bigoplus k\mathrm{d}x_i = \Omega_{\mathbb{A}^n}(0)
\end{array}$$

We have $J \to \Omega_{\mathbb{A}^n}(0)$ etc. By Leibniz rule it factorises as

$$J/\mathfrak{m}_P J \twoheadrightarrow \widetilde{J}/\mathfrak{m}J \to \widehat{J}/\mathfrak{m}_{\widehat{P}}\widehat{J}$$

and we are going to show they are both isomorphisms. Note if $f_1, \ldots, f_r$ generate J, they also generate $\widetilde{J}$ and $\widehat{J}$ so left to show injectivity.

To show $J/mJ \to \widetilde{J}/m\widetilde{J}$ is injective, let $f \in J$ such that $\frac{f}{1} \in \mathfrak{m}\widetilde{J}$. Werite

$$\frac{f}{1} = \sum \widetilde{a}_i \widetilde{g}_i$$

where $\widetilde{a}_i \in \mathfrak{m}$. We can find $u \in R \setminus \mathfrak{m}_P$ such that $u\widetilde{a}_i = a_i \in \mathfrak{m}_P, u$

To prove the injectivity of the second map we show exists N such that

$$\alpha_N : \widetilde{J}/\mathfrak{m}\widetilde{J} \to \widetilde{J} + \mathfrak{m}^N/\mathfrak{m}\widetilde{J} + \mathfrak{m}^N$$

is an isomorphism It is certainly surjective. Let $L_N = \ker \alpha_N$, $\widetilde{Q} = \widetilde{P}/\mathfrak{m}_{\widetilde{P}}\widetilde{J}$ with maximal ideal $\mathfrak{m}_{\widetilde{Q}}$. Let $\rho : \widetilde{P} \to \widetilde{Q}$ be the natural map. By Krull's intersection theorem

$$\bigoplus_N \mathfrak{m}_{\widetilde{Q}}^N = 0.$$

Then

$$\bigoplus (\mathfrak{m}\widetilde{J} + \mathfrak{m}^N) = \bigoplus \rho^{-1}(\mathfrak{m}_{\widetilde{Q}}^N) = \rho^{-1}(\bigoplus) = \mathfrak{m}\widetilde{J}.$$

The same proof shows that $\widehat{J}/\mathfrak{m}\widehat{J} \to \widehat{J} + \mathfrak{m}^N/\mathfrak{m}\widehat{J} + \mathfrak{m}^N$ is an isomorphism.

...

For the general dcase reduce to this by implicit function theorem and induction on rank $\Omega_{\mathbb{A}^n}(0) \to J\mathfrak{m}J$. ☐

Two things to remember:

1. Krull's intersection theorem: if A is a noetherian local ring or a noetherian domeain and I a proper ideal then $\bigcap_{N \geq 1} I^N = 0$. This implies for example $\mathcal{O}_{X,p} \to \widehat{\mathcal{O}}_{X,p}$ is injective.

2. $P/\mathfrak{m}J \to P/J$ should be "test small extension" but not in **Art**. By Krull intersection theorem, we can divide by $\mathfrak{m}_P^N$ without changing the kernel if $N \gg 0$.

3.2 Example of tangent and obstruction computations

As a general remark, for any functor $F : \mathbf{Sch}^{\text{op}} \to \mathbf{Set}$, we can consider its restriction to affine schemes. We can restrict it to **Art**, which then decomposes as coproduct of F_p for $p \in F(k)$.

Definition (flatness)**.** Let $\pi : X \to Y$ be a morphism, $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. Then $\mathcal{F}$ is *flat* over Y if for all $x \in X$, $\mathcal{F}_x$ is a flat $\mathcal{O}_{Y,\pi(x)}$-module.

Remark. If X is locally noetherian and $\mathcal{F}$ is coherent then $\mathcal{F}$ is flat (over X) if and only if it is locally free.

Let X be a projective scheme over k and $\mathcal{E}$ a coherent sheaf on X. Fix a very ample line bundle $\mathcal{O}_X(1)$ (i.e. an embedding $X \to \mathbb{P}_k^N$). Define a functor

$$\begin{aligned} Q_{\mathcal{E}} &: \mathbf{Sch}^{\text{op}} \to \mathbf{Set} \\ &S \mapsto \{\mathcal{F} \subseteq p_X^*\mathcal{E} : \mathcal{F} \text{ coherent}, p_x^*\mathcal{E}/\mathcal{F} \text{ flat over } S\}/\cong \end{aligned}$$

This implies that $\mathcal{F}$ is flat over S.

