Scuola Internazionale
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Geometry and Mathematical Physics

Derived Functors

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Contents

0 Introduction

Examples of derived functors:

- cohomology theory: eg sheaf cohomology, group cohomology, Lie algebra cohomology (the latter two are deeply related to extension problems).
- Ext functor, higher direct images.

They generalise to hyperderived functors and spectral sequences and ultimately, derived categories.

0.1 Categorical preliminaries

An *additive category* is an **Ab**-enriched category with all finite biproducts. An additive category is an *abelian category* if

- for every morphism $f : A \to B$, $\ker f$ and $\operatorname{coker} f$ exists.
- every monomorphism (resp. epimorphism) is a kernel (resp. cokernel).

Example. $\mathbf{Mod}_R$ is an abelian category. An example that is not abelian: $\mathbf{Vect}_X$, the category of vector bundles over X. Then "kernel" of f may not be a vector bundle.

For simplicity of argument, we will not be using generalised element to do diagram chasing in this course. Instead, we will pretend that our categories of interests are concretisable. This is justified by

> **Theorem 0.1** (Freyd-Mitchell)**.** *If A is a small abelian category there exists an unital ring R and an exact fully faithful functor $A \to \mathbf{Mod}_R$.*

0.2 Homological algebra preliminaries

Let $\mathbf{A}$ be an abelian category. A *short exact sequence* is the data of morphism $A'' \xrightarrow{i} A \xrightarrow{p} A'$ in which i is monic, p is epic and $\ker p = \operatorname{im} i$. We write

$$0 \longrightarrow A'' \xrightarrow{\ i\ } A \xrightarrow{\ p\ } A'' \longrightarrow 0$$

An additive functor $F : \mathbf{A} \to \mathbf{B}$ is *exact* if it preserves short exact sequences. More generally we have a left-exact sequence

$$0 \longrightarrow A \longrightarrow A' \longrightarrow \cdots$$

and similarly right-exact sequence.

Example. Let $\mathbf{A} = \mathbf{Mod}_R$. Suppose there is an exact sequence

$$0 \longrightarrow M' \longrightarrow M \longrightarrow M'' \longrightarrow 0$$

Fix an R-module N and define an endofunctor $F : M \mapsto M \otimes_A N$. Then in general we only have a sequence on the right

$$M' \otimes N \longrightarrow M \otimes N \longrightarrow M'' \otimes N \longrightarrow 0$$

Proposition 0.2 (snake lemma)**.** *Given a commutative diagram with exact rows*

$$\begin{tikzcd}
 & M' \arrow[r] \arrow[d, "f"] & M \arrow[r] \arrow[d, "g"] & M'' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & N' \arrow[r] & N \arrow[r] & N'' &
\end{tikzcd}$$

then exists a morphism $\ker h \to \operatorname{coker} f$ *making the snake sequence exact*

$$\begin{tikzcd}
 & \ker f \arrow[r] \arrow[d] & \ker g \arrow[r] \arrow[d] & \ker h \arrow[d] \arrow[dddll, dashed] & \\
 & M' \arrow[r] \arrow[d, "f"] & M \arrow[r] \arrow[d, "g"] & M'' \arrow[r] \arrow[d, "h"] & 0 \\
0 \arrow[r] & N' \arrow[r] \arrow[d] & N \arrow[r] \arrow[d] & N'' \arrow[d] & \\
 & \operatorname{coker} f \arrow[r] & \operatorname{coker} g \arrow[r] & \operatorname{coker} h &
\end{tikzcd}$$

Definition (differential object)**.** In an abelian cateogry $\mathbf{A}$, a *differential object* is a pair (A, d) where $A \in \mathrm{Ob}(\mathbf{A})$, $d : A \to A$ with $d^2 = 0$.

We define $Z(A) = \ker d$ to be *(co)cycles* and $B(A) = \operatorname{im} d$ to be *(co)boundaries*. Since $d^2 = 0$, $B(A)$ is a subject of $Z(A)$ and we call $H(A) = Z(A)/B(A)$ the *(co)homology* of A.

A *morphism of differential objects* $f : (A, d) \to (B, d')$ is a morphism $f : A \to B$ such that $d' \circ f = f \circ d$.

Denote by $\mathbf{Diff}(\mathbf{A})$ the category of differential objects in $\mathbf{A}$.

Given a morphism $f : (A, d) \to (B, d')$, we have $f(Z(A)) \subseteq Z(B)$, $f(B(A)) \subseteq B(B)$ (here "$\subseteq$" means being a subobject) and hence there is an induced morphism $H(f) : H(A) \to H(B)$. In other words, $H : \mathbf{Diff}(\mathbf{A}) \to \mathbf{A}$ is a functor.

long exact sequence of cohomology (triangle) Given a short exact sequence of differential objects

$$0 \longrightarrow A \xrightarrow{\ i\ } B \xrightarrow{\ p\ } C \longrightarrow 0$$

we call

$$\begin{tikzcd}
H(A) \arrow[r, "H(i)"] & H(B) \arrow[d, "H(p)"] \\
 & H(C) \arrow[ul, "\delta"]
\end{tikzcd}$$

an *exact triangle*. The construction of the morphism δ is a standard exercise.

1 Derived functors

1.1 Complexes

Definition (differential complex)**.** A *(differential) complex* in $\mathbf{A}$ is a collection of objects $\{A^n\}_{n\in\mathbb{N}}$ and $\{d_n : A^n \to A^{n+1}\}_{n\in\mathbb{N}}$ with $d_{n+1}d_n = 0$. Define $Z^n(A) = \ker d_n, B^n(A) = \operatorname{im} d_{n-1}$ and $H^n(A) = Z^n(A)/B^n(B)$.

A *morphism of complexes* $f : (A^\bullet, d) \to (B^\bullet, d')$ (of degree 0) is a collection $f = \{f_n : A^n \to B^n$ such that for each $n \in \mathbb{N}$, $d'_n \circ f_n = f_{n+1} \circ d_n$.

Denote by $\mathbf{Ch}(\mathbf{A})$ the category of complexes of objects in $\mathbf{A}$.

Note that given a morphism $f : A^\bullet \to B^\bullet$, we have morphisms $H^n(f) : H^n(A) \to H^n(B)$ for all $n \in \mathbb{N}$.

Example (de Rham cohomology)**.** Let X be a smooth manifold and $\Omega^k(X)$ the collection of differential k-forms on X. Then exterior derivative $\mathrm{d} : \Omega^k(X) \to \Omega^{k+1}(X)$ satisfies $\mathrm{d}^2 = 0$. We call the cohomology of the complex $(\Omega^\bullet(X), \mathrm{d})$ the de Rham cohomology of X, $H^k_{\mathrm{dR}}(X) = H^k(\Omega^\bullet(X), \mathrm{d})$.

Suppose $X = U \cup V$ where U, V are open. Then

$$\begin{tikzcd}
0 \arrow[r] & \Omega^k(X) \arrow[r] & \Omega^k(U) \oplus \Omega^k(V) \arrow[r] & \Omega^k(U \cap V) \arrow[r] & 0 \\
& \omega \arrow[r, mapsto] & (\omega|_U, \omega|_V) & & \\
& & (\nu, \tau) \arrow[r, mapsto] & \nu|_{U\cap V} - \tau|_{U\cap V} &
\end{tikzcd}$$

is exact for all k.

We say a sequence of morphisms of complexes $A^\bullet \to B^\bullet \to C^\bullet$ is exact if for each n the sequence $A^n \to B^n \to C^n$ is exact.

Theorem 1.1. *If*

$$0 \longrightarrow (A^\bullet, d') \longrightarrow (B^\bullet, d) \longrightarrow (C^\bullet, d'') \longrightarrow 0$$

is exact, we have a long exact sequence

$$\cdots \longrightarrow H^n(A) \longrightarrow H^n(B) \longrightarrow H^n(C) \longrightarrow H^{n+1}(A) \longrightarrow \cdots$$

Example (Mayer-Vietoris sequence)**.** The short sequence of de Rham complexes in the previous example gives

$$\begin{tikzcd}
0 \arrow[r] & H^0_{\mathrm{dR}}(X) \arrow[r] & H^0_{\mathrm{dR}}(U) \oplus H^0_{\mathrm{dR}}(V) \arrow[r] & H^0_{\mathrm{dR}}(U \cap V) \arrow[dll, out=-10, in=170] \\
& H^1_{\mathrm{dR}}(X) \arrow[r] & \cdots &
\end{tikzcd}$$

1.2 Homotopics of complexes

Definition (chain homotopy). Two chain maps $f, g : (A^\bullet, d) \to (B^\bullet, d')$ are *chain homotopic* if there exists $z_n : A^n \to B^{n-1}$ such that

$$f_n - g_n = z_{n+1} \circ d_n + d_{n-1} \circ z_n.$$

Proposition 1.2. *If f and g are chain homotopic then $H^n(f) = H^n(g)$.*

Definition (homotopic complex). Two complexes $A^\bullet$ and $B^\bullet$ are *homotopic* if there exists $f : A^\bullet \to B^\bullet, g : B^\bullet \to A^\bullet$ such that $g \circ f \simeq \mathrm{id}_{A^\bullet}, f \circ g \simeq \mathrm{id}_{B^\bullet}$.

Proposition 1.3. *Homotopic complexes have isomrphic cohomologies.*

Definition (quasi-isomorphism). We say a map between two complexes $f : A^\bullet \to B^\bullet$ is a *quasi-isomorphism* $H^n(f)$ is an isomorphic for all n.

Proposition 1.4. *Given a complex $A^\bullet$, if $\mathrm{id}_{A^\bullet} \simeq 0$ then $H^n(A) = 0$. In other words if the identity is* null-homotopic *then it is* acyclic*.*

More generally, if a complex $A^\bullet$ has null-homotopic identity from degree n_0 then $H^n(A) = 0$ for all $n \geq n_0$.

Proposition 1.5 (Poincaré lemma). $H^k_{\mathrm{dR}}(\mathbb{R}^n) = 0$ *for* $k > 0$.

Proof. Recall $H^k_{\mathrm{dR}}(M) = H^k((\Omega^\bullet(M), \mathrm{d}))$. We must find $z : \Omega^k(\mathbb{R}^n) \to \Omega^{k-1}(\mathbb{R}^n)$ such that. Given $\omega \in \Omega^k(\mathbb{R}^n)$, write

$$\omega = \sum_{i_1 < \cdots < i_k} \omega_{i_1 \cdots i_k} \mathrm{d}x^{i_1} \wedge \cdots \wedge \mathrm{d}x^{i_k}$$

Set

$$(z_k \omega)(x) = \sum_{i_1 < \cdots < i_k} \sum_{p=1}^{k} (-1)^{p-1} \left(\int_0^1 t^{k-1} \omega_{i_1 \cdots i_k}(txdt) \right) x^{i_p} dx^{i_1} \wedge \cdots \hat{\mathrm{d}x^{i_p}} \wedge \cdots \mathrm{d}x^{i_x}$$

Integration by parts gives the desired result. □

1.3 Left and right exact functors

Left and right derived functors are dual to each other so it suffices to deal with one of them.

Definition (left exact functor). An additive functor $F : \mathbf{A} \to \mathbf{B}$ between abelian categories is *left exact* if given an exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

in $\mathbf{A}$,

$$0 \longrightarrow F(A') \longrightarrow F(A) \longrightarrow F(A'')$$

is exact in $\mathbf{B}$.

Example.

1. Given $B \in \mathbf{A}$, $\mathrm{Hom}(B, -) : \mathbf{A} \to \mathbf{Ab}$ is left exact. So is $\mathrm{Hom}(-, B) : \mathbf{A}^{\mathrm{op}} \to \mathbf{Ab}$.

2. Given $M \in \mathbf{Mod}_R$, $- \otimes_R M$ is right exact.

1.4 Resolution

We will define right derived funcotrs $R^i F$ of a left exact functors F are defined using injective resolutions.

