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0 Introduction

Throughout this course we will highlight similarities between algebraic geometry and differential geometry.

Recall that a quasiprojective variety is a locally closed set in Zariski topology of a projective space $\mathbb{P}^n$. Let $X \subseteq \mathbb{P}^n$ be a quasiprojective variety and $\varphi : X \to \mathbb{P}^m_k$ be a map. φ is a *morphism* if φ is continuous and for all $U \subseteq \mathbb{P}^m_k$ open, for all regular function (which is defined below) f on U, $f \circ \varphi$ is regular on $\varphi^{-1}(U)$ is also regular.

Exercise.

1. Let X, Y be smooth manifolds, $\varphi : X \to Y$ a continuous map. Then φ is smooth if and only if for all $U \subseteq Y$ open, for all $f \in C^\infty(U)$, $f \circ \varphi \in C^\infty(\varphi^{-1}(U))$.

2. The same statement with smooth replaced by holomorphic.

Let $D(f) \subseteq \mathbb{A}^n_k$ be principal open. A function $g : D(f) \to k$ is said to be *regular* if exists $h \in k[x_1, \dots, x_n]$ and $r \geq 0$ such that $g = \frac{h}{f^r}$.

Suppose $X \subseteq \mathbb{A}^n_k$ is locally closed. $g : U \to K$ is regular if locally in the Zariski topology it extends to a regular function on a principal open in $\mathbb{A}^n_k$. For $X \subseteq \mathbb{P}^n$ locally closed, $g : X \to k$ is regular if $g|_{X \cap U_i}$ is regular for $i = 0, \dots, n$.

In each of the cases (smooth manifolds, complex manifolds, quasiprojective variety), we have a topological space and for each open set an associated set of regular functions.

The shift from varieties to schemes brings forth the possibility of nilpotent regular functions. Case in point: the intersection of a parabola and a tangent. In fact this is a familiar idea: suppose f is a continuous real function defined on a neighbourhood of the origin. Then we may expand

$$f(t) = a_0 + a_1 t + O(t^2)$$

where $O(t^2)$ denotes terms of order at least 2. The algebraic way to write this is to consider the function as an element of $\mathbb{R}[t]/t^2$.

1 Sheaves & Schemes

1.1 Sheaves

First notice that in each of the cases, regular function is a local notion: suppose $U \subseteq X$ open, $\{V_i\}_{i\in I}$ an open cover of U then $f \in C^\infty(U)$ if and only if $f|_{V_i} \in C^\infty(V_i)$ for all $i \in I$. In other words, given $f_i \in C^\infty(V_i)$ such that for all i, j, $f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j}$ then exists a unique $f \in C^\infty(U)$ such that $f|_{V_i} = f_i$.

Moreover if f, g are regular then so if $f + g$, fg and f/g if g does not vanish.

Definition ((pre)sheaf)**.** Let X be a topological space. A *sheaf of sets* $\mathcal{F}$ on X is

1. for every $U \subseteq X$ open, a set $\mathcal{F}(U)$,
2. for every $V \subseteq U \subseteq X$ open, a map $\mathcal{F}(U) \to \mathcal{F}(V)$ called the *restriction*, sometimes denoted r_{UV}, and sometimes we denote $r_{UV}(f)$ by $f|_V$ for $f \in \mathcal{F}(U)$,

such that

1. (presheaf) for every $W \subseteq V \subseteq U \subseteq X$ open, we have $r_{UW} = r_{VW} \circ r_{UV}$,
2. for $U \subseteq X$ open, for every open cover $\{V_i\}_{i\in I}$ of U, for all $f_i \in \mathcal{F}(V_i)$ such that $f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j}$ for all i, j, then exists a unique $f \in \mathcal{F}(V)$ such that $f|_{V_i} = f_i$.

A *sheaf of rings* is the same with every instance of "set" replaced by "ring", and "map" replaced by "ring homomorphism". Similarly for abelian groups, modules, algebras etc.

Exercise. If $\mathcal{F}$ is a sheaf on X then $\mathcal{F}(\emptyset)$ is the singleton set/zero ring/zero module etc.

Definition (ringed space)**.** A *ringed space* is a pair $(X, \mathcal{O}_X)$ where X is a topological space and $\mathcal{O}_X$ is a sheaf of rings.

Example.

- Let X be a topological space, and $\mathcal{O}_X$ the sheaf of continuous functions to $\mathbb{R}, \mathbb{C}$ or any topological ring.
- Let X be a smooth manifold, and $\mathcal{O}_X$ the sheaf of smooth functions. Similarly for complex manifolds and varieties.

A ringed space allows us to add, subtract and multiply functions. To do division, i.e. inverting a function, we must be able to tell its "vanishing set" and invert it away from the locus. However, unlike in the motivating examples, it makes no sense to talk about a function being "nonzero", or indeed taking any value at all. We have to do a bit more work.

Definition (stalk)**.** Let X be a topological space and $\mathcal{F}$ a presheaf of sets on X, $p \in X$. The *stalk* of $\mathcal{F}$ at p, denoted $\mathcal{F}_p$, is the quotient set

$$\{(U, f) : U \ni p \text{ open neighbourhood}, f \in \mathcal{F}(U)\}/\sim$$

where $(U, f) \sim (V, g)$ if and only if exists $W \subseteq U \cap V$ open neighbourhood of p such that $f|_W = g|_W$.

Exercise. If $\mathcal{F}$ is a sheaf of rings/modules etc then so is $\mathcal{F}$.

Definition (locally ringed space)**.** A *locally ringed space* is a ringed space $(X, \mathcal{O}_X)$ such that for every $p \in X$, the stalk $\mathcal{O}_{X,p}$ is a local ring.

Definition (residue field)**.** Let $(X, \mathcal{O}_X)$ be a locally ringed space. The *residue field* at p is

$$\kappa(p) = \mathcal{O}_{X,p}/\mathfrak{m}_{X,p}.$$

For U an open neighbourhood of p and $f \in \mathcal{O}_X(U)$, we define the *value of f at p* to be $f(p) = [(U, f)] \in \kappa(p)$.

Exercise. Show $D(f) = \{p \in U : f(p) \neq 0\}$ is open in U.

Example. Fix a field k. let D_k be the following ringed space: as a space D_k is a singleton and

$$\mathcal{O}_{D_k}(D_k) = k[t]/(t^2) \cong k \oplus kt.$$

This is a locally ringed space with $\kappa(\mathrm{pt}) = k$.

Definition (morphism of (pre)sheaves)**.** Let X be a topological space and $\mathcal{F}, \mathcal{G}$ presheaves on X. A *morphism of presheaves* $\varphi : \mathcal{F} \to \mathcal{G}$ is the data for aevery $U \subseteq X$ open a map $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ such that for all $V \subseteq U \subseteq X$ the following diagram commutes

$$\begin{array}{ccc} \mathcal{F}(U) & \xrightarrow{\varphi(U)} & \mathcal{G}(U) \\ \downarrow{\scriptstyle r_{UV}} & & \downarrow{\scriptstyle r_{UV}} \\ \mathcal{F}(V) & \xrightarrow{\varphi(V)} & \mathcal{G}(V) \end{array}$$

A *morphism of sheaves* is a morphism of presheaves between sheaves.

Example. Suppose M is a smooth manifold. Let $\mathcal{A}^p$ be the sheaf of p-forms. Then exterior derivative $\mathrm{d} : \mathcal{A}^p \to \mathcal{A}^{p+1}$ is a morphism of sheaves.

Exercise. If $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves, for all $p \in X$ it induces $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$.

Definition. A morphism of sheaves $\varphi : \mathcal{F} \to \mathcal{G}$ is *injective/surjective* if for every $p \in X$, the map $\varphi_p : \mathcal{F}_p \to \mathcal{G}_p$ is injective/surjective.

Lemma 1.1. *φ is injective if and only if for all $U \subseteq X$ open, $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is injective. φ is surjective if and only if for all $U \subseteq X$ open, for all $g \in \mathcal{G}(U)$, exists $\{V_i\}$ an open cover of U and $f_i \in \mathcal{F}(V_i)$ such that $\varphi(V_i)(f_i) = g|_{V_i}$.*

Proof. Exercise. □

Proposition 1.2. *φ is injective and surjective if and only if for all $U \subseteq X$ open, $\varphi(U) : \mathcal{F}(U) \to \mathcal{G}(U)$ is bijective.*

Proof. Need to show $\varphi(U)$ is surjective. Suppose $g \in \mathcal{G}(U)$ and take an open cover $\{V_i\}$ and $f_i \in \mathcal{F}(V_i)$ such that $\varphi(f_i) = g|_{V_i}$. Then

$$\varphi(f_i|_{V_i \cap V_j} - f_j|_{V_i \cap V_j}) = \varphi(f_i)|_{V_i \cap V_j} - \varphi(f_j)|_{V_i \cap V_j} = g|_{V_i \cap V_j} - g|_{V_i \cap V_j} = 0$$

so by injectivity $f_i|_{V_i \cap V_j} = f_j|_{V_i \cap V_j}$. Use sheaf axioms. □

Definition. Suppose $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphism of sheaves of abelian groups. Define

$$\begin{aligned}(\ker \varphi)(U) &= \ker \varphi(U) \\ (\operatorname{im} \varphi)(U) &= \{g \in \mathcal{G}(U) : \text{ exists cover } \{V_i\} \text{ of } U, \\ &\qquad f_i \in \mathcal{F}(V_i) \text{ such that } \varphi(f_i) = g|_{V_i}\}\end{aligned}$$

Exercise. $\ker \varphi$ and $\operatorname{im} \varphi$ are sheaves and $\ker \varphi_p = (\ker \varphi)_p, \operatorname{im} \varphi_p = (\operatorname{im} \varphi)_p$.

Example. $\ker(\mathrm{d} : \mathcal{A}^p \to \mathcal{A}^{p+1})$ is the sheaf of closed p-forms. By Poincaré lemma, $\operatorname{im} \mathrm{d}$ is the sheaf of closed $(p+1)$-forms.

Definition (sheafification)**.** Let X be a topological space and $\mathcal{F}$ a presheaf on X. There exists a sheaf $\mathcal{F}^+$ and a morphism of presheaves $\theta : \mathcal{F} \to \mathcal{F}^+$ such that for every sheaf $\mathcal{G}$ on X and every $\varphi : \mathcal{F} \to G$, exists a unique morphism $\varphi^+ : \mathcal{F}^+ \to \mathcal{G}$ such that $\varphi = \varphi^+ \circ \theta$. $\mathcal{F}^+$ is called the *sheafification* of $\mathcal{F}$.