For example let $X = \operatorname{Spec} k, \mathcal{E} = \mathcal{O}_X$. Then for $S = \operatorname{Spec} k[t]$, $\mathcal{F} = \mathcal{I}_0 \subseteq \mathcal{O}_{S\times X} = \mathcal{O}_S$ for some ideal sheaf $\mathcal{I}_0$.

Theorem 3.11. *Let $X = \operatorname{Spec} k, \mathcal{E} = V$, a finite-dimensional vector space. Then there is a canonical isomorphism*

$$Q_V(S) \cong \operatorname{Hom}(S, \coprod_{r=0}^{\dim V} \operatorname{Gr}(r, V)),$$

i.e. Q_V is represented by $\coprod_{r=0}^{\dim V} \operatorname{Gr}(r, V)$.

Theorem 3.12 (Grothendieck)**.** *$Q_{\mathcal{E}}$ splits as $\coprod_{P\in\mathbb{Q}[t]} Q_{\mathcal{E}}^P$ where for a scheme S*

$$Q_{\mathcal{E}}^P(S) = \{\mathcal{F} \in Q_{\mathcal{E}}(S) \text{ with Hilbert polynomial } P\}.$$

There exist projectives schemes $\operatorname{Quot}_X^P(\mathcal{E})$ representing $Q_{\mathcal{E}}^P$.

Fix now a point $\mathcal{F}_0 \hookrightarrow \mathcal{E}$ of $Q_{\mathcal{E}}(\operatorname{Spec} k)$ so we get a functor $Q_{\mathcal{F}_0} : \mathbf{Art} \to \mathbf{Set}$

$$Q_{\mathcal{F}_0}(A) = \{\mathcal{F} \subseteq p_X^*\mathcal{E} : \text{ coherent}, p_X^*\mathcal{E}/\mathcal{F} \text{ flat over } A, s_0^*\mathcal{F} = \mathcal{F}_0\}$$

where $s_0 : X \to X_A := X \times \operatorname{Spec} A$ is induced by $A \to k$. As a topological space X_A is just X and $\mathcal{O}_{X_A} = \mathcal{O}_X \otimes_k A$. Thus $\mathcal{F} \in \mathbf{Coh}(X_A)$ if and only if the pushforward of $\mathcal{F}$ along $X_A \to X$ is coherent, plus the structure of a sheaf of A-modules. Also $p_X^*\mathcal{E} = \mathcal{E} \otimes_k A$.

Let $A \to B$ be a small extension with kernel I, $\mathcal{F}_B \in Q_{\mathcal{F}_0}(B)$. Question: can we lift this to A?

Let $\mathcal{Q}_B$ be the quotient. As Q_B is flat, $- \otimes_B I$ is exact. ...

If so, this implies $Q_A \otimes_A B \to Q_B$ and $Q_A \otimes k \to Q_0$ are isomorphisms inducing a diagram with exact rows and columns.

Claim $\mathcal{F}_A$ is determined by its image in $\mathcal{E} \otimes A/\operatorname{im}\beta$ which is contained in $\mathcal{G} = \ker\alpha/\operatorname{im}\beta$.

Claim exact sequence

Proof. □

Proof. There is a bijection

$$\{\mathcal{F}_A \subseteq \mathcal{E} \otimes A : \text{ induces red diagram}\} \leftrightarrow \{\mathcal{F}_B \to \ker\alpha/\operatorname{im}\beta \text{ splitting } *\}$$

□

Proof. □

Lemma 3.13. *$\mathcal{F}_A$ so obtained gives rise to Q_A such that $Q_A \otimes_A B \xrightarrow{\cong} Q_B$ and $Q_A \otimes_A k \cong Q_0$.*

Proof. □

Proposition 3.14. *An A-module M is A-flat if adn only if $M \otimes_A B$ is B-flat and $(M \otimes B) \otimes_B I \to M$ is injective.*

Proof. Omitted. □

Corollary 3.15. *A lifting $\mathcal{F}_A$ of $\mathcal{F}_B$ exists if and only if $(*)$ splits. If it exists then the set of liftings is in bijection with the set of splittings.*

It is a standard fact that * as an extension defines an element of

$$\operatorname{Ext}^1_{\mathcal{O}_{X_B}}(\mathcal{F}_B, Q_0 \otimes_k I) \cong \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{F}_0, Q_0 \otimes_k I) \cong \operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{F}_0, Q_0) \otimes_k I.$$

and splits if and only if it is zero. If so then bijection with ...