Definition (resolution)**.** Suppose $A \in \mathbf{A}$. A *resolution* of A is a pair $(L^\bullet, \epsilon)$ where $L^\bullet \in \mathbf{Ch}(\mathbf{A})$ and $\epsilon : A \to L^0$ a morphism such that the sequence

$$0 \longrightarrow A \xrightarrow{\epsilon} L^0 \xrightarrow{d_0} L^1 \longrightarrow \cdots$$

is exact.

Example. By Poincaré lemma, $(\Omega^\bullet, \epsilon)$ is an resolution of $\mathbb{R}$, where ϵ is the inclusion of smooth functions.

In an abelian category $\mathbf{A}$, the following diagram is a pushout

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "g"] & B \arrow[d] \\ C \arrow[r] & B \oplus C / \mathrm{im}(f, -g) \end{tikzcd}$$

Proposition 1.6. *Given a short exact sequence in* $\mathbf{A}$

$$0 \longrightarrow A' \xrightarrow{i} A \xrightarrow{p} A'' \longrightarrow 0$$

then TFAE:

1. *i has a retraction,*

2. *p has a section,*

3. *$A \cong A' \oplus A''$, with i inclusion in the first factor and p projection onto the second factor.*

If any of these conditions hold then the short exact sequence is said to split*.*

Proposition 1.7. *Let $I \in \mathbf{A}$. Then TFAE:*

1. $\mathrm{Hom}(-, I)$ *is exact,*

2. *if $i : A' \to A$ is monic then for all $f : A' \to I$ exists $g : A \to I$ such*

that $g \circ i = f$

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r, "i"] \arrow[d, "f"] & A \arrow[dl, dotted, "g"] \\ & I & \end{tikzcd}$$

3. *every short exact sequence*

$$0 \longrightarrow I \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

splits.

If any of these conditions holds then I is called an injective object *of* $\mathbf{A}$*.*

Proof.

- $1 \iff 2$: given an exact sequence

$$0 \longrightarrow A' \xrightarrow{j} A \xrightarrow{q} A'' \longrightarrow 0$$

  $\mathrm{Hom}(A, I) \xrightarrow{i^*} \mathrm{Hom}(A', I) \to 0$ is exact if and only if 2 holds.

- $1 \implies 3$: applying the exact functor $\mathrm{Hom}(-, I)$ to the exact sequence, we can find $f \in \mathrm{Hom}(A, I)$ such that $f \circ i = \mathrm{id}_I$.

- $3 \implies 1$: assuming all such exact sequences split, we prove $j^* : \mathrm{Hom}(A, I) \to \mathrm{Hom}(A', I)$ is surjective. Form the pushout P

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r, "j"] \arrow[d, "f"] & A \arrow[d, "g"] \\ 0 \arrow[r, dashed] & I \arrow[r] & P \arrow[l, dashed, bend left, "h"] \end{tikzcd}$$

  Since the top row is monic, so is the bottom row. We can find a retraction $h : P \to I$. Then $f = h \circ g \circ j = j^*(h \circ g)$.

$\square$

As a first example, recall that an abelian group $A \in \mathbf{A}$ is *divisible* if for all $a \in G$ and all $n \in \mathbb{Z}$ there exists $h \in G$ such that $g = nh$.

Theorem 1.8. *Divisible groups are injectives in* $\mathbf{Ab}$*.*

Proof. Exercise. $\square$

Example. $\mathbb{Q}$ is divisible and

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Q} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0$$

is an *injective resolution* of $\mathbb{Z}$.

Definition (have enough injectives)**.** An abelian category $\mathbf{A}$ is said to *have enough injectives* if for all $A \in \mathbf{A}$ there exists a mono $A \to I$ where I is injective.

Exercise.

1. Show that every abelian group embeds in a divisible group, thus $\mathbf{Ab}$ has enough injectives.
2. Show the extension of scalars of an injective module is injective, so $\mathbf{Mod}_R$ has enough injectives.
3. $\mathbf{A}$ has enough injectives if and only if every object has an injective resolution.

Now we can start constructing right derived functors of a left exact functor. Suppose $F : \mathbf{A} \to \mathbf{B}$ is a left exact functor between abelian categories and we assume $\mathbf{A}$ has enough injectives. Given $A \in \mathbf{A}$, we take an injective resolution $A \to I^\bullet$. $F(I^\bullet)$ is a chain complex but not exact in general.

Definition (right derived functor)**.** We defined the *right derived functors* of F to be

$$R^n F(A) = H^n(F(I^\bullet))$$

for $n \geq 0$.

This really should be done in the derived category where there will be no ambiguity on choices. Unfortunately for now we will stick to the classical formalism and instead will show $R^n F$ is well-defined up to isomoprhism, because any two injective resolutions of an object A are homotopy equivalent.

Note that there is a natural isomorphism $R^0 F \cong F$. For example we will later see that the 0th sheaf cohomology is isomorphic to global sections.

Lemma 1.9. *Given an abelian category $\mathbf{A}$, two injective resolutions of an object $A \in \mathbf{A}$ are homotopy equivalent.*

This in turn follows from the lifting propery: suppose $A \to L^\bullet$ is an injective resolution and

$$B \xrightarrow{\eta} I^0 \xrightarrow{d_0} I^1 \longrightarrow \cdots$$

is a complex with I^*'s injective. Then a morphism $f : A \to B$ lifts to a morphism of complexes $L^\bullet \to I^\bullet$. Moreover any two such lifts are homotopy equivalent.

Given this, we can also finish the definition of $R^n F$ as a functor defining the map on morphisms: given a morphism $f : A \to B$ and injective resolutions $A \to I^\bullet, B \to J^\bullet$, f lifts to $g : I^\bullet \to J^\bullet$. Then we define

$$R^n F(f) = H^n(F(g)) : R^n F(A) = H^n(F(I^\bullet)) \to H^n(F(J^\bullet)) = R^n F(B).$$

Again this is well-defined.

Example. Suppose $\mathbf{A}$ is an abelian category with enough injectives and $A \in \mathbf{A}$. We have seen $\mathrm{Hom}_{\mathbf{A}}(A, -) : \mathbf{A} \to \mathbf{Ab}$ is left exact. We define the *Ext functors* to be

$$\mathrm{Ext}^i_{\mathbf{A}}(A, -) = R^i \,\mathrm{Hom}_{\mathbf{A}}(A, -).$$

The first Ext functor Ext^1 is related to the extension problem: given $A, C \in\in \mathbf{A}$, can we find $B \in \mathbf{A}$ and an exact sequence

$$0 \longrightarrow A \longrightarrow B \longrightarrow C \longrightarrow 0$$

In this case B is called an extension of C by A. One can show that there is a bijection between $\mathrm{Ext}^1(C, A)$ and the equivalence classes of extensions under the relation B and B' are equivalent if there is a commutative diagram

$$\begin{tikzcd} & & B \arrow[dd, "f"] \arrow[dr] & & \\ 0 \arrow[r] & A \arrow[ur] \arrow[dr] & & C \arrow[r] & 0 \\ & & B' \arrow[ur] & & \end{tikzcd}$$

By five lemma f is an isomorphism. This is an equivalence relation.

Example. Here is an example that illustrates the duality principle. Recall that $- \otimes_R M : \mathbf{Mod}_R \to \mathbf{Mod}_R$ is right-exact. The category of R-modules has enough projectives. Thus we define

$$\mathrm{Tor}_i(-, M) = F^i(- \otimes_R M).$$

In more detail, for N an R-module, take a projective resolution $P^\bullet \to N$ and we define $\mathrm{Tor}_i(N, M) = H^i(P^\bullet \otimes M)$.

Example: Lie algebra cohomology

Definition (Lie algebra and representation)**.** A *Lie algebra* $\mathfrak{g}$ over a ring R is a R-module with a skew bilinear operation $[\cdot, \cdot] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ satisfying the Jacobi identity

$$[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0$$

for all $x, y, z \in \mathfrak{g}$.

A *representation* of $\mathfrak{g}$ is a pair (M, ρ) where M is an R-module and $\rho : \mathfrak{g} \to \mathrm{End}_R(M)$ preserves the Lie bracket, i.e.

$$\rho[x, y] = [\rho(x), \rho(y)].$$

If $(M, \mathfrak{g})$ is representation, we define the *invariant submodule* to be

$$M^{\mathfrak{g}} = \{m \in M : \rho(x)(m) = 0 \text{ for all } x \in \mathfrak{g}\}.$$

$\mathbf{Rep}(\mathfrak{g})$ is abelian. If it has enough injectives then we can derive the left-exact functor $(-)^{\mathfrak{g}} : \mathbf{Rep}(\mathfrak{g}) \to \mathbf{Mod}_R$ and we can define *Lie algebra cohomology* to be

$$H^i(\mathfrak{g}, M) = R^i(-)^{\mathfrak{g}}(M) = R^i M^{\mathfrak{g}}.$$

To show $\mathbf{Rep}(\mathfrak{g})$ has enough injectives, define the *universal enveloping algebra* of $\mathfrak{g}$ to be

$$U(\mathfrak{g}) = T(\mathfrak{g})/I$$

where I is the two-sided ideal generated by elements of the form $x \otimes y - y \otimes x - [x, y]$. Then by the universal property of $U(\mathfrak{g})$, $\mathbf{Rep}(\mathfrak{g}) \cong \mathbf{Mod}_{U(\mathfrak{g})}$.

It can be computed using Chevalley-Eilenberg complex $(C^\bullet, d)$ where $C^i = \mathrm{Hom}_R(\Lambda^i \mathfrak{g}, M)$. When $\mathfrak{g}$ is free over R,

$$H^i_{\mathrm{CE}}(\mathfrak{g}, M) = H^i(\mathfrak{g}, M).$$

1.5 Long exact sequence of a derived functor

Suppose $F : \mathbf{A} \to \mathbf{B}$ is a left exact functors between abelian categories where $\mathbf{A}$ has enough injectives. Suppose we have an exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

We will show there exists a long exact sequence

$$\begin{tikzcd}
0 \arrow[r] & R^0F(A') \arrow[r] & R^0F(A) \arrow[r] & R^0F(A'') \arrow[dll, "\delta_0"'] \\
 & R^1F(A') \arrow[r] & R^1F(A) \arrow[r] & R^1F(A'') \arrow[dll, "\delta_1"'] \\
 & R^2F(A') \arrow[r] & \cdots &
\end{tikzcd}$$

Recall that $R^0F \cong F$, this makes precise the statement that right derived functors measure the failure the extent to which a left exact functor fails to be exact.

To derive this long exact sequence we want to construct a short exact sequence of complexes.

Lemma 1.10 (horseshoe lemma)**.** *Given a short exact sequence in* $\mathbf{A}$ *in which there is enough injectives, we can fit it into a commutative diagram*

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & A'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^0 \arrow[r] \arrow[d] & J^0 \arrow[r] \arrow[d] & K^0 \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & I^1 \arrow[r] \arrow[d] & J^1 \arrow[r] \arrow[d] & K^1 \arrow[r] \arrow[d] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

where the columns are injective resolutions and all rows are exact.

Proof. Take injective resolutions $A' \to I^\bullet$, $A'' \to J^\bullet$. Let $J^n = I^n \oplus K^n$. Clearly they are injective objects. We construct arrows $A \to J^0$ and $J^n \to J^{n+1}$ so that $J^\bullet$ is a chain complex making the above diagram commute, and it follows from the long exact sequence induced from short exact sequences that it is a resolution.

As usual induction on n. For $n = -1$ by injectivity of I^0 there is a lift $A \to I^0$ which composes to give $A \to J^0$.

$$\begin{tikzcd} 0 \arrow[r] & A' \arrow[r] \arrow[d] & A \arrow[r] \arrow[dl, dotted] & A'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & I^0 \arrow[r] & J^0 \arrow[r] & K^0 \arrow[r] & 0 \end{tikzcd}$$

By five lemma, the cokernels of the vertical maps also form a short exact sequence so apply the above the above procedure to the cokernels. ☐

Lemma 1.11. *If*

$$0 \longrightarrow B' \longrightarrow B \longrightarrow B'' \longrightarrow 0$$

is a split short exact sequence in $\mathbf{A}$ *and* $F : \mathbf{A} \to \mathbf{B}$ *is left exact then*

$$0 \longrightarrow F(B') \longrightarrow F(B) \longrightarrow F(B'') \longrightarrow 0$$

is short exact in $\mathbf{B}$.