Exercise. Follow the hint and prove the existence of $\mathcal{F}^+$: let $F = \coprod_{p \in X} \mathcal{F}_p$. There is a natural map $\pi : F \to X$. Define

$$\mathcal{F}^+(U) = \{s : U \to F : \pi \circ s = \mathrm{id}_U, \text{ locally induced by } \mathcal{F}\},$$

i.e. exists $\{V_i\}$ an open cover of U and $f_i \in \mathcal{F}(V_i)$ such that for all $p \in V_i$, $s(p) = (V_i, f_i)$.

Exercise. Let X be a topological space and A an abelian group. The *constant presheaf* is given by $U \mapsto A$ for all U open and id_A for all restrictions. Then its sheafification is the *constant sheaf* A, given by

$$A(U) = \{U \to A \text{ locally constant}\}.$$

Show further that $\mathcal{P}_A$ is the inverse image presheaf A on $Y = \{\mathrm{pt}\}$.

Definition (pushforward/pullback of sheaves)**.** Let $\alpha : X \to Y$ be a continuous map of topological spaces, $\mathcal{F}$ a sheaf on X, $\mathcal{G}$ a sheaf on Y. The *pushforward* $\alpha_*\mathcal{F}$ of $\mathcal{F}$ is defined by

$$(\alpha_*\mathcal{F})(V) = \mathcal{F}(\alpha^{-1}(V))$$

for $V \subseteq Y$ open. The *pullback* $\alpha^{-1}\mathcal{G}$ of $\mathcal{G}$ is defined by the sheafification of the presheaf given by

$$U \mapsto \varinjlim_{\substack{V \text{ open in } Y \\ V \supseteq \alpha(U)}} \mathcal{G}(V)$$

Exercise. $\alpha_*\mathcal{F}$ is a sheaf on Y and $\alpha^{-1}\mathcal{G}$ is a sheaf on X.

Proposition 1.3.

1. *For all $p \in X$, $(\alpha^{-1}\mathcal{G})_p$ is canonically isomorphic to $\mathcal{G}_{\alpha(p)}$.*
2. *There is a canonical bijection*

$$\mathrm{Hom}_{\mathbf{Sh}_X}(\alpha^{-1}\mathcal{G}, \mathcal{F}) \cong \mathrm{Hom}_{\mathbf{Sh}_Y}(\mathcal{G}, \alpha_*\mathcal{F}).$$

In other words α^{-1} is left adjoint to α_*.

Definition (morphism of ringed space)**.** A *morphism of ringed spaces* $(X, \mathcal{O}_X) \to (Y, \mathcal{O}_Y)$ is $(f, f^\#)$ where $f : X \to Y$ is a continuous map and $f^\# : \mathcal{O}_Y \to f_*\mathcal{O}_X$ is a morphism of sheaves.

Remark. A morphism of ringed spaces induces for every $p \in X$ a morphism of rings

$$\begin{aligned} \mathcal{O}_{Y,f(p)} &\to \mathcal{O}_{X,p} \\ [(V, g)] &\mapsto [(f^{-1}(V), f^\#(g))] \end{aligned}$$

Definition (morphism of locally ringed space)**.** A *morphism of locally ringed spaces* is a morphism of ringed spaces such that for all $p \in X$, the map $f^\# : \mathcal{O}_{Y,f(p)} \to \mathcal{O}_{X,p}$ is a local homomorphism, i.e. $f^\#(\mathfrak{m}_{f(p)}) \subseteq \mathfrak{m}_p$ or equivalently, it induces a homomorphism of residue fields $\kappa(f(p)) \to \kappa(p)$.

Remark. The category of locally ringed spaces is *not* a full subcategory of the category of ringed spaces.

Given a sheaf $\mathcal{F}$ on X and an open subset $i : U \to X$, we define $\mathcal{F}|_U = i^{-1}\mathcal{F}$. Stalks of $\mathcal{F}|_U$ and $\mathcal{F}$ are naturally isomorphic.

Lemma 1.4. *If $(X, \mathcal{O}_X)$ is a locally ringed space then $(U, \mathcal{O}_X|_U)$ is a locally ringed space for all $U \subseteq X$ open.*

If k is a field, a locally ringed k-space or a locally ringed space over k is a locally ringed space $(X, \mathcal{O}_X)$ such that $\mathcal{O}_X$ is a sheaf of k-algebras.

Exercise.

1. Let $(X, \mathcal{O}_X)$ be a locally ringed space. To give it a structure over k is the same as giving $\mathcal{O}_X(X)$ a structure of k-algebras.
2. If X is a locally ringed space over k then for all $p \in X$, the field $\kappa(p)$ is an extension of k.

Definition (spectrum)**.** Let k be a field. Define $\operatorname{Spec} k$, the *spectrum* of k, is the locally ringed space of a singleton with structure sheaf k.

Exercise. If $(X, \mathcal{O}_X)$ is a locally ringed space, to give it a structure of k-algebra is the same as giving a morphism $(X, \mathcal{O}_X) \to \operatorname{Spec} k$.

1.2 Affine schemes

Aim: associate to each ring A a locally ringed space $\operatorname{Spec} A$ such that for all locally ringed space $(X, \mathcal{O}_X)$ there exists a natural bijection

$$\operatorname{Hom}_{\mathbf{LRS}}(X, \operatorname{Spec} A) \cong \operatorname{Hom}_{\mathbf{Ring}}(A, \mathcal{O}_X(X)).$$

Then a *scheme* is defined to be a locally ringed space locally isomorphic to the spectrum of some ring.

Definition (spectrum)**.** Given a ring R, define a locally ringed space $\operatorname{Spec} R$ as follow. As a set, it is the set of prime ideals in R. The topology on $\operatorname{Spec} R$, the *Zariski topology*, is defined by requiring that a basis is given by principlal open sets, i.e. $D(f) = \{\mathfrak{p} \text{ prime ideals} : f \notin \mathfrak{p}\}$ for $f \in R$. The structure sheaf is given on the basis by $\mathcal{O}(D(f)) = R_f$ and restriction maps are localisations.

It is a standard result that given a basis $\mathcal{B}$ of a topological space X, the forgetful functor $\mathbf{Sh}_X \to \mathcal{B}\text{-}\mathbf{Sh}_X$ is an equivalence of categories. An explicit description of the structure sheaf is given by Corollary 5.3.

Lemma 1.5. *The stalk of* $\operatorname{Spec} R$ *at* $\mathfrak{p}$ *is* $\mathcal{O}_{\operatorname{Spec} R, \mathfrak{p}} \cong R_{\mathfrak{p}}$.

Example.

1. $(\operatorname{Spec} k, \mathcal{O}_{\operatorname{Spec} k})$ is a locally ringed space and we have recovered the previous definition.
2. $\operatorname{Spec} \mathbb{Z}/6\mathbb{Z} = \{(2), (3)\}$. The topology is discrete. The computation of the structure sheaf is left as an exercise.

   More generally, $\operatorname{Spec}(R_1 \times R_2) = \operatorname{Spec} R_1 \amalg \operatorname{Spec} R_2$ as a scheme.
3. $\operatorname{Spec} \mathbb{C}[t] = \{0\} \cup \{(t - \lambda) : t \neq 0\}$. Nonempty open sets have the form $U \cup \{0\}$ where U is a cofinite subset of $\mathbb{C}$.
4. For any ring R we define the affine n-space to be $\mathbb{A}^n_R = \operatorname{Spec} R[x_1, \dots, x_n]$.

1.3 Zariski topology

Let $X = \operatorname{Spec} A$ be an affine scheme. In this scection we discuss some properties of the topological space underlying $\operatorname{Spec} A$. By the definition of Zariski toplogy, a subset $U \subseteq X$ is open if and only if it is the union of $D(f)$'s. Conversely $C \subseteq X$ is closed if and only if exists $S \subseteq A$ such that

$$C = \bigcap_{f \in S} Z(f) = \{\mathfrak{p} : \mathfrak{p} \supseteq S\} = Z(S) = Z(I(S)).$$

Also note that for an ideal $I \subseteq A$, $Z(I) = Z(\sqrt{I})$.

Lemma 1.6. *There is a bijection between closed subsets of X and radical ideals in A.*

Proof. The assignments $I \mapsto \sqrt{I}, C \mapsto \bigcap_{\mathfrak{p} \in C} \mathfrak{p}$ are inverses to each other. □

Corollary 1.7. *If A is a noetherian ring then $X = \operatorname{Spec} A$ is a noetherian topological space.*

Recall that a topological space X is noetherian if every descending chain of closed subsets stabilises.

Corollary 1.8. *If A is noetherian then every open subset of X is quasi-compact.*

Proposition 1.9. *Let $\pi : A \to B$ be a ring homomophism. Let $X = \operatorname{Spec} A, Y = \operatorname{Spec} B$. Let $\varphi : Y \to X$ be the induced morphism. If π is injective then $\varphi(Y)$ is dense in X and $\varphi^\# : \mathcal{O}_X \to \varphi_* \mathcal{O}_Y$ is injective.*

Proof. Want to show for every $f \in A$ such that $D(f) \neq \emptyset$, $D(f) \neq \emptyset$ then $D(f) \cap \varphi(Y) \neq \emptyset$. But $\varphi^{-1}(D(f)) = D(\pi(f))$. $D(f) = D(\sqrt{(f)}) = \emptyset$ precisely when f is nilpotent. The second statement follows from the fact that $A_f \to B_{\pi(f)}$ is injective. □

Digression on non-closed points Let $X = \operatorname{Spec} R$ where R is a finitely generated k-algebra for some algebraically closed field k. Let $Y \subseteq X$ be the subset of closed points. Let $i : Y \to X$. Then $i^{-1} : \{\text{open subsets of } X\} \to \{\text{open subsets of } Y\}$ is a bijection that respects inclusion, intersection and open covers (the only nontrivial step is to prove injectivity, for which one uses noetherianness of R). As a corollary, $i_* : \mathbf{Sh}_Y \to \mathbf{Sh}_X$, and hence its adjoint i^{-1}, is an equivalence of categories.

We also note here the topological construction that reverses the above process. To go from X to Y one simply takes the closed points. Conversely, starting with a space Y, one can construct a space X by adding a point η_Z for each $Z \subseteq Y$ closed irreducible (that is not a singleton). The topology on X is determined by

1. the subspace topology on Y is Y,
2. $\overline{\{\eta_Z\}} \cap Y = Z$.

This in turn implies that $\overline{\{\eta_Z\}} = \{\eta_W : W \subseteq Z \text{ closed irreducible}\}$.

Exercise. Let X be a topological space and $x \in X$ such that $Z = \overline{\{x\}}$. Then Z is closed and irreducible. Such an x is called a *generic point* of Z.