In summary, one can prove (*without* knowing that $\operatorname{Quot}_X(\mathcal{E})$ is representable) that the induced functor **Art** $\to$ **Set** has tangent and obstruction space, represente by $\operatorname{Hom}_{\mathcal{O}_X}(\mathcal{F}_0, \mathcal{Q}_0)$ and $\operatorname{Ext}^1_{\mathcal{O}_X}(\mathcal{F}_0, \mathcal{Q}_0)$ respectively.

Remark. There are cases in which $\operatorname{Ext}^1(\mathcal{F}_0, \mathcal{Q}_0)$ is not minimal as obstruction space. For example let $X \subseteq \mathbb{P}^3_{\mathbb{C}}$ be a smooth quartic surface such that $(0,0,0,1) \notin X$. Let $\mathcal{E} = \mathcal{O}_X^{\oplus 3}$, $\mathcal{Q}_0 = \mathcal{O}_X(1)$ and $\mathcal{E} \to \mathcal{Q}_0$ given by (x,y,z).

Minimal obstruction is 0. c.f. "unobstructed"

There is an obstruction because it "obstructs" a different more general problem (relative deformation problem)

def $(X, \mathcal{E}, \mathcal{Q}) \to$ "forget" def $(X, \mathcal{E})$.

There are deformations of X on which $\mathcal{E}$ extends (it's trivial) buty $\mathcal{Q}$ doesn't by Hodge theory/topology (its first Chern class is not of type $(1,1)$).

Example. We give an example of a similar problem. Fix C, V. Define a functor

$$M : \mathbf{Sch}^{\mathrm{op}} \to \mathbf{Set}$$
$$S \mapsto \mathrm{Hom}(S \times C, V)$$

and morphisms are sent to compositions. If C, V are projective, claim M is an open subfunctor of $\mathrm{Hilb}(C \times V)$: given a morphism $\gamma : S \times C \to V$, consider its graph $\Gamma_\gamma \subseteq S \times C \times V$. $\Gamma_\gamma \to S \times C$ is an isomorphism so Γ_γ is flat over S. Conversely a closed subscheme $Z \subseteq S \times C \times V$ is a graph if and only if $Z \to S \times C$ is an isomorphism. Assuming Z is S-flat, claim

$$U = \{s \in S : Z_s \to \{s\} \times C \text{ is isomorphism}\}$$

is open in S and $(U \times C \times V) \cap Z \to U \times C$ is an isomorphism.

Thus M is representable. The corresponding functor $F_f : \mathbf{Art} \to \mathbf{Set}$ associated to $f : \operatorname{Spec} k \times C = C \to V$ always has tangent and obstruction space.

Easiest case

Proposition 3.16. *Suppose C is separated and V is smooth over k. Then F_f has tangent and obstruction space equal to $H^0(C, f^*\mathcal{T}_V)$ and $H^1(C, f^*\mathcal{T}_V)$.*

Sketch proof. We first consider case V, C affine. If V is affine n-space then... Next we consider smooth case.

Cover C, V by affines $\{C_i\}, \{V_i\}$ such that $f : C_i \to V_i$. We get obstruction in $H^1(C, f^*\mathcal{T}_V)$ as a Cech cocyle: giving $f_B : \operatorname{Spec} B \times C \to V$ is the same as $f_B^\# : f^{-1}\mathcal{O}_V \to \mathcal{O}_C \otimes_k B$. By the affine case locally we can lift to f_A^i and get them to agree on $C_i \cap C_j$. We get a cocycle by taking differences. One checks that changing liftings changes cocyle by a coboundary. □

A different type of functor. We want to study moduli of proper schemes. Define a module functor M sending S to isomorphism classes of $\{X_S \to S$ flat proper, plus assumptions on fibers that are open in S (for example dimension or types of singularity allowed)

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