Proof. Let $s : B'' \to B$ be a section of $p : B \to B''$. Then by functoriality $F(p) \circ F(s) = \mathrm{id}_{F(B'')}$ so $F(p)$ is epic. ☐

Given this, we can take a short exact sequence of injective resolutions

$$0 \longrightarrow I^\bullet \longrightarrow J^\bullet \longrightarrow K^\bullet \longrightarrow 0$$

which splits in $\mathbf{Ch}(\mathbf{A})$ as $I^\bullet$ is an injective object. This gives a short sequence in $\mathbf{ChB}$

$$0 \longrightarrow F(I^\bullet) \longrightarrow F(J^\bullet) \longrightarrow F(K^\bullet) \longrightarrow 0$$

It induces a long exact sequence and by definition $H^n(F(I^\bullet)) = R^nF(A')$ etc.

1.6 Acyclic resolution

Let $F : \mathbf{A} \to \mathbf{B}$ be a left exact functor between abelian categories and $\mathbf{A}$ has enough injectives.

Definition (acyclic object, acyclc resolution)**.** An object $C \in \mathbf{A}$ is F-*acyclic* if $R^iF(C) = 0$ for all $i > 0$. A resolution $L^\bullet$ of A is F-*acyclic* if all L^n's are F-acyclic.

Note that if $L^\bullet$ is any resolution of A, there always exist morphisms $H^n(F(L^\bullet)) \to R^nF(A)$.

Proposition 1.12. *If* $L^\bullet$ *is an acyclic resolution of* A *then* $H^n(F(L^\bullet)) \to R^nF(A)$ *is an isomorphism for all* $n \geq 0$. *Moreover this isomorphism is natural in the sense that if* $f : A \to A'$ *lifts to* F*-acyclic resolutions*

$g : L^\bullet \to H^\bullet$ then the following diagram commutes

$$\begin{tikzcd} H^n(F(L^\bullet)) \arrow[r, "H^n(F(g))"] \arrow[d, "\cong"] & H^n(F(H^\bullet)) \arrow[d, "\cong"] \\ R^nF(A) \arrow[r, "R^n(f)"] & R^nF(A') \end{tikzcd}$$

Proof. Write $Z^n = \ker(d_n : L^n \to L^{n+1}) \cong \operatorname{im}(d_{n-1} : L^{n-1} \to L^n)$. Then the acyclic resolution is equivalent to the following short exact sequences

$$0 \longrightarrow A \longrightarrow L^0 \longrightarrow Z^1 \longrightarrow 0$$

$$0 \longrightarrow Z^n \longrightarrow L^n \longrightarrow Z^{n+1} \longrightarrow 0$$

for $n \geq 0$. As L^n is acyclic, we have exact sequences

$$0 \longrightarrow F(A) \longrightarrow F(L^0) \longrightarrow F(Z^1) \longrightarrow R^1F(A) \longrightarrow 0$$

$$0 \longrightarrow R^{i-1}F(Z^1) \longrightarrow R^iF(A) \longrightarrow 0$$

for $i > 0$. Similarly the followings are exact

$$0 \longrightarrow F(Z^n) \longrightarrow F(L^n) \longrightarrow F(Z^{n+1}) \longrightarrow R^1F(Z^n) \longrightarrow 0$$

$$0 \longrightarrow R^i(Z^{n+1}) \longrightarrow R^{i+1}(Z^n) \longrightarrow 0$$

so

$$\begin{aligned} R^iF(A) &\cong R^{i-1}F(Z^1) \\ &\cong \cdots \\ &\cong R^1F(Z^{i-1}) \\ &\cong \frac{F(Z^n)}{\operatorname{im} F(L^{i-1})} \\ &\cong \frac{\ker(F(L^i) \to F(Z^{i+1}))}{\operatorname{im} F(L^{i-1})} \\ &\cong \frac{\ker(F(L^i) \to F(L^{i+1}))}{\operatorname{im} F(L^{i-1})} \\ &\cong H^i(F(L^\bullet)) \end{aligned}$$

Naturality can be checked by noting that all morphisms involved are functorial. □

Example.

1. A sheaf $\mathcal{F}$ on X is *flasque* if for all $U \subseteq X$ open, $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective. Flasque sheaves are acyclic so we may compute sheaf cohomology using flasque sheaves.

2. Free modules are projective so we may compute right derived functors using free resolutions.

1.7 δ-functors

Properties of (right) derived functors: the right derived functors $R^iF : A \to B$ of a left exact functor $F : A \to B$ has the following properties:

1. given a short exact sequence, there are connecting morphism δ's that fit into a long exact sequence.
2. naturality.

There are the properties we would like to extract. We thus define

Definition (δ-functor)**.** A *δ-functor* $\mathbf{A} \to \mathbf{B}$ is a collection $\{T^i\}_{i\in\mathbb{N}}$ of additive functors $T^i : \mathbf{A} \to \mathbf{B}$ such that

1. for any exact sequence

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

in $\mathbf{A}$, there are morphisms $\delta_i : T^i(A'') \to T^{i+1}(A')$ such that there is a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & T^0(A') \arrow[r] & T^0(A) \arrow[r] & T^0(A'') \arrow[dll, "\delta_0"] \\ & T^1(A') \arrow[r] & T^1(A) \arrow[r] & \cdots \end{tikzcd}$$

2. For every morphisms of exact sequences there are commutative diagrams

$$\begin{tikzcd} T^i(A'') \arrow[r, "\delta_i"] \arrow[d] & T^{i+1}(A') \arrow[d] \\ T^i(B'') \arrow[r, "\delta_i"] & T^{i+1}(B') \end{tikzcd}$$

Remark. T^0 is left exact.

Definition (effaceable)**.** A functor $F : \mathbf{A} \to \mathbf{B}$ of abelian categories is *effaceable* if for all $A \in \mathbf{A}$ there is a monomorphism $g : A \to A'$ such that $F(g) = 0$.

Lemma 1.13. *If $\mathbf{A}$ has enough injectives and $F(I) = 0$ for all injective objects $I \in \mathbf{A}$ then F is effaceable.*

Proof. Take a monomorphism $0 \to A \to I$ and apply F. $\square$

Definition (universal δ-functor)**.** A δ-functor $\{T^i, \delta_i\}$ is *universal* if for any other δ-functor $\{S^i, \sigma_i\}$ with a morphism $f^0 : T^0 \to S^0$, there are unique morphisms $f^i : T^i \to S^i$ such that

$$\begin{array}{ccc} T^i(A'') & \xrightarrow{\delta_i} & T^{i+1} \\ \downarrow & & \downarrow \\ S^i(A'') & \xrightarrow{\sigma_i} & S^{i+1}(A') \end{array}$$

commute for all exact sequences

$$0 \longrightarrow A' \longrightarrow A \longrightarrow A'' \longrightarrow 0$$

Theorem 1.14. *A δ-functor $\{T^i, \delta_i\}$ such that T^i is effaceable for all $i > 0$ is universal.*

It follows that unviersal δ-functors $\{T^i, \delta_i\}$ and $\{S^i, \sigma_i\}$ such that $T^0 \cong S^0$ are isomorphic via a unique isomorphism.

Corollary 1.15. *Let $F : \mathbf{A} \to \mathbf{B}$ be a left exact functor. Assume $\mathbf{A}$ has enough injectives. Then $\{R^iF, \delta_i\}$ is a universal δ-functor. In particular if $\{T^i, \sigma_i\}$ is a universal δ-functor such that $T^0 \cong F$ then $T^i \cong R^iF$.*

Proof. From the observation at the beginning of the section $\{R^iF, \delta_i\}$ is a δ-functor. It is also effaceable since $0 \to I \to I \to 0$ is an injective resolution. $\square$

Applications: on paracompact space Čech cohomology is the same as sheaf cohomology: proof by showing Čech cohomology is a universal δ-functor, and 0th Čech cohomology is exactly global sections.

2 Sheaves

2.1 Presheaves

If X is a topological space, form the category of open sets $\mathbf{Op}(X)$ of X. Its objects are open subsets of X and the morphisms are inclusions. A *presheaf* of abelian groups on X is a contravariant functor $\mathcal{P} : \mathbf{Op}(X) \to \mathbf{Ab}$. A *morphism of presheaves* is a morphism of functors.

Example.

1. Constant presheaf: fix an abelian group G and define the constant presheaf by $U \mapsto G$ and all morphisms to be id_G.
2. The sheaf of continuous functions.

For $x \in X$, the set of open neighbourhoods ordered by reverse inclusion is a directed set. Thus if $\mathcal{P}$ is a presheaf on X, we define the *stalk* of $\mathcal{P}$ at x to be

$$\mathcal{P}_x = \varinjlim_{U \ni x} \mathcal{P}(U).$$

Elements of $\mathcal{P}_x$ are called *germs* (of sections of $\mathcal{P}$ at x).

2.2 Sheaves

Definition (sheaf)**.** A *sheaf* $\mathcal{F}$ is a presheaf satisfying the sheaf axioms

1. if U is an open set, $\{V_i\}$ is an open cover of U and $s \in \mathcal{F}(U)$ is such that $s|_{V_i} = 0$ for all i then $s = 0$.
2. if U is an open set, $\{V_i\}$ is an open cover of U and $s_i \in \mathcal{F}(V_i)$ are such that $s_i|_{U_i \cap U_j} = s_j|_{U_i \cap U_j}$ for all i, j such that $U_i \cap U_j \neq \emptyset$ then exists $s \in \mathcal{P}(U)$ such that $s|_{V_i} = s_i$.

Remark. If both axioms hold then S1 implies that the s in S2 is unique.

Example.

1. The constant presheaf does not satisfy S2.
2. Let X be a differentiable manifold. Define a presheaf by $U \mapsto H^1_{\mathrm{dR}}(U)$. Choose U such that $H^1_{\mathrm{dR}}(U) \neq 0$. We can cover U by contractible V_i's. Thus we can find nonzero $\xi \in H^1_{\mathrm{dR}}(U)$ such that $\xi|_{V_i} = 0$ for all i. Thus it does not satisfy S1. Note the stalk at every point is 0. We will see its associated sheaf is 0.

Exercise. If $\mathcal{F}$ is a sheaf on X such that $\mathcal{F}_x = 0$ for all $x \in X$ then $\mathcal{F} = 0$.

2.3 Sheafification

Given a presheaf $\mathcal{P}$ on X, we construct the *étalé space* $\underline{\mathcal{P}} = \coprod_{x \in X} \mathcal{P}_x$ together with a natural map $\pi : \underline{\mathcal{P}} \to X$. If $U \subseteq X$ open and $s \in \mathcal{P}(U)$ then

$$\begin{aligned} \underline{s} : U &\to \underline{\mathcal{P}} \\ x &\mapsto s_x \end{aligned}$$

defines a section of π. Topologise $\underline{\mathcal{P}}$ by giving a basis $\{\underline{s}(U) : U \subseteq X \text{ open}, s \in \mathcal{P}(U)\}$. Then π is continuous (in fact this topology is the weakest topology for which π is continuous). We then define the *sheafification* of P, $P^\natural$, to be the presheaf

$$U \mapsto \{\text{continuous sections } U \to \underline{\mathcal{P}}\},$$

which is a sheaf since continuity is a local property.

The natural map $i : \mathcal{P} \to \mathcal{P}^\natural$ induces an isomorphism on all stalks.

Example.

1. The sheafification of the presheaf $U \mapsto H^1_{\mathrm{dR}}(U)$ in the previous section is the zero sheaf as all stalks are zero.