Even if the topological data can be recovered from the closed point in the case of a k-finitely generated algebra, it is not a good idea to just with the closed points as there may be different structure sheaves. Indeed, that is the whloe point of inventing locally ringed space. For example, note that for any local ring R that is a domain, $\operatorname{Spec} R$ contains only two points: a generic point and a closed point. If we only look at the closed point then there is no data other than a singleton. For example let $R = k[t]_{(t)}, S = k[[t]]$. There is a natural map $R \to S$ that induces a homeomorphism $\operatorname{Spec} S \to \operatorname{Spec} R$ (which is of course not an isomorphism of ringed spaces).

Exercise. Determine $\operatorname{Spec} k[t]/(t^n)$ and show that $\varprojlim_n k[t]/(t^n) = k[[t]]$.

Theorem 1.10. *Let $(X, \mathcal{O}_X)$ be a locally ringed space and A be a ring. Then the naural map*

$$\operatorname{Hom}_{\mathbf{LRS}}(X, \operatorname{Spec} A) \to \operatorname{Hom}_{\mathbf{Ring}}(A, \mathcal{O}_X(X))$$

is a bijection.

Proof. We will construct an inverse. Suppose given a ring map $\overline{\varphi}^\# : A \to \mathcal{O}_X(X)$, we want to construct a morphism $(\varphi, \varphi^\#) : X \to \operatorname{Spec} A$ such that $\varphi^\#(\operatorname{Spec} A) = \overline{\varphi}^\#$. For any $x \in X$ the composition

$$A \to \mathcal{O}_X(X) \to \mathcal{O}_{X,x} \to \kappa(x)$$

has image a domain so the kernel is a prime ideal $\mathfrak{p}$. We define $\varphi(x)$ to be $\mathfrak{p}$. To show φ is continuous, suffice to show $\varphi^{-1}(D(f))$ is open in X. For $x \in X$, $\varphi(x) \in D(f)$ if and only if $\overline{\varphi}^\#(f) \neq 0 \in \kappa(x)$, since the bottom row of this commutative diagram is an injection

$$\begin{array}{ccc} A & \xrightarrow{\overline{\varphi}^\#} & \mathcal{O}_X(X) \\ \downarrow & & \downarrow \\ \kappa(\mathfrak{p}) & \hookrightarrow & \kappa(x) \end{array}$$

To define $\varphi^\#$, by an argument similar to the construction of the structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$ before, we only need to give it on $D(f)$. Let $g = \overline{\varphi}^\#(f)$. Then $\varphi^{-1}(D(f)) = D(g)$. Thus we need to define a ring map $\mathcal{O}_{\operatorname{Spec} A}(D(f)) = A_f \to \mathcal{O}_X(D(g))$. We have the following claim whose proof is left as an exercise: let $(X, \mathcal{O}_X)$ be a locally ringed space, $U \subseteq X$ open and $g \in \mathcal{O}_X(U)$. Then g is invertible on U if and only if for all $x \in U$, $[(U, g)]$ is invertible in $\mathcal{O}_{X,x}$. Since $\overline{\varphi}^\#(f)$ is invertible in $\mathcal{O}_X(D(g))$, the diagonal map in the following commutative diagram factors through A_f by universal property of localisation.

$$\begin{array}{ccc} A & \xrightarrow{\overline{\varphi}^\#} & \mathcal{O}_X(X) \\ \downarrow & \searrow & \downarrow \\ A_f & \dashrightarrow & \mathcal{O}_X(D(g)) \end{array}$$

□

Corollary 1.11. *If A, B are rings then there exists a natural bijection*

$$\mathrm{Hom}_{\mathbf{Ring}}(A,B) \cong \mathrm{Hom}_{\mathbf{LRS}}(\operatorname{Spec} B, \operatorname{Spec} A).$$

Example.

1. Suppose X is a k-locally ringed space. Then

$$\begin{aligned}\mathrm{Hom}_{k\text{-}\mathbf{LRS}}(X, \mathbb{A}^n_k) &= \mathrm{Hom}_{k\text{-}\mathbf{LRS}}(X, \operatorname{Spec} k[x_1,\dots,x_n]) \\ &\cong \mathrm{Hom}_k(k[x_1,\dots,x_n], \mathcal{O}_X(X)) \\ &\cong \mathcal{O}_X(X)^n\end{aligned}$$

   so $\mathbb{A}^n_k$ represents n-tuples of global sections. Comparing this with smooth manifold: $C^\infty(M,\mathbb{R}^n) \cong C^\infty(M)^n$ (or holomorphic functions for complex manifold), we see that $\mathbb{A}^n_k$ plays the same role as $\mathbb{R}^n$ (or $\mathbb{C}^n$).

2. For a finitely generated k-algebra $A = k[x_1,\dots,x_n]/I$, we can interpret the bijection as

$$\begin{aligned}&\mathrm{Hom}_{k\text{-}\mathbf{LRS}}(X, \operatorname{Spec} A) \\ \cong\ &\mathrm{Hom}_k(A, \mathcal{O}_X(X)) \\ =\ &\{\varphi \in \mathrm{Hom}_k(k[x_1,\dots,x_n], \mathcal{O}_X(X)) : \varphi(I) = 0\} \\ =\ &\{(g_1,\dots,g_n) \in \mathcal{O}_X(X) : f(g_1,\dots,g_n) = 0 \text{ for all } f \in I\}\end{aligned}$$

   Since A is noetherian, we can express these using finitely many conditions.

3. $\operatorname{Spec} k[t]/(t^2)$ represents $X \mapsto \{g \in \mathcal{O}_X(X) : g^2 = 0\}$.

Definition (scheme)**.** An *affine scheme* is a locally ringed space isomorphic to $\operatorname{Spec} R$ for some ring R. A *scheme* is a locally ringed space with an open cover by affine schemes.

Definition ((locally) of finite type)**.** If k is a scheme, a k-scheme is *locally of finite type* over k if it has an open cover by affines which are spectra of finitely generated k-algebras. It is *of finite type* if there exists a finite such cover.

Definition ((locally) noetherian)**.** A scheme is *locally noetherian* if it has an open cover by spectra of noetherian rings. It is *noetherian* if there exists a finite such cover.

Lemma 1.12. *An affine scheme is (quasi)compact.*

Proof. We may start with an open cover by principal opens. Let $X = \operatorname{Spec} A$ be an affine scheme. We want to know given $S \subseteq A$, when is $\{D(f)\}_{f \in S}$ an open cover of X. $X = \bigcup_{f \in S} D(f)$ if and only if

$$\emptyset = \bigcap_{f \in S} Z(f) = \bigcap_{f \in S} \{\mathfrak{p} \in \operatorname{Spec} A : \mathfrak{p} \ni f\} = \{\mathfrak{p} : \mathfrak{p} \supseteq S\},$$

i.e. there does not exist a prime ideal containing $I(S)$, the ideal generated by S, if and only if $I(S) = A$, i.e. exists $f_1, \dots, f_n \in S, a_1, \dots, a_n \in A$ such that

$$1 = \sum a_i f_i.$$

Thus

$$X = \bigcup_{f \in S} D(f) = \bigcup_{i=1}^{n} D(f_i) = X.$$

□

> **Lemma 1.13.** *Let $X = \operatorname{Spec} A$ be an affine scheme, $U = \operatorname{Spec} B$ an open affine subscheme of A. Then exist $f_1, \dots, f_n \in A$ such that $U = \bigcup_{i=1}^n D_X(f_i)$ and for each i, $D_X(f_i) = D_U(f_i|_U)$.*

Proof. $U \subseteq X$ open can be written as a union of principal opens. Since U is quasicompact this can be written as a finite union. This proves the first statement. As $D_X(f) = \{x \in X : f(x) \neq 0\}$,

$$D_X(f) \cap U = \{x \in U : f(x) \neq 0\} = \{x \in U : f|_U(x) \neq 0\} = D_U(f|_U).$$

□

Exercise (Hartshorne, Ex II.2.16)**.** Let X be a scheme, $f \in \mathcal{O}_X(X)$. By the general result on locally ringed space we know $f|_{D(f)}$ is invertible. Thus we have a map $\alpha : \mathcal{O}_X(X)_f \to \mathcal{O}_X(D(f))$. Show

1. if X is quasicompact then α is injective.
2. if X has a finite affine cover $\{U_i\}$ such that $U_i \cap U_j$ is quasicompact for all i, j then α is surjective.

Proof.

1. Suppose $\frac{g}{f^r} \in \ker \alpha$ so $g|_{D(f)} = 0$. We would like to show that exists $s \geq 0$ such that $f^N = 0 \in \mathcal{O}_X(X)$, i.e. $f^N g = 0 \in \mathcal{O}_{X,x}$ for all $x \in X$. If $x \in D(f)$ then $g|_{D(f)} = 0$ so $g = 0 \in \mathcal{O}_{X,x}$. If $x \notin D(f)$ then let $U = \operatorname{Spec} A$ be an affine neighbourhood of x in X. Then $g|_{D_U(f|_U)} = 0$. As $\mathcal{O}_U(D_U(f|_U)) = A_{f|_U}$, this means exists n such that $f|_U^n = 0$ on U. By quasicompactness we can find a finite affine cover $\{U_1, \dots, U_m\}$ of X, $n_1, \dots, n_m \geq 0$ such that $f|_{U_i}^{n_i} \cdot g = 0$ on U_i. Take $N = \max\{n_1, \dots, n_m\}$ so $f^N g = 0$.

2. Given $g \in \mathcal{O}_X(D(f))$, we seek $N \geq 0$ such that $f^N|_{D(f)} \cdot g$ is the restriction to $D(f)$ of an element of $\mathcal{O}_X(X)$. On each U_i this is true, i.e. exists $m_i \geq 0$ such that $f^{m_i} g \in U_i \cap D(f)$ is the restriction of an element of $\mathcal{O}_X(U_i)$. Let $m = \max\{m_i\}$ so exists $h_i \in \mathcal{O}_X(U_i)$ such that $h_i = f^m g \in \mathcal{O}_X(U_i \cap D(f))$. Pass to intersections $U_{ij} = U_i \cap U_j$, we have

$$h_i = f^m g = h_j \in \mathcal{O}_X(U_{ij} \cap D(f))$$

so by injectivity exists $n_{ij} \geq 0$ such that $f^{n_{ij}} h_i = f^{n_{ij}} h_j$ on U_{ij}. Let $N = \max\{n_{ij}\}$ so $f^N h_i = f^N h_j$ on U_{ij}. Exists $\tilde{h} \in \mathcal{O}_X(X)$ such that $\tilde{h}|_{U_i} = f^N h_i$. $\tilde{h} = f^{N+m} g$ on $U_i \cap D(f)$. As $\{U_i\}$ is a cover of X, $\{U_i \cap D(f)\}$ is an open cover of $D(f)$ so $\tilde{h}|_{D(f)} = f^{N+m}|_{D(f)} \cdot g$.