2. The sheafification of a constant preseheaf is the *constant sheaf*, whose sections are locally constant functions.

Proposition 2.1. *For a presheaf $\mathcal{P}$ on X, sheafification $i : \mathcal{P} \to \mathcal{P}^\natural$ has the following universal property: for any sheaf $\mathcal{F}$ on X, any morphism of presheaves $f : \mathcal{P} \to \mathcal{F}$ factorises uniquely through i.*

$$\begin{tikzcd}
\mathcal{P} \arrow[r, "f"] \arrow[d, "i"] & \mathcal{F} \\
\mathcal{P}^\natural \arrow[ur, dashed, "\exists!"] &
\end{tikzcd}$$

Thus given a morphism of presheaves $f : \mathcal{P} \to \mathcal{Q}$, there is a natural map $f^\natural : \mathcal{P}^\natural \to \mathcal{Q}^\natural$, so sheafification defines a functor.

2.4 Exact sequences of presheaves and sheaves

If $\mathbf{A}$ is an abelian category then for any category $\mathbf{B}$, $[\mathbf{B}, \mathbf{A}]$ is naturally an abelian category. One readily checks that a sequence of presheaves

$$0 \longrightarrow \mathcal{P}' \longrightarrow \mathcal{P} \longrightarrow \mathcal{P}'' \longrightarrow 0$$

is short exact if and only if

$$0 \longrightarrow \mathcal{P}'(U) \longrightarrow \mathcal{P}(U) \longrightarrow \mathcal{P}''(U) \longrightarrow 0$$

is short exact for all U.

Given a morphism of sheaves $f : \mathcal{F} \to \mathcal{G}$, the presheaf $\ker f$ is always a sheaf but this need not be the case for $\operatorname{coker} f$. For example let X be a complex manifold. Then

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathcal{O}_X \xrightarrow{\exp} \mathcal{O}_X^*$$

is not exact on the right as presheaves, as $\exp(f)(z) = \exp^{2\pi i f(z)}$ is not surjective on some open sets $U \subseteq X$. However every $x \in X$ has a neighbourhood U on which exp is surjective, so the sequence is exact on the stalks.

Definition (exact sequence of sheaves)**.** A sequence of sheaves

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is *exact* if it is exact on all stalks.

Example. Let X be a smooth manifold. Let Ω_X^p be the sheaf of p-forms on X. Then exterior derivative $\mathrm{d} : \Omega_X^p \to \Omega_X^{p+1}$ defines sheaf morphisms and gives rise to a complex

$$C^\infty \longrightarrow \Omega_X^1 \longrightarrow \cdots \longrightarrow \Omega_X^n \longrightarrow 0$$

This is called the *de Rham complex*. It is exact in positive degrees by Poincaré lemma. This gives a resolution of the constant sheaf $\mathbb{R}$. Note that as *presheaves*,

$$H^k(\Omega_X^\bullet)(U) = H_{\mathrm{dR}}^k(U).$$

2.5 Some constructions of sheaves

Definition (pushforward)**.** Let $f : X \to Y$ be a continuous map. We define functors

- $f_* : \mathbf{Sh}_X \to \mathbf{Sh}_Y$ by
$$f_*\mathcal{F} : U \mapsto \mathcal{F}(f^{-1}(U)).$$
  It is left exact.
- $f^{-1} : \mathbf{Sh}_Y \to \mathbf{Sh}_X$ by defiing $f^{-1}\mathcal{G}$ to be the sheafification of the presheaf
$$U \mapsto \varinjlim_{V \supseteq f(V)} \mathcal{G}(V).$$

It preserves stalks so it is exact.

Proposition 2.2. *f^{-1} is left adjoint to f_*, i.e. there are natural isomorphisms*

$$\mathbf{1}_{\mathbf{Sh}_Y} \to f_*f^{-1}, f^{-1}f_* \to \mathbf{1}_{\mathbf{Sh}_X},$$

or equivalently, there are natural isomorphisms

$$Hom_{\mathbf{Sh}_X}(f^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathbf{Sh}_Y}(\mathcal{G}, f_*\mathcal{F}).$$

2.6 Sheaf of modules

Definition (ringed space)**.** A *ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings on X.

A morphism of ringed spaces $(f, f^\#) : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ where $f : X \to Y$ is a continuous map and $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves of rings.

Definition (sheaf of module)**.** Let $(X, \mathcal{O}_X)$ be a sheaf of modules. An $\mathcal{O}_X$-*module* is a sheaf of abelian groups $\mathcal{M}$ on X such that each $\mathcal{M}(U)$ is an $\mathcal{O}_X(U)$-module and restrictions are compatible with module maps.

A morphism of sheaf of $\mathcal{O}_X$-modules $\phi : \mathcal{M} \to \mathcal{N}$ is a morphism of sheaves ϕ such that $\phi_U : \mathcal{M}(U) \to \mathcal{N}(U)$ is a morphism of $\mathcal{O}_X(U)$-modules compatible with restriction maps.

Proposition 2.3. $\mathbf{Mod}_{\mathcal{O}_X}$ *is an abelian category.*

Example. Consider the ringed space $(X, \mathcal{O}_X)$ where X is a complex manifold and $\mathcal{O}_X$ is the sheaf of holomorphic functions on X. Let $\pi : E \to X$ be a holomorphic vector bundle. Then we can define the sheaf $\mathcal{E}$ of sections of E by

$$U \mapsto \{s : U \to E : s \text{ holomorphic}, \pi \circ s = \mathrm{id}_U\}.$$

Then $\mathcal{E}$ is an $\mathcal{O}_X$-module.

Example. Given any topological space X, consider the ringed space $(X, \mathbb{Z}_X)$ where $\mathbb{Z}_X$ is the constant sheaef $\mathbb{Z}$. Then an $\mathbb{Z}_X$-module is exactly a sheaf on X.

Definition. Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. We define functors

- $f_* : \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_Y}$ by

$$f_*\mathcal{M} : U \mapsto \mathcal{M}(f^{-1}(U))$$

  where RHS is regarded as an $\mathcal{O}_Y(U)$-module via $\mathcal{O}_Y(U) \to f_*\mathcal{O}_X(U)$. It is left exact.

- $f^* : \mathbf{Mod}_{\mathcal{O}_Y} \to \mathbf{Mod}_{\mathcal{O}_X}$ by defining $f^*\mathcal{N}$ to be the sheafification of the presheaf

$$U \mapsto f^{-1}\mathcal{N}(U) \otimes_{(f^{-1}\mathcal{O}_Y)(U)} \mathcal{O}_X(U)$$

  where $f^{-1}\mathcal{O}_Y \to \mathcal{O}_X$ is obtained from $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ using adjunction. It is right exact. f is said to be *flat* if f^* is exact.

(? Claims f is flat if and only if $\mathcal{O}_{X,x}$ is a flat $\mathcal{O}_{Y,f(x)}$-module for all x. Check this)

Proposition 2.4. *Let $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ be a morphism of ringed spaces. Then f^* is left adjoint to f_*.*

Sketch proof. This follows from tensor-Hom adjunction: let R, S be rings, Y an R-module, Z an S-module and X and (R, S)-bimodule. Then there is a bijection

$$\mathrm{Hom}_S(Y \otimes_R X, Z) \cong \mathrm{Hom}_R(Y, \mathrm{Hom}_S(X, Z)).$$

Globalise to get

$$\begin{aligned}\mathrm{Hom}_X(f^*\mathcal{G},\mathcal{F}) &= \mathrm{Hom}_X(f^{-1}\mathcal{G}\otimes_{f^{-1}\mathcal{O}_Y}\mathcal{O}_X,\mathcal{F})\\ &= \mathrm{Hom}_Y(f^{-1}\mathcal{G},\mathrm{Hom}_X(\mathcal{O}_X,\mathcal{F}))\\ &= \mathrm{Hom}_Y(f^{-1}\mathcal{G},\mathcal{F})\\ &= \mathrm{Hom}_Y(\mathcal{G},f_*\mathcal{F})\end{aligned}$$

□

Let $(X,\mathcal{O}_X)$ be a ringed space. For $\mathcal{O}_X$-sheaves $\mathcal{M}$ and $\mathcal{N}$, we define a preshaef $\mathcal{H}om_X(\mathcal{M},\mathcal{N})$ by

$$U\mapsto \mathrm{Hom}_U(\mathcal{M}|_U,\mathcal{N}|_U).$$

This is a sheaf of $\mathcal{O}_X$-module.

Exercise. Let $\mathcal{M}$ be an $\mathcal{O}_X$-module. Show

$$\begin{aligned}\mathrm{Hom}_X(\mathcal{M},-):\mathbf{Mod}_{\mathcal{O}_X}\to\mathbf{Ab}\\ \mathcal{H}om_X(\mathcal{M},-):\mathbf{Mod}_{\mathcal{O}_X}\to\mathbf{Mod}_{\mathcal{O}_X}\end{aligned}$$

are both left exact $\mathrm{Hom}_X(\mathcal{M},-)=\Gamma(X,-)\circ\mathcal{H}om_X(\mathcal{M},-)$. Upon showing the category of sheaves has enough injectives, we can then define right derived functors

$$\begin{aligned}\mathrm{Ext}^i_X(\mathcal{M},-)&=R^i\,\mathrm{Hom}_X(\mathcal{M},-)\\ \mathcal{E}xt^i_X(\mathcal{M},-)&=R^i\mathcal{H}om_X(\mathcal{M},-)\end{aligned}$$

The two are thus related by a spectral sequence.

Let $\mathcal{M},\mathcal{N}$ be $\mathcal{O}_X$-modules, We define $\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{N}$ to be the sheaf of $\mathcal{O}_X$-module obtained by sheafifying

$$U\mapsto\mathcal{M}(U)\otimes_{\mathcal{O}_X(U)}\mathcal{N}(U).$$

The functor $-\otimes_{\mathcal{O}_X}\mathcal{M}:\mathbf{Mod}_{\mathcal{O}_X}\to\mathbf{Mod}_{\mathcal{O}_X}$ is right exact.

Proposition 2.5. *We have*

$$(\mathcal{M}\otimes_{\mathcal{O}_X}\mathcal{N})_x\cong\mathcal{M}_x\otimes_{\mathcal{O}_{X,x}}\mathcal{N}_x$$

for all $x\in X$.

3 Čech cohomology

Let X be a topological space and $\mathcal{P}$ a presheaf on X. Let $\mathcal{U} = \{U_i\}$ be an open cover of X where I is totally ordered. Suppose $i_0 < \cdots < i_p$, we use the notation

$$U_{i_0 \cdots i_p} = U_{i_0} \cap \cdots \cap U_{i_p}.$$

For $p \geq 0$, we define the *Čech complex* $(C^\bullet(\mathcal{U}, \mathcal{P}), \delta)$ where

$$C^p(\mathcal{U}, \mathcal{P}) = \prod_{i_0 < \cdots < i_p} \mathcal{P}(U_{i_0 \cdots i_p})$$

and $\delta : C^p(\mathcal{U}, \mathcal{P}) \to C^{p+1}(\mathcal{U}, \mathcal{P})$ is given by

$$(\delta\alpha)_{i_0 \cdots i_{p+1}} = \sum_{k=0}^{p+1} (-1)^k \alpha_{i_0 \cdots \hat{i}_k \cdots i_{p+1}}|_{U_{i_0 \cdots i_{p+1}}}.$$

One can check that $\delta^2 = 0$. We then define *Čech cohomology* of $\mathcal{U}$ with coefficients in $\mathcal{P}$ to be

$$H^k(\mathcal{U}, \mathcal{P}) = H^k((C^\bullet(\mathcal{U}, \mathcal{P}), \delta)).$$

Proposition 3.1. *If $\mathcal{F}$ is a sheaf then $H^0(\mathcal{U}, \mathcal{F}) \cong \mathcal{F}(X)$.*

Proof. Suppose $\alpha \in H^0(\mathcal{U}, \mathcal{F}) = \ker(\delta : C^0(\mathcal{U}, \mathcal{F}) \to C^1(\mathcal{U}, \mathcal{F}))$. Then $\alpha_i|_{U_{ij}} = \alpha_j|_{U_{ij}}$ for all i, j so exists a unique $\tilde{\alpha} \in \mathcal{F}(X)$ such that $\tilde{\alpha}|_{U_i} = \alpha_i$. This defines a map $H^0(\mathcal{U}, \mathcal{F}) \to \mathcal{F}(X)$. It has an obvious inverse. $\square$

Exercise. Compute $H^*(\mathcal{U}, \mathbb{R})$ for $X = S^1$ where $\mathcal{U}$ consists of three arcs that intersect pairwise but has empty triple intersection. The result is isomorphic to $H^*_{\mathrm{dR}}(X)$.

Suppose

$$0 \longrightarrow \mathcal{P}' \longrightarrow \mathcal{P} \longrightarrow \mathcal{P}'' \longrightarrow 0$$

is a short exact sequence of presheaves on X. Then for all coverings $\mathcal{U}$ we have a short exact sequence of complexes

$$0 \longrightarrow C^\bullet(\mathcal{U}, \mathcal{P}') \longrightarrow C^\bullet(\mathcal{U}, \mathcal{P}) \longrightarrow C^\bullet(\mathcal{U}, \mathcal{P}'') \longrightarrow 0$$

which gives rise to a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & H^0(\mathcal{U}, \mathcal{P}') \arrow[r] & H^0(\mathcal{U}, \mathcal{P}) \arrow[r] & H^0(\mathcal{U}, \mathcal{P}'') \arrow[dll] \\ & H^1(\mathcal{U}, \mathcal{P}') \arrow[r] & H^1(\mathcal{U}, \mathcal{P}) \arrow[r] & \cdots \end{tikzcd}$$

We would like to remove the dependency on the choice of open covering. Partially order the open covers of X by refinement so they form a filtered category. Then define

$$\check{H}^i(X, \mathcal{P}) = \varinjlim_{\mathcal{U}} H^i(\mathcal{U}, \mathcal{P}).$$

Exercise. A presheaf is *separated* if it satisfies sheaf axiom 1. Show that if $\mathcal{P}$ is a separated presheaf then $\check{H}^0(X,\mathcal{P}) = \mathcal{P}^\natural(X)$.

As filtered colimit commutes with finite limits (and certainly colimits) in **Ab**, taking direct limit is exact so we have a long exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \check{H}^0(X,\mathcal{P}') \arrow[r] & \check{H}^0(X,\mathcal{P}) \arrow[r] & \check{H}^0(X,\mathcal{P}'') \arrow[dll, out=-5, in=175] \\ & \check{H}^1(X,\mathcal{P}') \arrow[r] & \check{H}^1(X,\mathcal{P}) \arrow[r] & \cdots \end{tikzcd}$$

3.1 Čech cohomology for paracompact spaces

Lemma 3.2. *Let X be a paracompact topological space and $\mathcal{P}$ a presheaf on X such that $P^\natural = 0$. Let $\mathcal{U}$ be an open cover of X and $\alpha \in C^k(\mathcal{U},\mathcal{P})$. Then there is a refinement $\mathcal{V}$ of $\mathcal{U}$ such that $\tau(\alpha) = 0$ where $\tau : C^\bullet(\mathcal{U},\mathcal{P}) \to C^\bullet(\mathcal{V},\mathcal{P})$.*

Proof. Hirzebruch, Topological methods in algebraic geometry, lemma 2.9.2. □

For any presheaf $\mathcal{P}$, the morphism $\mathcal{P} \to \mathcal{P}^\natural$ induces a morphism

$$\check{H}^\bullet(X,\mathcal{P}) \to \check{H}^\bullet(X,\mathcal{P}^\natural). \tag{$*$}$$

Proposition 3.3. *Let $\mathcal{P}$ be a sheaf on a paracompact space X, then $(*)$ is an isomorphism.*

Proof. Let Q_1, Q_2 be the kernel and cokernel of the presheaf morphism $P \to P^\natural$. Then $Q_1^\natural = Q_2^\natural = 0$ so $\check{H}^\bullet(X,Q_1) = \check{H}^\bullet(X,Q_2) = 0$. We have two short exact sequences

$$0 \longrightarrow \mathcal{Q}_1 \longrightarrow \mathcal{P} \longrightarrow \mathcal{Z} \longrightarrow 0$$

$$0 \longrightarrow \mathcal{Z} \longrightarrow \mathcal{P}^\natural \longrightarrow \mathcal{Q}_2 \longrightarrow 0$$

By considering the long exact seqeuence of cohomology we get

$$\check{H}^\bullet(X,\mathcal{P}) \cong \check{H}^\bullet(X,\mathcal{Z}) \cong \check{H}^\bullet(X,\mathcal{P}^\natural).$$

□

Corollary 3.4. *Let X be a paracompact space. Then a short exact sequence of sheaves*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

induces a long exact sequence of Čech cohomology groups

$$\begin{tikzcd} 0 \arrow[r] & \check{H}^0(X,\mathcal{F}') \arrow[r] & \check{H}^0(X,\mathcal{F}) \arrow[r] & \check{H}^0(X,\mathcal{F}'') \arrow[dll, out=0, in=180] \\ & \check{H}^1(X,\mathcal{F}') \arrow[r] & \check{H}^1(X,\mathcal{F}) \arrow[r] & \cdots \end{tikzcd}$$

Proof. Let Q be the presheaf cokernel of $\mathcal{F}' \to \mathcal{F}$ so $Q^\natural \cong \mathcal{F}''$. Apply $\check{H}^\bullet(X,Q) \cong \check{H}^\bullet(X,\mathcal{F}'')$ to the long exact sequence of cohomology groups of presheaves. □

Theorem 3.5. *If X is paracompact then $\check{H}^i(X,-) : \mathbf{Sh}_X \to \mathbf{Ab}$ together with the connecting morphisms previously defined is a δ-functor.*

Proof. We need to prove that for any commutative diagrams with exact rows

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F}' \arrow[r] \arrow[d] & \mathcal{F} \arrow[r] \arrow[d] & \mathcal{F}'' \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & \mathcal{G}' \arrow[r] & \mathcal{G} \arrow[r] & \mathcal{G}'' \arrow[r] & 0 \end{tikzcd}$$

we have for every $i \geq 0$ commutative diagrams

$$\begin{tikzcd} \check{H}^i(X,\mathcal{F}'') \arrow[r, "\delta_i"] \arrow[d] & \check{H}^{i+1}(X,\mathcal{F}') \arrow[d] \\ \check{H}^i(X,\mathcal{G}'') \arrow[r, "\delta_i"] & \check{H}^{i+1}(X,\mathcal{G}') \end{tikzcd}$$

Again we replace $\mathcal{F}''$ and $\mathcal{G}''$ by the respectively presheaf cokernels $\mathcal{P}$ and $\mathcal{Q}$. Then we have a morphism of long exact sequences of Čech cohomology groups. In particular

$$\begin{tikzcd} \check{H}^i(X,\mathcal{F}'') \arrow[r, "\cong"] & \check{H}^i(X,\mathcal{P}) \arrow[r] \arrow[d] & \check{H}^{i+1}(X,\mathcal{F}') \arrow[d] \\ \check{H}^i(X,\mathcal{G}'') \arrow[r, "\cong"] & \check{H}^i(X,\mathcal{Q}) \arrow[r] & \check{H}^{i+1}(X,\mathcal{G}') \end{tikzcd}$$

□

Definition (skyscraper sheaf)**.** Let X be a topological space, $x \in X$ and G an abelian group. The *skyscraper sheaf* of G at x is

$$G(x) : U \mapsto \begin{cases} G & x \in U \\ 0 & x \notin U \end{cases}$$

Exercise. $G(x)_y = 0$ if $y \notin \overline{\{x\}}$.

Proposition 3.6. $\mathbf{Mod}_{\mathcal{O}_X}$ *has enough injectives.*

Proof. Let $\mathcal{F}$ be an $\mathcal{O}_X$-module. Then $\mathcal{F}_x$ is an $\mathcal{O}_{X,x}$-module so we can find an injective $\mathcal{O}_{X,x}$-module I_x such that $\mathcal{F}_x \hookrightarrow I_x$. For $x \in X$, let $j_x : \{x\} \to X$. Define

$$\mathcal{I} = \prod_{x \in X} j_{x,*} I_x.$$

By adjunction we have

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, j_{x,*} I_x) \cong \mathrm{Hom}_{\mathcal{O}_{X,x}}(\mathcal{F}_x, I_x)$$

so we obtain an injective morphism $g : \mathcal{F} \hookrightarrow \mathcal{I}$.

Now we show $\mathcal{I}$ is an injective $\mathcal{O}_X$-module. Let $S_x : \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_{X,x}}, \mathcal{F} \mapsto \mathcal{F}_x$ which is exact. Then

$$\mathrm{Hom}_{\mathcal{O}_X}(\mathcal{F}, \mathcal{I}) = \prod_{x \in X} \mathrm{Hom}_{\mathcal{O}_{X,x}}(S_x(\mathcal{F}), I_x).$$

As

$$Hom_{\mathcal{O}_X}(-, \mathcal{I}) = \prod_{x \in X} \mathrm{Hom}_{\mathcal{O}_{X,x}}(-, I_x) \circ S_x$$

is a composition of exact functors it is also exact. $\square$

In particular if we equip a topological space X with the constant structure sheaf Z_X then $\mathbf{Sh}_X$ has enough injectives.

Note that in general, given a ringed space $(X, \mathcal{O}_X)$, the categories $\mathbf{Mod}_{\mathcal{O}_X}$ and $\mathbf{Sh}_X$ have different injective objects.

$$\begin{tikzcd}
\mathbf{Mod}_{\mathcal{O}_X} \arrow[r, "\Gamma"] \arrow[d] & \Gamma(X, \mathcal{O}_X)\text{-}\mathbf{Mod} \arrow[d] \\
\mathbf{Sh}_X \arrow[r, "\Gamma"] & \mathbf{Ab}
\end{tikzcd}$$

We look for Γ-acyclic sheaves whose definition is the same in both, which will be flasque sheaves. After that we show any injective $\mathcal{O}_X$-module is flasque and hence acyclic so the right derived functors of these two coincide.

Definition (flasque sheaf)**.** A sheaf $\mathcal{F}$ is *flasque* if for all $U \subseteq X$ open, $\mathcal{F}(X) \to \mathcal{F}(U)$ is surjective.

Do flasque sheaves exist in general? Recall that $\mathcal{F} \cong \mathcal{F}^\natural$, which is defined as the sheaf of continuous sections of its étalé space $\underline{\mathcal{F}}$. We define the *flasque envelope* of $\mathcal{F}$ to be the sheaf $G_0(\mathcal{F})$ of *all* sections of $\mathcal{F}$, which is flasque. One then obtains the *Godemont resolution* of $\mathcal{F}$ as follow. Let $\mathcal{Q}_0 = G_0(\mathcal{F})/\mathcal{F}$. Take the flasque envelope of $\mathcal{Q}_0$ and call it $G_1(\mathcal{F})$. Then there is a morphism

$G_0(\mathcal{F}) \to G_1(\mathcal{F})$. Continuing this way one get a flasque resolution of $\mathcal{F}$.

$$\begin{tikzcd}
 & & & 0 \arrow[d] & & \\
0 \arrow[r] & \mathcal{F} \arrow[r] & G_0(\mathcal{F}) \arrow[r] \arrow[dr] & \mathcal{Q}_0 \arrow[r] \arrow[d] & 0 & \\
 & & & G_0(\mathcal{Q}_0) = G_1(\mathcal{F}) \arrow[d] \arrow[dr] & & \\
 & & 0 \arrow[r] & \mathcal{Q}_1 \arrow[r] & G_0(\mathcal{Q}_1) = G_2(\mathcal{F}) \arrow[dr] & \\
 & & & & & \ddots
\end{tikzcd}$$

We would like to show that flasque sheaves are Γ-acyclic

Lemma 3.7. *If*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is an exact sequence of sheaves and $\mathcal{F}'$ is flasque then

$$0 \longrightarrow \mathcal{F}'(U) \longrightarrow \mathcal{F}(U) \longrightarrow \mathcal{F}''(U) \longrightarrow 0$$

is exact for every open U.