□

Proposition 1.14 (Hartshorne, Ex II.2.17)**.** *Let X be a scheme. Assume $f_1, \dots, f_n \in \mathcal{O}_X(X)$ such that $(f_1, \dots, f_n) = \mathcal{O}_X(X)$ and such that $D(f_i)$ is affine for all i. Then X is affine.*

Proof. Let $U_i = D(f_i)$. Note

$$U_i \cap U_j = \{x \in U_i : f_j(x) \neq 0\} = D_{U_i}(f_j|_{U_i})$$

is affine. Thus by exercise for all $f \in \mathcal{O}_X(X)$, $\mathcal{O}_X(X) \cong \mathcal{O}_X(D(f))$. Let $A = \mathcal{O}_X(X)$ so $\mathcal{O}_X(D(f_i)) \cong A_{f_i}$. The identity map $A \to \mathcal{O}_X(X)$ corresponds to a morphism $\varphi : X \to \operatorname{Spec} A$. For all i, let $\varphi^{-1}(D_{\operatorname{Spec} A}(f_i)) = D_X(f_i)$. The morphism $\varphi|_{U_i} : U_i \to V_i$, where $V_i = D_{\operatorname{Spec} A}(f_i)$ between affine schemes correspond to a map

$$\varphi^\# : \mathcal{O}_{V_i}(V_i) \to \mathcal{O}_{U_i}(U_i).$$

But LHS is $\mathcal{O}_{\operatorname{Spec} A}(V_i) = A_{f_i}$ and RHS is $\mathcal{O}_X(D_X(f_i)) = A_{f_i}$. One can check that by our construction of the maps, $\varphi^\#$ is exactly the map α in the above exercise. Thus $\varphi|_{U_I} : U_i \to V_i$ is an isomorphism. The result can then be deduced from the following exercise: let $\varphi : X \to Y$ be a morphism of locally ringed spaces. If exists an open cover $\{V_i\}$ of Y such that $\varphi : \varphi^{-1}(V_i) \to V_i$ is an isomorphism then φ is an isomorphism. □

Definition (affine morphism)**.** Let $\varphi : X \to Y$ be a morphism of schemes. We say φ is *affine* if for every $V \subseteq Y$ open affine, $\varphi^{-1}(V)$ is affine.

Exercise.

1. Composition of affine morphisms is affine.
2. Let X be a scheme, $f \in \mathcal{O}_X(X)$, $U = D(f)$ and $i : U \to X$ the inclusion. Then i is affine.

Proposition 1.15. *Let $\varphi : X \to Y$ be a morphism of schemes. Then φ is affine if and only if there exists an affine open cover $\{V_i\}$ of Y such that for every i, $\varphi^{-1}(V_i)$ is affine.*

Proof. Only need to prove if. Let $V \subseteq Y$ affine and let $U = \varphi^{-1}(V)$. Fix $i \in I$ and we can write $V \cap V_i \subseteq V_i$ as a union of principal opens in V_i. $U_i = \varphi^{-1}(V_i)$ is affine so we can cover V with open affines $\{W_j\}$ whose inverse images are also affines. Cover each W_j with open affines which are principal in both V and in W_j. Thus we can find a cover of V by principal affines, which can be taken to be a finite collection $\{D(f_m)\}$, such that each $\varphi^{-1}(D(f_m))$ is affine. Let $g_m = \varphi^\#(f_m)$ where $\varphi^\# : \mathcal{O}_Y(V) \to \mathcal{O}_X(U)$. Since $\sum a_m f_m = 1$ for some a_m, $\sum \varphi^\#(a_m) g_m = 1$. U is affine by the proposition we have just shown. □

This shows that the property of a morphism being affine is *local on the base*. This generalises to Grothendieck topologies.

1.4 Subschemes

Lemma 1.16. *If $(X, \mathcal{O}_X)$ is a scheme, $U \subseteq X$ open then $(U, \mathcal{O}_X|_U)$ is a scheme.*

Proof. Suppose $\{V_i\}$ is an affine open cover of X. Then U has an open cover $\{V_i \cap U\}$, so it suffices to do the case where X is affine. If $X = \operatorname{Spec} A$ is affine then U has an open cover by principal open subsets. But for $f \in A$, $(D(f), \mathcal{O}_X|_{D(f)}) \cong \operatorname{Spec} A_f$ which is affine. □

Definition (open and closed embedding)**.** A morphism $\varphi : Y \to X$ is an *open embedding* if $\varphi(Y)$ is open in X and $\varphi : Y \to \varphi(Y)$ is an isomorphism.

A morphism $\varphi : Y \to X$ is a *closed embedding* if $\varphi(Y)$ is closed, φ is a homeomorphism onto its image and $\varphi^\# : \mathcal{O}_X \to \varphi_* \mathcal{O}_Y$ is surjective.

Our aim of this section is to show that closed immersions are locally modelled on morphisms induced by surjective ring maps. Of course we have to first prove that they are indeed closed embeddings.

Lemma 1.17. *Let $X = \operatorname{Spec} A, Y = \operatorname{Spec} B, \pi : A \to B$ a morphism inducing the morphism $\varphi : Y \to X$. If π is surjective then φ is a closed embedding.*

Proof. Let $I = \ker \pi$ and let $C = \phi(Y)$. By ideal correpondence φ is continuous and bijective. Thus to show it is a homeomorphism onto its image suffice to show it is open. Given $g \in B$, by surjectivity of π we can find $f \in A$ a preimage of g. Then

$$\varphi(D(g)) = \{\mathfrak{p} \in X : \mathfrak{p} \supseteq I, g \notin \pi(\mathfrak{p})\} = \{\mathfrak{p} \in X : \mathfrak{p} \supseteq I, f \notin \mathfrak{p}\} = D(f) \cap C$$

which is open in C.

To show $\varphi^\#$ is surjective it suffices to show that for all $f \in A$, $\varphi^\#(D(f)) : A_f \to B_{\pi(f)}$ is surjective. Given $\frac{b}{\pi(f)^n} \in B_{\pi(f)}$, take a such that $\pi(a) = b$. Then $\pi(\frac{a}{f^n}) = \frac{b}{\pi(f)^n}$. □

Exercise.

1. If $\varphi : Y \to X$ and $\varphi : Z \to Y$ are closed embeddings then so is $\varphi \circ \psi : Z \to X$.

2. If $\varphi : Y \to X$ is an isomorphism then it is both a closed and open embedding.

3. Let $\varphi : Y \to X$ be a morphism. Then φ is a closed embedding if and only if exists an open cover $\{U_i\}$ of X such that $\varphi|_{V_i} : V_i \to U_i$ is a closed embedding for all i where $V_i = \varphi^{-1}(U_i)$. (hint: $C \subseteq X$ is closed if and only if exists an open cover $\{U_i\}$, $C \cap U_i$ is closed in U_i. Also if $\varphi : Y \to X$ is a morphism of locally ringed space then $\varphi^{\#}$ is surjective if and only if exists an open cover V_i such that $\varphi^{\#}|_{V_i}$ is surjective).

4. Prove 3 by replacing "exists" by "for every", i.e. this is a property local on basis.

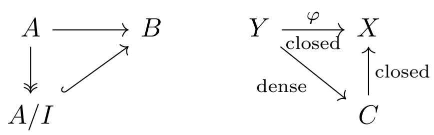
Proposition 1.18. *Let $X = \operatorname{Spec} A$ be an affine scheme and $\varphi : Y \to X$ a closed embedding. Then*

1. *if $Y = \operatorname{Spec} B$ then $\varphi^{\#} : A \to B$ is surjective.*

2. *Y is affine.*

Thus the only closed subscheme structures on X are induced by surjective ring maps.

Proof. First suppose $Y = \operatorname{Spec} B$. Let $\pi : A \to B$ be the induced map with kernel I. Let $C = \operatorname{Spec} A/I$. Then $\varphi(Y) \subseteq C$ and by injectivity of $A/I \to B$, is dense in C. $\varphi(Y)$ being closed in X and dense in C and C being closed in Y imply that $\varphi(Y) = C$ so $Y \to C$ is a closed embedding.



Thus we may assume that π is injective, i.e. $C = X$ and $\varphi : Y \to X$ is a homeomorphism. $\varphi^{\#} : \mathcal{O}_X \to \varphi_* \mathcal{O}_Y$ is surjective since φ is a closed embedding, and is injective by a result proven earlier. Thus φ is an isomoprhism.

To show Y is affine, let $I = \ker(I \to \mathcal{O}_Y(Y))$ and let $C = \operatorname{Spec} A/I$. Then we have the same factorisation $Y \to C \to X$ of φ and by the same argument C is the image of φ. Cover Y by open affines, whose images under φ are opens in C, which can be covered by principal opens in X. Thus we can find a finite open cover of principal opens $\{D(f_i)\}$ of X such that $\varphi^{-1}(D(f_i)) = D(\varphi^{\#}(f_i))$ is open affine in Y for all i. Then $\varphi^{\#}(f_i)$ generate $\mathcal{O}_Y(Y)$ so Y is affine by our affine criterion. □

Theorem 1.19. *Let $\varphi : Y \to X$ be a morphism of schemes. Then TFAE:*

1. *φ is a closed embedding.*

2. *φ is affine and for all $U = \operatorname{Spec} A$ open affines in X, let $B = \mathcal{O}_Y(\varphi^{-1}(U))$. Then $\varphi^{\#} : A \to B$ is surjective.*

3. *φ is affine and we can cover Y by open affines U_i such that $\varphi^\#(U)$: $\mathcal{O}_X(U) \to \mathcal{O}_Y(\varphi^{-1}(U))$ is surjective.*

Proof.

- 1 $\implies$ 2: closed embedding is local on base by exercise 3.
- 2 $\implies$ 3: obvious.
- 3 $\implies$ 1: being a topological closed embedding is local on base.

□

Note that it is possible to have non-isomorphic closed subschemes with the same closed subset, for example $\operatorname{Spec} k[t]/(t^n) \to \mathbb{A}^1_k$. This leads to the question whether any of them is more "natural" than others.

Definition (reduced scheme)**.** A scheme X is *reduced* if for all $U \subseteq X$ open the ring $\mathcal{O}_X(U)$ is has no nilpotents.

Exercise.

1. Let $X = \operatorname{Spec} A$. Then X is reduced if and only if A is reduced.
2. A scheme X is reduced if and only if there exists/for every affine open cover $\{U_i\}$, each U_i is reduced.

Theorem 1.20. *Let X be a scheme. Then exists a closed embedding i : $X_{\text{red}} \to X$ such that i is a homeomorphism and X_{red} is reduced. It has the universal property that all morphisms $\varphi : Y \to X$ from reduced scheme Y factorise through X_{red}.*

Proof. If $X = \operatorname{Spec} A$ then $X_{\text{red}} = \operatorname{Spec} A/\sqrt{(0)}$. For a general scheme X cover X by open affines and glue.