Proof. Only need to show the last map is surjective. Suppose $s \in \mathcal{F}''(U)$. Consider the collection

$$\{(V,t) : t \in \mathcal{F}(V), g(t) = s|_V\}$$

partially ordered by $(V,t) \leq (V',t')$ if $V \subseteq V'$ and $t'|_V = t$. As $\mathcal{F}_x \to \mathcal{F}''_x$ is surjective for any $x \in U$ it is nonempty and by Zorn's lemma there is a maximal element (V,t). Now we show $V = U$. Suppose not, then let $x \in U \supsetneq V$. By surjectivity on stalks can find W a neighbourhood of x and $t' \in \mathcal{F}(W)$ such that $g(t') = s|_W$. Then $t|_{W\cap V} - t'|_{W\cap V} \in \ker g$ so it can be written as $f(r)$ for some $r \in \mathcal{F}'(W \cap V)$. Since $\mathcal{F}'$ is flasque, r extends to a $\tilde{r} \in \mathcal{F}'(W)$. Then t and $f(\tilde{r}) + t'$ glue to a section of $\mathcal{F}(W \cup V)$, contradicting maximality of (V,t). $\square$

Lemma 3.8. *A quotient of flasque sheaves is flasque.*

Proof. For any $U \subseteq X$ we have a commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{F}'(X) \arrow[r] \arrow[d] & \mathcal{F}(X) \arrow[r] \arrow[d, two heads] & \mathcal{F}''(X) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{F}'(U) \arrow[r] & \mathcal{F}(U) \arrow[r] & \mathcal{F}''(U) \arrow[r] & 0
\end{tikzcd}$$

The composition $\mathcal{F}(X) \to \mathcal{F}(U) \to \mathcal{F}''(U)$ is surjective and hence $\mathcal{F}''(X) \to \mathcal{F}''(U)$ is surjective too. $\square$

Proposition 3.9. *Every injective $\mathcal{O}_X$-module is flasque.*

We introduce a construction that will be used in the proof. Suppose $U \subseteq X$ open, $\mathcal{F}$ an $\mathcal{O}_U$-module. Let $j : U \hookrightarrow X$. We define *extension by zero* $j_!\mathcal{F}$ an $\mathcal{O}_X$-module by

$$V \mapsto \begin{cases} \mathcal{F}(V) & V \subseteq U \\ 0 & V \nsubseteq U \end{cases}$$

Proof. Let $\mathcal{O}_{(U)} = j_!\mathcal{O}_U$ as an $\mathcal{O}_X$-module. Suppose $\mathcal{I}$ is an injective $\mathcal{O}_X$-module and suppose $V \subseteq U$. Then we have an injection $\mathcal{O}_{(V)} \hookrightarrow \mathcal{O}_{(U)}$. Apply $\mathrm{Hom}_{\mathcal{O}_X}(-, \mathcal{I})$, we get

$$\mathcal{I}(U) = \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_{(U)}, \mathcal{I}) \twoheadrightarrow \mathrm{Hom}_{\mathcal{O}_X}(\mathcal{O}_{(V)}, \mathcal{I}) = \mathcal{I}(V).$$

$\square$

Remark. In particular injective sheaves are flasque.

Theorem 3.10. *Flasque sheaves are acyclic for sheaf cohomology.*

Proof. Suppose $\mathcal{F}$ is a flasque $\mathcal{O}_X$-module. Inject $\mathcal{F}$ into an injective $\mathcal{O}_X$-module $\mathcal{I}$ and let $\mathcal{Q} = \mathcal{I}/\mathcal{F}$ which is flasque. Taking sheaf cohomology, we get an exact sequence

$$0 \longrightarrow H^0(X, \mathcal{F}) \longrightarrow H^0(X, \mathcal{I}) \longrightarrow H^0(X, \mathcal{Q}) \longrightarrow H^1(X, \mathcal{F}) \longrightarrow 0$$

and $H^i(X, \mathcal{Q}) \cong H^{i+1}(X, \mathcal{F})$ for $i \geq 1$. As $\mathcal{F}, \mathcal{I}$ and $\mathcal{Q}$ are flasque the first three terms are already exact so $H^1(X, \mathcal{F}) = 0$. This holds for all flasque sheaves $\mathcal{F}$ so inductively $H^i(X, \mathcal{F}) = 0$ for all $i > 0$. $\square$

Corollary 3.11. *If $\mathcal{F}$ is any $\mathcal{O}_X$-module and*

$$0 \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^0 \longrightarrow \mathcal{F}^1 \longrightarrow \cdots$$

is a flasque resolution then

$$H^i(\Gamma(X, \mathcal{F}^\bullet)) \cong H^i(X, \mathcal{F})$$

for all $i \geq 0$.

In particular we can take any Godemant resolution $G^\bullet(\mathcal{F})$.

3.2 Higher direct image

Suppose $f : (X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is a morphism of ringed spaces. We can take the right derived functors of $f_* : \mathbf{Mod}_{\mathcal{O}_X} \to \mathbf{Mod}_{\mathcal{O}_Y}$, called *higher direct images* and denoted $R^i f_*$. Similarly we can consider the higher direct image between categories of sheaves.

Suppose $\mathcal{F}$ is an $\mathcal{O}_X$-module. Then we define a sheaf of $\mathcal{O}_Y$-modules $S^i_f(\mathcal{F})$ on Y by the associated sheaf of

$$U \mapsto H^i(f^{-1}(U), \mathcal{F}).$$

Exercise.

1. $\{S^i_f\}$ defines a δ-functor.
2. This δ-functor is universal (this follows immediately from effaceability).
3. Show $S^0_f(\mathcal{F}) = f_*\mathcal{F}$ so $S^i_f \cong R^i f_*$.

3.3 Comparison theorem of Čech cohomology

Let $\mathcal{U}$ be an open cover of X and $\mathcal{F}$ a sheaf on X. We construct morphisms $H^i(\mathcal{U},\mathcal{F}) \to H^i(X,\mathcal{F})$. The strategy is to promote Čech cochain to a complex of sheaves, and then lift the identity on the global sections to a morphism to injective resolution.

Let $j_{i_0\cdots i_p} : U_{i_0\cdots,i_p} \hookrightarrow X$. We define a sheaf

$$\check{\mathcal{C}}^p(\mathcal{U},\mathcal{F}) = \prod_{i_0<\cdots<i_p} (j_{i_0\cdots i_p})_*\mathcal{F}|_{U_{i_0\cdots i_p}}$$

and define $\delta : \check{\mathcal{C}}^p(\mathcal{U},\mathcal{F}) \to \check{\mathcal{C}}^{p+1}(\mathcal{U},\mathcal{F})$ in the usual way. We then get the Čech sheaf complex $(\check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F}),\delta)$.

Proposition 3.12.

1. *For $V \subseteq X$ open we have*

$$\check{\mathcal{C}}^p(\mathcal{U},\mathcal{F})(V) = \prod_{i_0<\cdots<i_p} \mathcal{F}(V \cap U_{i_0\cdots i_p})$$

and in particular $\Gamma(X, \check{\mathcal{C}}^p(\mathcal{U},\mathcal{F})) = C^p(\mathcal{U},\mathcal{F})$.

2. *There is a a sheaf morphism $\epsilon : \mathcal{F} \to \check{\mathcal{C}}^0(\mathcal{U},\mathcal{F})$ defined by $\mathcal{F}(V) \ni t \mapsto (t|_{V\cap U_{i_0}})$.*
3. *We have a resolution of $\mathcal{F}$*

$$0 \longrightarrow \mathcal{F} \xrightarrow{\epsilon} \check{\mathcal{C}}^0(\mathcal{U},\mathcal{F}) \xrightarrow{\delta_0} \check{\mathcal{C}}^1(\mathcal{U},\mathcal{F}) \longrightarrow \cdots$$

Proof. ϵ is injective by the first sheaf axiom. $\ker\delta_0 = \operatorname{im}\epsilon \cong \mathcal{F}$ by the second sheaf axiom. To show the complex is exact in positive degree if suffice to check exactness on stalks by exhibiting a null homotopy. Define $K : \check{\mathcal{C}}^p(\mathcal{U},\mathcal{F})_x \to \check{\mathcal{C}}^{p-1}(\mathcal{U},\mathcal{F})_x$ as follows. Suppose $s_x \in \check{\mathcal{C}}^p(\mathcal{U},\mathcal{F})$ is represented by (V,s) where wlog $V \subseteq U_j$ for some j. Then define

$$(Ks_x)_{i_0,\dots,i_{p-1}} = (-1)^\sigma s_{i_0\cdots j\cdots i_{p-1}}$$

where σ is the sign of the permutation $(j, i_0, \dots, i_{p-1}) \mapsto (i_0,\dots,j,i_{p-1})$. $\square$

Now take an injective resolution $0 \to \mathcal{F} \to \mathcal{I}^\bullet$ and we can extend identity to a morphism of resolutions

$$\begin{tikzcd} 0 \arrow[r] & \mathcal{F} \arrow[r] \arrow[d, equal] & \check{\mathcal{C}}^\bullet(\mathcal{U},\mathcal{F}) \arrow[d] \\ 0 \arrow[r] & \mathcal{F} \arrow[r] & \mathcal{I}^\bullet \end{tikzcd}$$

Taking global sections, taking cohomology and passing to direct limit we thus get $\check{H}^\bullet(X,\mathcal{F}) \to H^\bullet(X,\mathcal{F})$.

In general this morphism is not an isomorphism. But we can say in general

- in degree 0 both groups are $\Gamma(X,\mathcal{F})$, so an isomorphism,
- in degree 1 it is always an isomorphism,
- in degree 2 it is always an injection.

Example (from Tohoku paper)**.** Let X be an irreducible topological space. Let Y_1, Y_2 be two irreducible closed subsets such that $Y_1 \cap Y_2 = \{p,q\}$. Take the constant sheaf k_Y on $Y_1 \cup Y_2$. Then $\check{H}^2(Y,k_Y) \neq H^2(Y,k_Y)$.

What we want to show is

1. if X is paracompact then for all $\mathcal{F}$, $\check{H}^\bullet(X,\mathcal{F}) \cong H^\bullet(X,\mathcal{F})$.
2. if X is a noetherian separated scheme and $\mathcal{F}$ a quasicoherent $\mathcal{O}_X$-module then for $\mathcal{U}$ an affine open cover $H^\bullet(\mathcal{U},\mathcal{F}) \cong H^\bullet(X,\mathcal{F})$.

The second can be be used to show $\mathrm{Pic}(X) \cong H^1(X,\mathcal{O}_X^*)$.

Lemma 3.13. *If $\mathcal{F}$ is flasque and $\mathcal{U}$ is an open cover of X then $H^i(\mathcal{U},\mathcal{F}) = 0$ for $i > 0$ and so $\check{H}^i(X,\mathcal{F}) = 0$.*

Proof. If $\mathcal{F}$ is flasque then the sheaves $\check{\mathcal{C}}^p(\mathcal{U},\mathcal{F})$ are flasque. □

Theorem 3.14. *When X is paracompact then δ-functor $\check{H}^i(X,-) : \mathbf{Sh}_X \to \mathbf{Ab}$ is universal.*

Proof. Injective sheaves are flasque so $\check{H}^i(X,\mathcal{I}) = 0$ for all $\mathcal{I}$ injective and all $i > 0$ so $\check{H}^i(X,-)$ is effaceable. □

Corollary 3.15. *When X is paracompact the natural morphisms $\check{H}^i(X,-) \to H^i(X,-)$ are isomorphisms.*

Lemma 3.16. *If X is a noetherian affine scheme and*

$$0 \longrightarrow \mathcal{F}' \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}'' \longrightarrow 0$$

is an exact sequence of $\mathcal{O}_X$-modules and $\mathcal{F}'$ is quasicoherent then the sequence is exact also as a sequence of presheaves.