Note that if $i : X_{\text{red}} \to X$ has been constructed and $U \subseteq X$ open, then $U_{\text{red}} = (U, \mathcal{O}_{X_{\text{red}}}|_U)$ with morphism $j : U_{\text{red}} \to U$ given by identity on topological space and $j^\# : \mathcal{O}_U \to \mathcal{O}_{U_{\text{red}}}$ given by restricting $i^\#$ to U. □

1.5 Locally closed embedding

Definition (locally closed embedding)**.** A morphism of schemes $f : X \to Y$ is a *locally closed embedding* if it factors as $X \xrightarrow{i} U \xrightarrow{j} Y$ with i a closed embedding and j an open embedding.

2 Gluing in algebraic geometry

There are two types of gluing. We start from gluing sheaves.

Exercise. Let X and Y be topological spaces. For $U \subseteq X$ open, define $\mathcal{F}_Y(U) = \{\varphi : U \to Y \text{ cont}\}$ with restriction of functions. Show that $\mathcal{F}_Y$ is a sheaf of sets.

Lemma 2.1. *Let X be a toplogical space and $\mathcal{A}, \mathcal{B}$ sheaves of rings on X. Then*

$$\mathcal{F}(U) = \mathrm{Hom}_{\mathbf{ShRing}}(\mathcal{A}|_U, \mathcal{B}|_U)$$

is a sheaf of sets on X.

Proof. Restriction is restriction of sheaf morphisms and it is easy to see $\mathcal{F}$ is a presheaf. Let $\{U_i\}$ be an open cover and let $U_{ij} = U_i \cap U_j$. Assume $\varphi_i : \mathcal{A}|_{U_i} \to \mathcal{B}|_{U_i}$ is such that $\varphi_i|_{U_{ij}} = \varphi_j|_{U_{ij}}$ then we want to show there exists a unique $\varphi : \mathcal{A} \to \mathcal{B}$ restricting to φ_i. For all $V \subseteq X$ open, define $\varphi(V) : \mathcal{A}(V) \to \mathcal{B}(V)$ as follow. Since $\varphi|_{U_i} = \varphi_i$, we set

$$\varphi(a)|_{U_i \cap V} = \varphi_i(a|_{U_i \cap V}) =: b_i$$

Let $V_i = U_i \cap V$ and $\{V_i\}$ is an open cover of V. Since $V_{ij} = V \cap U_{ij}$,

$$b_i|_{V_{ij}} = \varphi_i(a|_{V_{ij}}) = \varphi_j(a|_{V_{ij}}) = b_j|_{Vij}.$$

Thus we are forced to set $\varphi(a) = b$ where $b \in \mathcal{B}(V)$ is uniquely specified by $b|_{V_i} = b_i$. Check that φ is compatible with restriction and $\varphi(V)$ is a ring map using sheaf axioms. $\square$

Exercise. Show that if $\mathcal{A}, \mathcal{B}$ are sheaves of sets (resp. R-modules) then $\mathcal{H}\wr\Updownarrow(\mathcal{A}, \mathcal{B})$ is a sheaf of sets (resp. R-modules).

Corollary 2.2. *Let X, Y be locally ringed spaces. Then the assignment*

$$\mathcal{F}_Y(U) = \mathrm{Hom}_{\mathbf{LRS}}(U, Y)$$

defines a sheaf of sets on X.

Proof. Exercise. Need $\varphi : U \to Y$ continuous map and $\varphi^{\#} : \varphi^{-1}\mathcal{O}_Y \to \mathcal{O}_U$ local homomorphism of sheaves of rings. $\square$

There is another kind of gluing, namely gluing like stacks. Suppose X is a topological space with open cover $\{U_i\}_i$. Suppose $\mathcal{A}_i$ is a sheaf of rings on U_i. Assume that for all i, j there is we are given $\varphi_{ij} : \mathcal{A}_i|_{U_{ij}} \to \mathcal{A}_j|_{U_{ij}}$ an isomorphism of sheaves of rings. We ask

1. is there a sheaf of rings $\mathcal{A}$ on X and isomorphisms $\varphi_i : \mathcal{A}|_{U_i} \to \mathcal{A}_i$ such that $\varphi_{ij} = \varphi_j|_{U_{ij}} \circ \varphi_j|_{U_{ij}}^{-1}$?
2. is $(\mathcal{A}, \{\varphi_i\})$ is unique?

For Q1, there is a clear necessary conidition. Let $U_{ijk} = U_i \cap U_j \cap U_k$ be the triple intersection. Then on U_{ijk} we need

$$\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}.$$

For Q2, there is a trivial way in which it is not unique: just take any $\tilde{\mathcal{A}}$ isomorphic to $\mathcal{A}$ with $\tilde{\varphi}_i$ composition with the isomorphism. But this is as but as it goes.

Theorem 2.3. *Let $\mathcal{A}_i, \varphi_{ij}$ be as before. Suppose for all i, j, k we have on U_{ijk}*

$$\varphi_{jk} \circ \varphi_{ij} = \varphi_{ik}$$

and $\varphi_{ii} = \mathrm{id}$ (called the cocycle condition*). Then exists a $(\mathcal{A}, \{\varphi_i\})$ as required. Furthermore it is unique in the sense that if $(\tilde{\mathcal{A}}, \{\tilde{\varphi}_i\})$ is another solution then exists a unique isomorphism $\alpha : \tilde{\mathcal{A}} \to \mathcal{A}$ such that $\tilde{\varphi}_i = \varphi_i \circ \alpha|_{U_i}$.*

Proof. Exercise. To get start with, define

$$\mathcal{A}(V) = \{(a_i) \in \prod \mathcal{A}_i(V \cap U_i) : \varphi_{ij}(a_i) = a_j\}.$$

Then use cocycle condition to show it is a sheaf. □

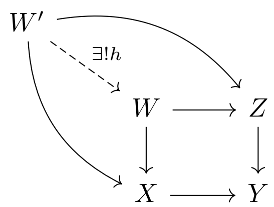
3 Fibre products

Note that in general fibre products do not exist: suppose $X = \{0\}, Y = \mathbb{R}^2, Z = \mathbb{R}, f(x) = x, g(x, y) = xy$, then $X \times_Z Y$ is not a manifold. It turns out this works if at least one of f, g is a submersion (to prove this need structure theorem for submersions).

Definition (cartesian diagram)**.** A commutative diagram

$$\begin{CD} W @>q>> Z \\ @VpVV @VVgV \\ X @>f>> Y \end{CD}$$

is *cartesian* if for every W' making the outer square commute there is a unique $W' \to W$ making the diagram commute



Exercise. In the category of sets the above diagram is cartesian if and only if $W \to X \times_Y Z = \{(x, z) \in X \times Z : f(x) = g(z)\}$ is bijective. Similar for topological spaces where $X \times_Y Z$ is given the subspace topology and h a homeomorphism.

Theorem 3.1. *Let $f : X \to Y, g : Z \to Y$ be a morphism of schemes. Then exists W a cartesian diagram of schemes.*

Definition (fibred product)**.** We say W is a *fibred product* of X and Z over Y and denote it by $X \times_Y Z$. Some other names inclue pullback, cartesian diagram, or that p is obtained from g via base change f.

We first show that if $f : X \to Y$ is an open immersion, let $U = f(X) \subseteq Y$, then $W = g^{-1}(U) \subseteq Z$ with $q : W \to Z$ the inclusion and $p = f^{-1} \circ g : W \to X$ is cartesian. Hint: show that

$$\begin{aligned} \operatorname{Hom}(S, X) &\to \{\varphi \in \operatorname{Hom}(S, Z) : \varphi(S) \subseteq U\} \\ \varphi &\mapsto f \circ \varphi \end{aligned}$$

is a bijection for all schemes S.

Next, we show the special case $X = \operatorname{Spec} A, Y = \operatorname{Spec} B, Z = \operatorname{Spec} C$, set $W = \operatorname{Spec} A \otimes_B C$. Use that the diagram

$$\begin{CD} B @>>> C \\ @VVV @VVV \\ A @>>> A \otimes_B C \end{CD}$$

is co-cartesian in the categroy of rings and the adjunction between global sections and Spec.

Proposition 3.2. *Let $f : X \to Y, g : Z \to Y$ be morphisms of schemes. Let $\{U_i\}$ be an open cover of X. Assume $U_i \times_Y Z$ exists for all i then $X \times_Y Z$ exists.*

Sketch proof. Define $W_i = U_i \times_Y Z, W_{ij} = U_{ij} \times_Y Z$ and let $\varphi_{ij} : W_{ij} \to W_{ji}$ be the isomorphism. Use cocycle conditions to glue. ☐

In the same spirit we have

Proposition 3.3. *Let $f : X \to Y$ and $g : Z \to Y$ be morphisms of schemes. Assume there exists a cover $\{U_i\}$ of Y such that $X_i \times_{Y_i} Z_i$ exists for all i where $X_i = f^{-1}(U_i), Z_i = g^{-1}(U_i)$. Then $X \times_Y Z$ exists.*

The we can show the fibre product $X \times_Y Z$ exists by showing the existence in the increasingly more general cases:

1. X, Y, Z are affine.
2. Y, Z are affine: cover X by open affines. Use 1 and glue.
3. Y affine: cover Z by open affines. Use 2 and glue.
4. general case: cover Y by open affines. Use 3 and glue.

We will see a more concrete constructions for schemes locally of finite type over an algebraically closed field.

Property local on base A property P of morphism of schemes is *local on base* if for all morphisms of schemes $f : X \to Y$, f has P if and only only for all open cover $\{U_i\}$ of Y, for all i, $g^{-1}(U_i) \to U_i$ has P.

Exercise. Show that if $f : X \to Y$ has a property which is local on base then for all $U \subseteq Y$, $g^{-1}(U) \to U$ has the same property.

Definition (property stable under base change)**.** Let P be a propery of morphisms. We say P is *stable under base change* if for any $g : Z \to Y$ with P and any morphism $f : X \to Y$, $q : X \times_Y Z \to X$ has P.

Example. Examples of properties stable under base change include: open immersion, affine morphism, closed immersion. For example for closed immersion, affine morphism is stable under base change, and tensor product is right exact so preserves surjection of rings, which are local models of closed immersions.

Note that if a property P is local on base and stable under base change then a morphism $f : X \to Y$ has P if and only if exists an open cover $\{U_i\}$ of Y such that for all i, $f^{-1}(U_i) \to U_i$ has P.

4 Separated and proper schemes

To motivate the definition of separated and proper schemes, prove the following statements:

1. X is a Hausdorff if and only if $\Delta_X : X \to X \times X$ is closed.

2. X compact if and only if for all Y, $X \times Y \to Y$ is closed.

Let $f : X \to Y$ be a morphism of schemes. Then the commutative diagram

$$\begin{array}{ccc} X & \longrightarrow & X \\ \downarrow & & \downarrow f \\ X & \xrightarrow{f} & Y \end{array}$$

induces a morphism $\Delta_X : X \to X \times_Y X$.

Exercise. If X is a manifold and $f : X \to *$ then $\Delta_X : X \to X \times X$ is the diagonal map.

Lemma 4.1. *Let $\varphi : X \to Y$ be a morphism of affine schemes. Then $\Delta : X \to X \times_Y X$ is a closed embedding.*

Proof. Let $X = \operatorname{Spec} B, Y = \operatorname{Spec} A$. Then $\Delta^\# : B \otimes_A B \to B, b_1 \otimes b_2 \mapsto b_1 b_2$ is surjective. □

Corollary 4.2. *Let $\varphi : X \to Y$ be a morphism of schemes. Then $\Delta : X \to X \times_Y X$ is a locally closed embedding.*

Proof. First show as an exercse that exists an affine cover $\{U_i\}$ of X and an affine cover $\{V_i\}$ of Y such that $\varphi(U_i) \subseteq V_i$ for all i. Let $W_i = U_i \times_Y U_i = U_i \times_{V_i} U_i$. Then $\Delta^{-1}(W_i) = U_i$. Let $W = \bigcup W_i \subseteq Y$ open. Then Δ factors as

$$X \xrightarrow{\alpha} W \hookrightarrow X \times_Y X.$$

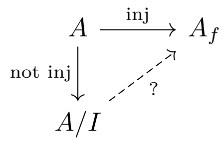
α is a closed embedding since $\alpha^{-1}(W_i) = U_i$ is affine and $U_i \to W_i$ is a closed embedding by the lemma. □

Definition (separated morphism)**.** A morphism $\varphi : X \to Y$ is *separated* if Δ is a closed embedding.

Definition. Let X be a scheme and $U \subseteq X$ open. U is *dense* in X if there does not exists a proper closed subscheme such that $U \hookrightarrow X$ factors through Z.

Example. Let $X = \operatorname{Spec} A$ where A is a domain. Then every nonempty $U \subseteq X$ is dense: indeed suffice to show this for nonempty $D(f)$, i.e. $f \neq 0$. Let $Z =$

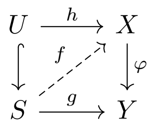
Spec A/I be a proper subscheme where $I \neq 0$. But there does not exists a map $A/I \to A_f$ making the following diagram commute



The same argument shows that for any reduced scheme X, an open subset $U \subseteq X$ is dense if and only if it is topologically dense.

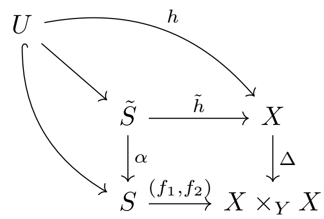
Example. Let $A = k[x, y]/(xy, y^2)$. $X = \operatorname{Spec} A$ is the line with thickening at the origin. Let $B = A_x = k[x, x^{-1}]$. Then $U = D(x) \subseteq X$ is not dense as it factors through $Z = \operatorname{Spec} A/(y)$. Note however U is dense as a topological space.

Proposition 4.3. *Let $\varphi : X \to Y$ be separated. Let S be a scheme and $U \subseteq S$ dense. Let $f_1, f_2 : S \to X$ such that $\varphi \circ f_1 = \varphi \circ f_2$ and $f_1|_U = f_2|_U$ then $f_1 = f_2$. In other words, in the following commutative square*



there exists at most one f making the diagram commute.

Proof. Consider



where the lower square is cartesian. Δ is a closed embedding and so is α. Since U is dense in S, α is an isomophism so $f_1 = \tilde{h} \circ \alpha^{-1} = f_2$. □

Example. Let $k = \overline{k}, U_1 = \operatorname{Spec} k[t], U_2 = \operatorname{Spec} k[s]$. Let $U_{12} = D(t) \subseteq U_1 = \operatorname{Spec}[t, t^{-1}], U_{21} = D(s) \subseteq U_2 = \operatorname{Spec} k[s, s^{-1}]$. Let $\varphi_{12} : U_{12} \to U_{21}$ be the isomorphism given by $\varphi_{12}^{\#}(s) = t$. Glue U_1 and U_2 along φ_{12} to obtain X.

$X \times_k X$ is covered by four copies of $\mathbb{A}_k^2$, $W_{ij} = U_i \times_k U_j$. Δ_X is closed in $W_{11} \cup W_{22}$, but the closure of Δ_X in $X \times_Y X$ is $\Delta_X \cup \{(0_1, 0_2), (0_2, 0_1)\}$ (do it locally on W_{12}, then W_{21} by symmetry). Thus $X \to \operatorname{Spec} k$ is not separated.

This is also witnessed by the lifting property: consider $U = \mathbb{A}_k^1 \setminus \{0\} \subseteq S = \mathbb{A}_k^1$. Then we can set $f_1(0) = 0_1, f_2(0) = 0_2$.

Note that the example works for any choice of schemes $U_1 \cong U_2$ and $U_{12} \subseteq U_1$ open dense subscheme. In fact, this can be used to test separatedness.

Theorem 4.4 (valuative criterion for separatedness)**.** *Let $\varphi : X \to Y$ be a morphism of schemes. Then φ is separated if and only if for every commutative square*

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & \overset{f}{\nearrow} & \downarrow \varphi \\ S & \longrightarrow & Y \end{array}$$

there exists at most one f making the diagram commute. Here S is the spectrum of a valuation ring and $U \subseteq S$ its generic point.

There is another version that holds for φ of finite type over an algebraically closed field k, where S is a reduced irreducible nonsingular curve over k and $U = X \setminus \{p\}$ where p is a closed point.

We will prove the second version later. Note that it can be written as $S = \operatorname{Spec} \mathcal{O}_{C,p}$ with C a nonsingular reduced irreducible curve over k and $p \in C$ a closed point, $U = \operatorname{Spec} K(\mathcal{O}_{C,p})$ the quotient field. $\mathcal{O}_{C,p}$ is a (discrete) valuation ring.

Exercise. Use the valuative criterion to show that composition of separated schemes is separated, and $\varphi : X \to Y$ is separated if and only if $\varphi_{\text{red}} : X_{\text{red}} \to Y_{\text{red}}$ is separated.

Another way to show that the composition $X \to Y \to Z$ of two separated morphisms is separated is to show that the square in the below diagram is cartesian

$$\begin{array}{ccccc} X & \xrightarrow{\Delta_f} & X \times_Y X & \longrightarrow & X \times_Z X \\ & & \downarrow & & \downarrow \\ & & Y & \xrightarrow{\Delta_g} & Y \times_Z Y \end{array}$$

Lemma 4.5. *Suppose $f : X \to Y, g : Y \to Z$ are morphisms of schemes and $h = g \circ f$. If h and g are separated then f is separated.*

Proof. Using the same diagram, since the base change of Δ_h is a closed embedding, Δ_f is closed so a closed embedding. □

Example.

1. Construct the differential geometric analogue of the line with two origins using $U_1 = U_2 = \mathbb{R}, U_{12} = U_{21} = \mathbb{R} \setminus \{0\}$ and $\varphi_{12} = \text{id}$. Show the resulting space X is a manifold except that it is not Hausdorff.

2. Show that if X is a manifold except perhaps not Hausdroof then X is Hausdorff if and only if every sequence $\{x_n\}_n$ has at most one limit. Rephrase to look like the valuative criterion with $Y = *, U = \{\frac{1}{n} : n \in \mathbb{N}\} \subseteq \mathbb{R}$ and $S = \{0\} \cup U$.

Definition (universally closed morphism)**.** A morphism of schemes is *universally closed* if every base change is closed.

Example. Let $X = \operatorname{Spec} k[t]$ where $k = \overline{k}$. Let $Y = \operatorname{Spec} k$. $\varphi : X \to Y$ induced by $k \hookrightarrow k[t]$. Then φ is closed but not universally closed. Take $Z = \operatorname{Spec} k[s]$ and $\psi : Z \to Y$ induced by $k \hookrightarrow k[s]$. $U = \operatorname{Spec} k[s,t]/(st-1)$ is closed in $X \times_Y Z$ but $q(U) = Z \setminus \{0\}$ is not closed.

Let $i : U \to Z$ be the restriction of q. Show i is an open embedding. Let $h : U \to X$ be the restriction of $p : X \times_Y Z \to X$. Show that there does not exist $f : Z \to X$ making the diagram commute

$$\begin{array}{ccc} U & \xrightarrow{h} & X \\ \downarrow{\scriptstyle i} & \overset{f}{\nearrow} & \downarrow{\scriptstyle \varphi} \\ Z & \longrightarrow & Y \end{array}$$

Theorem 4.6 (valuative criterion for universally closed morphism)**.** *Let $\varphi : X \to Y$ be a morphism of schemes. Then φ is separated if and only if for every commutative diagram*

$$\begin{array}{ccc} U & \longrightarrow & X \\ \downarrow & \overset{f}{\nearrow} & \downarrow{\scriptstyle \varphi} \\ S & \longrightarrow & Y \end{array}$$

there exists a morphism f making the diagram commute. Here S is the spectrum of a valuation ring and $U \subseteq S$ its generic point.

There is another version that holds for φ of finite type over an algebraically closed field k, where S is a reduced irreducible nonsingular curve over k and $U = X \setminus \{p\}$ where p is a closed point.

Exercise. Again with the manifold analogy, let $U = \{\frac{1}{n}\}, S = U \cup \{0\}$. Show that a smooth map $\varphi : X \to Y$ of manifolds is proper in the topological sense if for all commutative diagrams of continuous maps

$$\begin{array}{ccc} U & \xrightarrow{h} & X \\ \big\downarrow & & \downarrow{\scriptstyle \varphi} \\ S & \xrightarrow{g} & Y \end{array}$$

there exist an infinite subset $U_1 \subseteq U$ and a continuous map $f : S_1 := U_1 \cup \{0\} \to X$ such that $\varphi \circ f = g_{S_1}, \varphi|_{U_1} = h|_{U_1}$.

$$\begin{array}{ccccc} U_1 & \longrightarrow & U & \xrightarrow{h} & X \\ \downarrow & & \overset{f}{\big\downarrow} & \nearrow & \downarrow{\scriptstyle \varphi} \\ S_1 & \longrightarrow & S & \xrightarrow{g} & Y \end{array}$$

Note that in algebraic geometry there is also a valuative criterion that looks like this.