3.4 Leray's theorem

Let X be a topological space and $\mathcal{F}$ a sheaf on X. Then the theorem roughly says that $H^i(\mathcal{U},\mathcal{F}) \cong H^i(X,\mathcal{F})$ for all $i \geq 0$ whenever $\mathcal{U}$ is “fine enough” to ensure that $\mathcal{F}$ has no cohomology on the intersections $U_{i_0 \cdots i_p}$.

Theorem 3.17. *Assume there is an integer n such that $H^i(U_{i_0\cdots i_p}, \mathcal{F}) = 0$ for all $1 \leq i \leq n$ for all intersections $U_{i_0\cdots i_p}$. Then $H^i(\mathcal{U}, \mathcal{F}) \cong H^i(X, \mathcal{F})$ for all $1 \leq i \leq n$.*

Since for a separated scheme intersections of affines are affine and quasicohernet sheaves on noetherian affine schemes are acyclic, the second comparison theorem thus follows.

3.5 Comparison with de Rham cohomology

Theorem 3.18. *If X is a differentiable manifold then*

$$H^i(X, \mathbb{R}) \cong H^i_{\mathrm{dR}}(X)$$

for all $i \geq 0$.

Definition (support of a sheaf, support of a section)**.** If $\mathcal{F}$ is a sheaf on X then its *support* is

$$\mathrm{supp}(\mathcal{F}) = \{x \in X : \mathcal{F}_x \neq 0\}.$$

If $s \in \mathcal{F}(U)$ then its *support* is

$$\mathrm{supp}(s) = \{x \in U : s_x \neq 0\}.$$

Exercise.

1. Show $\mathrm{supp}(s)$ is closed.
2. Find an example where $\mathrm{supp}(\mathcal{F})$ is not closed (hint: consider $j_!\mathcal{F}$).

Definition (fine sheaf)**.** A sheaf of rings $\mathcal{F}$ on a topological space X is *fine* if it admits a partition of unity subordinated to any locally finite open cover, i.e. if $\{U_i\}_{i\in I}$ is a locally finite cover then there exists a family $\{\rho_i \in \mathcal{F}(X)\}$ such that

1. $\mathrm{supp}(\rho_i) \subseteq U_i$ for all i,
2. $\sum_{i\in I} \rho_i = 1$.

Theorem 3.19. *If $\mathcal{F}$ is a fine sheaf of rings on a paracompact topological space X and $\mathcal{M}$ is an $\mathcal{F}$-module then $H^i(\mathcal{U}, \mathcal{M}) = 0$ for $i > 0$ when $\mathcal{U}$ is a locally finite open cover. Then $H^i(X, \mathcal{M}) = 0$.*

In other words on a paracompact space, sheaves of module over a fine sheaf of rings are acyclic.

Proof. The second statement follows from the first since paracompactness implies that locally finite covers are cofinal in all covers. For the first we show

$C^\bullet(\mathcal{U}, \mathcal{M})$ is acyclic by constructing a null homotopy. Suppose $\alpha \in C^p(\mathcal{U}, \mathcal{M})$, we set

$$(K\alpha)_{i_0\cdots i_{p-1}} = \sum_{k=0}^{p} (-1)^k \sum_{i_{k-1}<j<i_k} \rho_j \alpha_{i_0\cdots i_{k-1} j i_{k+1}\cdots i_{i-1}}.$$

□

Recall the de Rham complex

$$0 \longrightarrow \mathbb{R} \longrightarrow \Omega_X^0 \longrightarrow \Omega_X^1 \longrightarrow \cdots \longrightarrow \Omega_X^n \longrightarrow 0$$

which, by the result we have just shown, is an acyclic resolution of $\mathbb{R}$ so it follows that

$$H^i(X, \mathbb{R}) \cong H^i((\Omega^\bullet(X), \mathrm{d})) = H^i_{\mathrm{dR}}(X).$$

There is a parallel result in complex geometry using the *Dolbeault complex*. Let X be a complex manifold. Then there is a sheaf morphism $\overline{\partial} : \Omega_X^{p,q} \to \Omega_X^{p,q+1}$ such that $\overline{\partial}^2 = 0$. For a fixed $p \geq 0$, the $\overline{\partial}$-Poincaré lemma says that

$$0 \longrightarrow \Omega_X^p \longrightarrow \Omega_X^{p,0} \xrightarrow{\overline{\partial}} \Omega_X^{p,1} \xrightarrow{\overline{\partial}} \Omega^{p,2} \longrightarrow \cdots$$

is exact, where Ω_X^p is the sheaf of holomorphic p-forms. More over each $\Omega_X^{p,0}$ is a C_X^∞-module so acyclic. Thus

$$H^q(X, \Omega_X^p) \cong H^q((\Omega^{p,\bullet}(X), \overline{\partial})) = H^{p,q}_{\overline{\partial}}(X),$$

where RHS is the *Dolbeault cohomology*.

3.6 Cohomology of a good cover

Definition (good cover)**.** Let X be a differentiable manifold. An open cover $\{U_i\}$ of X is *good* if all nonempty intersections $U_{i_0\cdots i_p}$ are homeomorphic to $\mathbb{R}^n$ (where $n = \dim X$).

All differentiable manifold admits a good cover. To see this one picks a Riemannian metric on X and takes geodesic balls of sufficiently small radius For details see volume II of Kobayashi-Narimen.

For a good cover $\{U_i\}$,

$$H^k(U_{i_0\ldots i_p}, \mathbb{R}) = H^k_{\mathrm{dR}}(U_{i_0\cdots i_p}) = 0$$

for $k > 0$ so

$$H^k(\mathcal{U}, \mathbb{R}) \cong H^k(X, \mathbb{R}) \cong H^k_{\mathrm{dR}}(X)$$

for all $k \geq 0$ by Leray's theorem. Note that since $U_{i_0\ldots i_p}$ is connected, the information associated to this cover is essentially combinatorial.

4 Spectral sequences

We consider a collection of complexes $\{(E_k, d_k)\}_{k\in\mathbb{Z}}$ where E_{k+1} is the cohomology of (E_k, d_k). When this goes well the sequence stabilises so $E_k \cong E_{k+1}$ for all $k \geq k_0$.

For simplicity we work in the category of R-module, but the discussion applies to all abelian categories with all coproducts.

4.1 Filtered complex

Suppose $\{K^n\}_{n\in\mathbb{N}}$ is a complex with differential d. Assume there is a filtration in subcomplexes

$$K = K_0 \supseteq K_1 \supseteq K_2 \supseteq \cdots,$$

meaning that $d(K_p) \subseteq K_p$. Write $K_p^n = K_p \cap K^n$. Consider the associated graded module $\mathrm{Gr}(K) = \bigoplus_{p\in\mathbb{Z}} K_p/K_{p+1}$ (we set $K_p = K$ for $p \leq 0$).

A double complex gives a natural filtered complex. What we consider below is called first quadrant double complex. Consider $\{K^{p,q}\}_{p,q\in\mathbb{N}}$ with two differentials $\delta_1 : K^{p,q} \to K^{p+1,q}, \delta_2 : K^{p,q} \to K^{p,q+1}$ which anticommute, i.e. $\delta_1\delta_2 + \delta_2\delta_1 = 0, \delta_1^2 = \delta_2^2 = 0$. Consider the *total complex* associated to the double complex

$$(T^n = \bigoplus_{p+q=n} K^{p,q}, d = \delta_1 + \delta_2).$$

Example (Čech-de Rham double complex)**.** Let X be a differentiable manifold and $\mathcal{U}$ an open cover. Let $K^{p,q} = C^p(\mathcal{U}, \Omega_X^q)$. Let $\delta : K^{p,q} \to K^{p+1,q}, d : K^{p,q} \to K^{p,q+1}$. They commute so we set $\delta_1 = \delta, \delta_2 = (-1)^p d$.

The total complex of a double complex has two natural filtrations: the filtration by columns is given by $T_p = \bigoplus_{\substack{n\geq p\\ q\geq 0}} K^{n,q}$. Then $\mathrm{Gr}(T)_p = \bigoplus_{q\geq 0} K^{p,q}$. Analogously we have filtration by rows $T'_q = \bigoplus_{\substack{p\geq 0\\ n\geq q}} K^{p,n}$ and $\mathrm{Gr}(T')_q = \bigoplus_{p\geq 0} K^{p,q}$.

> **Definition** (regular filtration)**.** A filtration $K_\bullet$ of a complex $(K^\bullet, d)$ is *regular* if for every n the filtration
>
> $$K^n = K_0^n \supseteq K_1^n \supseteq K_2^n \supseteq \cdots$$
>
> is finite, i.e. there is a number $\ell(n)$ such that $K_p^n = 0$ for $p > \ell(n)$.

Both filtration of T^n by rows and columns are regular since the double complex is first quadrant.

Now we begin the construction of the spectral sequence. We forget for a second the grading. Let (K, d) be a complex and consider a filtration

$$K = K_0 \supseteq K_1 \supseteq \ldots$$

Let $G = \bigoplus_{p\in\mathbb{Z}} K_p$, which is naturally a differential object. The inclusion $K_{p+1} \to K_p$ induces a morphism $i : G \to G$ with $\operatorname{coker} i \cong \mathrm{Gr}(K) =: E$. Then we have an exact sequence of differential modules

$$0 \longrightarrow G \xrightarrow{i} G \longrightarrow E \longrightarrow 0$$

giving an exact triangle

$$\begin{tikzcd} H(G) \arrow[r, "i"] & H(G) \arrow[d, "j"] \\ & H(E) \arrow[ul, "k"] \end{tikzcd}$$

We assume the filtration is regular, i.e. of finite length ℓ. We have the following sequence (not exact!)

$$0 \longrightarrow H(K_\ell) \longrightarrow H(K_{\ell-1}) \longrightarrow \cdots \longrightarrow H(K_1) \longrightarrow H(K) \longrightarrow H(K_{-1}) \longrightarrow \cdots$$

whose direct sum we call G_1. As the filtration is compatible with the differential, $i : G \to G$ induces $H(K_p) \to H(K_{p+1})$ which we also call i. Then in the following sequence

$$0 \longrightarrow i(H(K_\ell)) \longrightarrow \cdots \longrightarrow i(H(K_1)) \longrightarrow H(K) \longrightarrow H(K_{-1}) \longrightarrow \cdots$$

$i(H(K_1)) \to H(K)$ is an inclusion so injective. We call the direct sum G_2. Iteratively we apply i and at each step we call G_k the sum of the terms in the sequence. Eventually we get

$$0 \longrightarrow i^\ell(H(K_\ell)) \longrightarrow \cdots \longrightarrow i(H(K_1)) \longrightarrow H(K) \longrightarrow H(K_{-1}) \longrightarrow \cdots$$

where all arrows are injective so can be regarded as a decreasing filtration $F_p = i^p(H(K_p))$ of $H(K)$. The sequence then stabilises: $G_{r+1} \cong G_r$ whenever $r \geq \ell + 1$. We thus define $G_\infty = G_{\ell+1} = \bigoplus_{p\in\mathbb{Z}} F_p$.

Let $E_1 = H(E)$ and we can rewrite the above exact triangle as

$$\begin{tikzcd} G_1 \arrow[r, "i_1"] & G_1 \arrow[d, "j_1"] \\ & E_1 \arrow[ul, "k_1"] \end{tikzcd}$$

We define a differential $d_1 : E_1 \to E_1$ by $d_1 = j_1 \circ k_1$. Let $E_2 = H(E_1, d_1)$. Recall that G_2 is the image of $i_1 : G_1 \to G_1$. We define morphisms such that the triangle

$$\begin{tikzcd} G_2 \arrow[r, "i_2"] & G_2 \arrow[d, "j_2"] \\ & E_2 \arrow[ul, "k_2"] \end{tikzcd}$$

is exact:

- i_2 is induced by i_1 by letting $i_2(i_1(x)) = i_1(i_1(x))$.
- j_2 is induced by j_1 by letting $j_2(i_1(x)) = [j_1(x)]_{E_2}$.
- k_2 is induced by k_1 by letting $k_2([y]) = k_1(y)$.

Check these are well-defined and give the desired exact triangle.

Iterate this process and we get a sequence of *derived triangles*. For $r \geq \ell+1$ the morphism i_r becomes injective so E_r stabilises and $k_r = 0$. As i_∞ is injective, the exact triangle

$$\begin{tikzcd} G_\infty \arrow[r, "i_\infty"] & G_\infty \arrow[d, "j_\infty"] \\ & E_\infty \arrow[ul, "k_\infty"] \end{tikzcd}$$

becomes a short exact sequence

$$0 \longrightarrow G_\infty \xrightarrow{i_\infty} G_\infty \xrightarrow{j_\infty} E_\infty \longrightarrow 0$$

so $E_\infty = \bigoplus_{p \leq \ell} F_p/F_{p+1}$.

Definition (spectral sequence)**.** A sequence of differential modules $\{(E_k, d_k)\}$ such that $H(E_k, d_k) \cong E_{k+1}$ is called a *spectral sequence*. If the modules E_k becomes stationary, we call the stationary value E_∞. If E_∞ is isomorphic to the graded module associated to some filtered module H, we say that the spectral sequence *converges* to H.

Thus in our motivating example the spectral sequence converges to $H(K)$.

Remark. If $\{K_\bullet\}$ has length ℓ but $\{E_\bullet\}$ stablises at some $k_0 < \ell$, we say that the spectral sequence *degenerates* at r_0. In particular $d_r = 0$ for $r \geq r_0$.

The next step is to switch the grading on. Everything we have done so far is compatible with grading so we just need to carefully track them through the calculation.

Theorem 4.1. *Let $(K^\bullet, d)$ be a complex with a compatible regular filtration. Then there is a spectral sequence $\{(E_k, d_k)\}$ where each E_k is graded, which converges to the graded module associated to some filtration on $H^\bullet(K, d)$.*

Proof. G_r is graded by degree: $G_r = \bigoplus_{n\in\mathbb{Z}} G_r^n$ where $G_r^n = \bigoplus_{p\in\mathbb{Z}} i^{r-1}(H^n(K_p))$. For every n if $r \geq \ell(n)+1$ then $i_r : G_r^n \to G_r^n$ is injective, so $k_r : G_r^n \to G_r^{n+1}$ is zero. Thus in every degree the derived triangle stabilises and we get a short exact sequence

$$0 \longrightarrow G_\infty^n \xrightarrow{i_\infty} G_\infty^n \longrightarrow E_\infty^n \longrightarrow 0$$

$F_p^n = i^{\ell(n)}(H^n(K_p))$ gives a filtration of $H^n(K_p)$ and $E_\infty^n = \bigoplus_p F_p^n/F_{p+1}^n$. $\square$

4.2 Bidegree

In our previous example we can write

$$K_p/K_{p+1} = \bigoplus_{q\in\mathbb{Z}} K_p^q/K_{p+1}^q = \bigoplus_{q\in\mathbb{Z}} K_p^{p+q}/K_{p+1}^{p+q}$$

then E is actually bigraded:

$$E = \bigoplus_{p\in\mathbb{Z}} K_p/K_{p+1} = \bigoplus_{p,q\in\mathbb{Z}} K_p^{p+q}/K_{p+1}^{p+q}.$$

We let $E_0^{p,q} = K_p^{p+q}/K_{p+1}^{p+q}$. The natural grading on E can then be seen as the grading of the total complex of the double complex by total degree, i.e.

$$\bigoplus_{p+q=n} E_0^{p,q} = \bigoplus_{p+q=n} K_p^{p+q}/K_{p+1}^{p+q} = \bigoplus_p K_p^n/K_{p+1}^n = E^n.$$

As $d : K_p^{p+q} \to K_p^{p+q+1}$, it descends to $d_0 : E_0^{p,q} \to E_0^{p,q+1}$. We let

$$E_1^{p,q} = H^q(E_0^{p,\bullet}, d_0) = H^{p+q}(K_p/K_{p+1})$$

Then by construction

$$E_1 = H(E_0, d_0) = \bigoplus_{p,q} E_1^{p,q}$$

so it is also bigraded.

Claim that $d_1 : E_1^{p,q} \to E_1^{p+1,q}$: for $x \in E_1^{p,q} = H^{p+q}(K_p/K_{p+1})$, write $x = [y]$ for $y \in K_p/K_{p+1}$. Then

$$d_1(x) = j_1(k_1(x)) = j_1(k(y)) \in H^{p+q+1}(K_{p+1}/K_{p+2}) = E_1^{p+1,q}$$

since k shifts degree by 1 (so $k_1 : H^{p+q}(K_p/K_{p+1}) \to H^{p+q+1}(K_{p+1})$).

Set $E_2^{p,q} = H^p(E_1^{\bullet,q}, d_1)$ and $d_2 : E_2^{p,q} \to E_2^{p+2,q-1}$: denote by $[-]_r$ the cohomology class in E_r of a cocycle in E_{r-1}... (?computation postponed)

Theorem 4.2 (five term exact sequence)**.** *Assume that $K_p^n = 0$ for $p > n$. Then there is an exact sequence*

$$0 \longrightarrow E_2^{1,0} \longrightarrow H^1(K) \longrightarrow E_2^{0,1} \xrightarrow{d_2} E_2^{2,0} \longrightarrow H^2(K)$$

The remaining three nontrivial arrows are called edge morphisms*.*

The existence of the edge morphisms is the content of the following lemmas.

Lemma 4.3. *For every $r \geq 1$ there are canonical morphisms $H^q(K) \to E_r^{0,q}$.*

Proof. $K_0 = K$ so

$$E_\infty^{0,q} = F_0^q/F_1^q = H^q(K)/F_1^q$$

so there is a surjective morphism $H^q(K) \to E_\infty^{0,q}$.

On the other hand for $r \geq 1$ a nonzero class in $E_2^{0,q}$ cannot be a boundary as $E_r^{-r,q+r-1} = 0$, so cohomology classes are cycles. But cohomology classes are elements in $E_{r+1}^{0,q}$ so $E_r^{0,q} \supseteq E_{r+1}^{0,q} \supseteq \cdots \supseteq E_\infty^{0,q}$. Composing these two maps give the desired morphism. ☐

Lemma 4.4. *Assume $K_p^n = 0$ for $p > n$. Then for every $r \geq 2$ there is a morphism $E_r^{p,0} \to H^p(K)$.*

Proof. $K_p^n = 0$ for $p > n$ implies that for $q \leq 0$,

$$E_0^{p,q} = K_p^{p+q}/K_{p+1}^{p+q} = K_p^{p+q}$$

so for $r \geq 1$, $E_0^{p,q} =$ for $q < 0$. Thus for $r \geq 2$ each element in $E_r^{p,0}$ is a boundary so we have surjections $E_r^{p,0} \twoheadrightarrow E_{r+1}^{p,0} \twoheadrightarrow \cdots \twoheadrightarrow E_\infty^{p,0}$. Again due to the fact $K_p^n = 0$ for $p > n$, $H^n(K_p) = 0$ so

$$E_\infty^{p,0} = F_p^p/F_{p+1}^p = F_p^p$$

so $E_\infty^{p,0} \subseteq H^p(K)$. Combining these two gives the desired morphism. □

4.3 Spectral sequences associated with a double complex

Let K be a (first quadrant) double complex with two differentials $\delta_1 : K^{p,q} \to K^{p+1,q}, \delta_2 : K^{p,q} \to K^{p,q+1}$ that anticommute. Let $(T, d = \delta_1 + \delta_2)$ be the total complex graded by total degree. The filtration by columns is given by

$$T_p = \bigoplus_{n \geq p, q \in \mathbb{Z}} K^{n,q}.$$

As before from this filtered graded complex we form

$$G^n = \bigoplus_p T_p^n = \bigoplus_p \bigoplus_{j=0}^{n-p} K^{n-j,j}.$$

In fact G is bigraded: if we set

$$G^{p,q} = T_p^{p+q} = \bigoplus_{j=0}^{q} K^{p+q-j,j}$$

then

$$\bigoplus_{p+q=n} G^{p,q} = G^n.$$

One can check

$$E_0^{p,q} = T_p^{p+q}/T_{p+1}^{p+q} = K^{p,q}$$

and $d_0 : E_0^{p,q} \to E_0^{p,q+1}$ is δ_2, so $E_1^{p,q} = H^q(E_0^{p,\bullet}, \delta_2)$. $E_1^{p,q} = H^{p+q}(T_p/T_{p+1})$. On the other hand $T_p/T_{p+1} = \bigoplus_q K^{p,q}$. So (?) $d_1 = \delta_1$. Thus $E_2^{p,q} = H^p(E_1^{\bullet,q}, \delta_1)$.

If we use filtration by rows then $'K^{p,q} = K^{q,p}$ and

$$'E_1^{p,q} = H^q(K^{\bullet,p}, \delta_1), 'E_2^{p,q} = H^p('E_1^{\bullet,q}, \delta_2).$$

The upshot is that under our assumptions, both spectral sequences converge to $H^\bullet(T)$.

Example (Čech-de Rham theorem)**.** Let X be a differentiable manifold with a good cover $\mathcal{U}$. Define a double complex by

$$K^{p,q} = C^p(\mathcal{U}, \Omega_X^q) = \prod_{i_0 < \cdots < i_p} \Omega_X^q(U_{i_0 \cdots i_p}),$$

δ_1 the Čech differential and $\delta_2 = (-1)^p \mathrm{d}$. This gives $E_0^{p,q} = K^{p,q}$. Apply column filtration first, we have

$$E_1^{p,q} = H^q(C^p(\mathcal{U}, \Omega_X^\bullet), \mathrm{d}) = \prod_{i_0 < \cdots < i_p} H_{\mathrm{dR}}^q(U_{i_0 \dots i_p}) = \begin{cases} C^p(\mathcal{U}, \mathbb{R}) & q = 0 \\ 0 & q > 0 \end{cases}$$

where in the last equality we used $\mathcal{U}$ is a good cover. Thus in E_1, everything other than the first row is 0. Thus to compute E_2 suffices to compute

$$E_2^{p,0} = H^p(E_1^{\bullet,0}, \delta) = H^p(C^\bullet(\mathcal{U}, \mathbb{R}), \delta) = H^p(\mathcal{U}, \mathbb{R}).$$

As $d_2 = 0$, the spectral sequence degenerates at the second page so $E_\infty = E_2$. As E_∞^n has only one nonzero graded piece $E_\infty^{n,0}$, so does $H^n(T)$ so $H^n(T) \cong E_\infty^{n,0} = H^p(\mathcal{U}, \mathbb{R})$.

Now take the row filtration,

$${}'E_1^{p,q} = H^q(K^{\bullet,p}, \delta) = H^q(C^\bullet(\mathcal{U}, \Omega_X^p)) = \begin{cases} \Omega^p(X) & q = 0 \\ 0 & q > 0 \end{cases}$$

since Ω^p is acyclic and

$${}'E_2^{p,0} = H^p(\Omega^\bullet(X), \mathrm{d}) = H_{\mathrm{dR}}^p(X)$$

and again the spectral sequence degenerates at the second page so ${}'E_\infty = {}'E_2 = H(T)$. Thus

$$H^\bullet(\mathcal{U}, \mathbb{R}) \cong H^\bullet(T) \cong H_{\mathrm{dR}}^\bullet(X).$$

Now take a direct limit of good coers, LHS also equals to $H^\bullet(X, \mathbb{R})$. Later we will generalise this to Čech spectral sequence.

Index