Proposition 4.7. *Let $f : X \to Y, g : Y \to Z$ be morphisms of schemes and $h = g \circ f$. If h is unviersally closed and g is separated then f is universally closed.*

Proof. Let $\varphi : S \to Y$ and $\psi = g \circ \varphi : S \to Z$. We have a commutative diagram with cartesian squares

$$\begin{array}{ccccc} & & X \times_Y S & \longrightarrow & Y \\ & & \downarrow & & \downarrow \Delta_g \\ X & \longleftarrow & X \times_Z S & \xrightarrow{(f,\varphi)} & Y \times_Z Y \\ \downarrow h & & \downarrow & & \\ Z & \underset{\psi}{\longleftarrow} & S & & \end{array}$$

□

Exercise. To remember the hyposthesis and conclusion of the proposition, show that any continuous map from a (quasi)compact topological space X to a Hausdorff space Y is universally closed.

Definition (proper morphism)**.** A morphism $\varphi : X \to Y$ of schemes is *proper* if it is separated and universally closed.

Lemma 4.8. *Let $f : X \to Y, g : Y \to Z$ be morphisms of schemes and $h = g \circ f$. If h is proper and g is separated then f is proper.*

In complex manifolds, the best examples of compact ones that are closed submanifolds of $\mathbb{CP}^n$. Fact: they are all algebraic varieties!

Definition (projective space)**.** Let A be a ring and $n \geq 1$. We define the *projective n-space* $\mathbb{P}^n_A$ as follows. Let $U_i = \operatorname{Spec} A[x^i_0, \dots, x^i_n]/(x^i_i - 1) \cong \mathbb{A}^n_A$ for $0 \leq i \leq n$. Let $U_{ij} \subseteq U_i$ be $D(x^i_j)$. Let $\varphi_{ij} : U_{ij} \to U_{ji}$ by

$$\varphi^\#_{ij}(x^j_m) = \frac{x^i_m}{x^i_j}$$

for $0 \leq m \leq n$. Note $\varphi^\#_{ij}(x^j_j) = 1, \varphi^\#_{ij}(x^j_i) = \frac{x^i_i}{x^i_j} = \frac{1}{x^i_j}$ is invertible so $\varphi^\#_{ij}$ is well-defined. The data $\{U_i, \varphi_{ij}\}$ satisfies the cocycle conditions and the glued scheme is called $\mathbb{P}^n_A$.

By convention we define $\mathbb{P}^0_A = \operatorname{Spec} A = U_0 = \operatorname{Spec} A[x^0_0]/(x^0_0 - 1)$.

Exercise. Let A be a ring and $n \in \mathbb{N}$. Define a morphism $\pi : \mathbb{P}^n_A \to \operatorname{Spec} A$ by defining $\pi_i : U_i \to \operatorname{Spec} A$ to be the morphism given by $\pi^\#_i : A \hookrightarrow A[x^i_0, \dots, x^i_n]/(x^i_i - 1)$. Show that π_i's are compatible and hence glue to give π.

Definition. Let X a scheme. We define

$$\mathbb{P}^n_X = \mathbb{P}^n_{\mathbb{Z}} \times_{\operatorname{Spec} \mathbb{Z}} X$$

and $\pi : \mathbb{P}^n_X \to X$ the projection.

Exercise. Show there is a canonical isomorphism $\mathbb{P}^n_{\operatorname{Spec} A} \cong \mathbb{P}^n_A$ that is compatible with π.

5 Sheaves of modules

The differential geometric analogue of the subject of this chapter is vector bundle. For E a vector bundle over a manifold M, let $\mathcal{E}$ be the sheaf of sections of E. Then for all $U \subseteq M$ open, $\mathcal{E}(V)$ is a sheaf over $C^\infty(U)$. Furthermore the module map is compatible with restrictions of sheaves.

Definition (sheaf of modules)**.** Let $(X, \mathcal{O}_X)$ be a ringed space. A *sheaf of $\mathcal{O}_X$-module* is a sheaf of abelain groups $\mathcal{F}$ on X plus $\mathcal{O}_X(U)$-module structure on $\mathcal{F}(U)$ for all $U \subseteq X$ open, such that if $V \subseteq U$ then for all $f \in \mathcal{O}_X(U), s \in \mathcal{F}(U)$, $(f \cdot s)|_V = f|_V \cdot s|_V$.

Example. $\mathcal{O}_X$ is a sheaf of $\mathcal{O}_X$-modules. If $\mathcal{F}, \mathcal{G}$ are sheaf of $\mathcal{O}_X$-modules then so is $\mathcal{F} \oplus \mathcal{G}$. In particular $\mathcal{O}_X^{\oplus r}$ is a sheaf of $\mathcal{O}_X$-module.

Definition (morphism of sheaf of modules)**.** A *morphism of $\mathcal{O}_X$-modules* $\varphi : \mathcal{F} \to \mathcal{G}$ is a morphisms of sheaves such that for all $U \subseteq X$ open, $\mathcal{F}(U) \to \mathcal{G}(U)$ is a morphism of $\mathcal{O}_X(U)$-modules.

Example. If φ is a morphism of sheaf of $\mathcal{O}_X$-modules then $\ker \varphi, \operatorname{im} \varphi$ and $\operatorname{coker} \varphi$ are sheaves of $\mathcal{O}_X$-modules.

Remark. If $(X, \mathcal{O}_X)$ is a ringed spae and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-module, then for all $U \subseteq X$ open $\mathcal{F}|_U$ is a sheaf of $\mathcal{O}_X|_U$-module.

Exercise. Let X be a ringed space, $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-modules. Show that the map

$$\begin{aligned} \operatorname{Hom}_{\mathcal{O}_X}(\mathcal{O}_X, \mathcal{F}) &\to \mathcal{F}(X) \\ \varphi &\mapsto \varphi(X)(1) \end{aligned}$$

is bijective.

Definition ((locally) free sheaf)**.** Let $(X, \mathcal{O}_X)$ be a locally ringed space, $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-module. $\mathcal{F}$ is *free of rank r* if it is isomorphic to $\mathcal{O}_X^{\oplus r}$. It is *locally free of rank r* if exists an open cover $\{U_i\}$ of X such that $\mathcal{F}|_{U_i}$ is free of rank r for all i.

Exercise. Let M be a manifold and E a rank r vector bundle on M. Then

1. $\mathcal{E}$ is locally free of rank r.
2. If $\mathcal{E}$ is locally free of rank r then exists E a vector bundle of rank r on M such that $\mathcal{E} \cong \mathcal{F}$.
3. $\mathcal{E}$ is free of rank r if and only if E is trivial.

The aim of this chapter is to show

Theorem 5.1. *Let A be a ring, M an A-module. Then there exists a sheaf of abelian groups $\widetilde{M}$ on the topological space of $X = \operatorname{Spec} A$ such that*

1. $\widetilde{M}(D(f)) \cong M_f$,
2. $\widetilde{M}_{\mathfrak{p}} \cong M_{\mathfrak{p}}$,
3. $\widetilde{M}$ *is a sheaf of $\mathcal{O}_X$-modules,*
4. $\mathbf{Mod}_A \to \mathbf{Mod}_{\mathcal{O}_X}, M \mapsto \widetilde{M}$ *defines a fully faithul exact functor.*

Let A be a ring and $X = \operatorname{Spec} A$ as a topological space. Given M an A-module, we define a presheaf $\mathfrak{M}$ by

$$U \mapsto \prod_{\mathfrak{p} \in U} M_{\mathfrak{p}}$$

and the restriction maps projections.

Exercise. Show that $\mathfrak{M}$ is a sheaf of A-modules on X.

Definition. We define the presheaf (which is a sheaf of $\mathcal{O}_X$-modules) $\widetilde{M}$ to be

$$U \mapsto \left\{ \begin{array}{c} (m_{\mathfrak{p}}) \in \mathfrak{M}(U)\text{: for all } \mathfrak{p} \in U \text{, exists } m \in M, f \in A \\ \text{such that } f \notin \mathfrak{p} \text{ and for all } \mathfrak{q} \in U \cap D(f)\text{, } m_{\mathfrak{q}} = \frac{m}{f} \end{array} \right\}.$$

Exercise.

1. Show $\widetilde{M}(U)$ is an A-submodule of $\mathfrak{M}(U)$.
2. Show the restriction map $\mathfrak{M}(U) \to \mathfrak{M}(V)$ maps $\widetilde{M}(U)$ to $\widetilde{M}(V)$.
3. Show $\widetilde{M}$ is a subsheaf of A-module of $\mathfrak{M}$. In particular $\widetilde{M}$ is a sheaf.
4. Let $g \in A$ and $B = A_g, Y = D(g) = \operatorname{Spec} B, N = M_g$. Show that there is a natural isomorphism $\widetilde{M}|_Y \to \widetilde{N}$ induced by $M_{\mathfrak{p}} \cong N_{\mathfrak{p}}$.
5. There exists a homomorphism of A-modules $M \to \widetilde{M}(U), m \mapsto (m_{\mathfrak{p}})$ where $m_{\mathfrak{p}} = \frac{m}{1}$ which is compatible with restriction.

Theorem 5.2.

1. *The natural map $M \to \widetilde{M}(X)$ is an isomorphism of A-modules.*
2. *For all $g \in A$, exists a natural isomorphism of A_g-modules $M_g \to \widetilde{M}(D(g))$ that is compatible with the map in 1.*
3. *For all $\mathfrak{q} \in X$,*

$$\begin{aligned} \widetilde{M}_{\mathfrak{q}} &\to M_{\mathfrak{q}} \\ (U, (m_{\mathfrak{p}})) &\mapsto m_{\mathfrak{q}} \end{aligned}$$

is an isomorphism of A-modules.

Proof.

1. Let the map in 1 be α. Suppose $m \in \ker \alpha$, so for all $\mathfrak{p}$ exists $f \notin \mathfrak{p}$ such that $fm = 0$. Consider

$$I = \{f \in A : fm = 0\} = \mathrm{Ann}(m)$$

Then $I \nsubseteq \mathfrak{p}$ for all $\mathfrak{p}$ so $I = A$ and it follows that $m = 0$.

To show surjectivity, let $(m_\mathfrak{p}) \in \widetilde{M}(X)$. We can cover X by opens $D(f_1), \dots, D(f_n)$ such that exists $m_1, \dots, m_n \in M$ with $m_\mathfrak{p} = \frac{m_i}{f_i}$ for all $\mathfrak{p} \in D(f_i)$. For all i, j, $\frac{m_i}{f_i} = \frac{m_j}{f_j} \in M_{f_i f_j}$ (by injectivity of α) so can find s large enough such that

$$f_j^s f_i^s (f_i m_j - f_j m_i) = 0$$

in M for all i, j. Let $g_i = f_i^{s+1}, \tilde{m}_i = f_i^s m_i$ so $g_i \tilde{m}_j = g_j \tilde{m}_i$ and $\frac{m_i}{f_i} = \frac{\tilde{m}_i}{g_i}$. Note $D(g_i) = D(f_i)$, and $D(f_i)$'s cover X so exists $a_1, \dots, a_n$ such that $\sum a_i g^i = 1$. Let $m = \sum a_i \tilde{m}_i$. Then

$$g_i m = \sum a_j g_i \tilde{m}_j = \sum a_j g_j \tilde{m}_i = \tilde{m}_i$$

so $\alpha(m)$ is the given section.

2. Follows from exercise 4.

3. Let the map in 3 be β. We construct an inverse γ: for $\frac{m}{f} \in M_\mathfrak{q}$, let $\gamma(\frac{m}{f}) = (D(f), \alpha(\frac{m}{f}))$. $\beta(\gamma(\frac{m}{f})) = \frac{m}{f}$ by definition. Let $u = (D(f), (m_\mathfrak{p})) \in \widetilde{M}_\mathfrak{q}$. Exist g such that $\mathfrak{q} \in D(g)$ and $u = (D(fg), \alpha(\frac{m}{f^r g^r}))$ so

$$\gamma(\beta(u)) = \gamma(\frac{m}{f^r g^r}) = (D(fg), \alpha(\frac{m}{f^r g^r})) = u.$$

□

Exercise. Show that $\widetilde{A}$ is a sheaf of A-algebra.

Corollary 5.3. *We define the structure sheaf $\mathcal{O}_X$ to be $\widetilde{A}$.*

Proposition 5.4. *For every A-module M, $\widetilde{M}$ is a sheaf of $\mathcal{O}_X$-modules.*

Definition (quasicoherent sheaf)**.** A sheaf $\mathcal{F}$ of $\mathcal{O}_X$-modules on a scheme X is *quasicoherent* if for every $U = \operatorname{Spec} A \subseteq X$ affine open, there exists an A-module M such that $\mathcal{F}|_U \cong \widetilde{M}$.

Lemma 5.5. *Let X be a scheme, $\mathcal{F}$ a quasicoherent sheaf of $\mathcal{O}_X$-module. Let $Y \subseteq X$ be open. Then $\mathcal{F}|_Y$ is also quasicoherent.*

Proposition 5.6. *Let $X = \operatorname{Spec} A$ be affine, $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-module. Let $M = \mathcal{F}(X)$. Then exists a natural morphism $\widetilde{M} \to \mathcal{F}$.*

Sketch proof. It is enough to define the morphism $\varphi : \widetilde{M} \to \mathcal{F}$ on $D(f)$. Suppose $u = \frac{m}{f^r} \in \widetilde{M}(D(f))$. Let $\varphi(u) = \frac{1}{f^r} \cdot m|_{D(f)}$. □

Corollary 5.7. *Let X be a scheme, $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-module. If exists an open cover $\{U_i\}$ of X such that each $\mathcal{F}|_{U_i}$ is quasicoherent then $\mathcal{F}$ is quasicoherent.*

Proof. Suffice to show for the case $X = \operatorname{Spec} A$ and $U_i = D(f_i)$. Let $M = \ker(\prod \mathcal{F}(D(f_i)) \rightrightarrows \prod \mathcal{F}(D(f_i f_j)))$. Claim that $\widetilde{M} \cong \mathcal{F}$: by sheaf axiom we have an exact sequence

$$0 \longrightarrow \mathcal{F}(D_f) \longrightarrow \prod \mathcal{F}(D_f \cap D(f_i)) \rightrightarrows \prod \mathcal{F}(D_f \cap D(f_i) \cap D(f_j))$$

But since each $\mathcal{F}|_{D(f_i)}$ is quasicoherent, $\mathcal{F}(D_f \cap D(f_i)) = \mathcal{F}(D(f_i))_f$ so $\mathcal{F}(D_f) \cong M_f$. □

Proposition 5.8. *Let $\mathcal{F}, \mathcal{G}$ be quasicoherent sheaves on a scheme X. Then $\mathcal{F} \oplus \mathcal{G}$ is quasicoherent. Moreover for any morphism $\alpha : \mathcal{F} \to \mathcal{G}$, $\ker \alpha, \operatorname{im} \alpha, \operatorname{coker} \alpha$ are quasicoherent and the functor $\tilde{\cdot} : \mathbf{Mod}_A \to \mathbf{Mod}_{\mathcal{O}_{\operatorname{Spec} A}}$ is exact.*

Proof. It is enough to assume $X = \operatorname{Spec} A$. Then the first statement follows from

$$\widetilde{M} \oplus \widetilde{N} = \widetilde{M \oplus N}.$$

For the second statement, let $\varphi : M \to N$ be induced by α. Let $K = \ker \varphi$. Then the natural homomorphism $\widetilde{K} \to \ker \varphi$ is an isomorphism since it is an isomorphism on all stalks. Same for image and cokernel. □

Corollary 5.9. *Every locally free sheaf is quasicoherent.*

Proposition 5.10. *Let X be a scheme and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-module. Then $\mathcal{F}$ is quasicoherent if and only if for all $U \subseteq X$ open and for all $f \in \mathcal{O}_X(U)$, $\mathcal{F}(U)_f \to \mathcal{F}(V)$ is an isomorphism where $V = \{x \in U : f(x) \neq 0\}$.*

Proof.

- $\Longleftarrow$: for $U \subseteq X$ open affine, let $M = \mathcal{F}(U)$. Then $\widetilde{M}(D(f)) \cong \mathcal{F}(D(f))$ is an isomorphism for all $f \in \mathcal{O}_X(U)$ so $\mathcal{F}|_U \cong \widetilde{M}$.

- $\Longrightarrow$: let $U \subseteq X$ open and take an open affine cover $\{U_i\}$ of U. Let $\{W_{ijk}\}$ be an open affine cover of $U_i \cap U_j$. Sheaf axiom says that

$$0 \longrightarrow \mathcal{F}(U) \longrightarrow \prod_i \mathcal{F}(U_i) \rightrightarrows \prod_{i,j,k} \mathcal{F}(W_{ijk})$$

is exact. Localisation is exact so the sequence remains exact after applying $- \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X(U)_f$.

□

Proposition 5.11. *Let $X = \operatorname{Spec} A$ be an affine scheme. Then $\tilde{\cdot}$ gives a fully faithful essentially surjective exact functor $\mathbf{Mod}_A \to \mathbf{Qcoh}(X)$.*

Proof. The map

$$\begin{aligned} \alpha : \operatorname{Hom}(M, N) &\to \operatorname{Hom}(\widetilde{M}, \widetilde{N}) \\ \varphi &\mapsto ((m_\mathfrak{p}) \mapsto (\varphi_\mathfrak{p}(m_\mathfrak{p}))) \end{aligned}$$

has a left inverse β by taking global section. To show β is injective, if $\beta(\psi) = 0$ then for all $f \in X$, $\psi(D(f)) : M_f \to N_f$ is zero so ψ is zero. Exactness can be checked on stalks. □

Definition (coherent sheaf)**.** Let X be a locally noetherian scheme. A quasicoherent sheaf $\mathcal{F}$ on X is *coherent* if and only if exists an open cover $\{U_i\}$, where $U_i = \operatorname{Spec} A_i$ with A_i noetherian, such that $\mathcal{F}|_{U_i} \cong \widetilde{M_i}$ where M_i is a finitely generated A_i-module.

Note that we only defined coherent sheaf on a locally noetherian scheme here. There is a more general definition of coherent sheaf for ringed spaces (see e.g. EGA), which reduces to our definition in locally noetherian scheme case. The reason we avoid mentioning this definition here is that they don't behave well in the nonnoetherian case, and we rarely, if ever, encounter such a scheme in this course.

Proposition 5.12. *Let $X = \operatorname{Spec} A$ where A is noetherian and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-module. Then $\mathcal{F}$ is coherent if and only if $\mathcal{F} = \widetilde{M}$ with M a finitely generated A-module.*

Proposition 5.13. *Let $X = \operatorname{Spec} A$ and $\mathcal{F}$ a sheaf of $\mathcal{O}_X$-module, $M = \mathcal{F}(X)$ and $\{U_i = D(f_i)\}$ a finite open cover. If $\mathcal{F}|_{U_i}$ is quasicoherent for all i then $\mathcal{F}$ is quasicoherent. If A is Noetherian and $\mathcal{F}|_{U_i}$ is coherent for all i then $\mathcal{F}$ is coherent.*

Proof. 1 has been proven in proposition Proposition 5.10. For 2 let $M_i = \mathcal{F}(U_i)$. Since the cover is finite and each M_i is finitely generated, we can find finitely many elements in M that generate all M_i. Then they generate all stalks. □

Lemma 5.14. *Let $X = \operatorname{Spec} A$ where A is noetherian. Let M be an A-module. Then $\widetilde{M}$ is coherent if and only if M is finitely generated.*

Proof. For the nontrivial direction, need to show $\widetilde{M}(D(f))$ is finitely generated as A_f-module for all f. Let $A^{\oplus r} \twoheadrightarrow M$. Use exactness of localisation. □

Exercise. Let X be a locally noetherian scheme. Assume we have an exact sequence

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

of quasicoherent sheaves. Then if any two of them are coherent then so is the third.

Let $\varphi : X \to Y$ be a morphism of schemes. Suppose $\mathcal{F}$ be a sheaf of $\mathcal{O}_X$-module. Then $\varphi^\#$ makes $\varphi_*\mathcal{F}$ a sheaf of $\mathcal{O}_Y$-modules.

Proposition 5.15. *In this setup if $\mathcal{F}$ is quasicoherent then so is $\varphi_*\mathcal{F}$.*

Proof. Let $\mathcal{G} = \varphi_*\mathcal{F}$. Let $V \subseteq Y$ open and $f \in \mathcal{O}_Y(V)$. Let $V_1 = \{y \in V : f(y) \neq 0\}$. We would like to show $\mathcal{G}(V)_f \to \mathcal{G}(V_1)$ is an isomorphism. Let $U = \varphi^{-1}(V)$ and $U_1 = \{x \in U : g(x) \neq 0\}$ where $g = \varphi^\#(f)$. Then $U_1 = \varphi^{-1}(V_1)$. We have a commutative diagram

$$\begin{array}{ccc} \mathcal{F}(U)_g & \xrightarrow{\cong} & \mathcal{F}(U_1) \\ \big\downarrow = & & \big\downarrow = \\ \mathcal{G}(V)_f & \longrightarrow & \mathcal{G}(V_1) \end{array}$$

where for the vertical arrow on the left we use the fact if M a B-module and $A \to B$ is a ring map such that $f \mapsto g$, then

$$\begin{aligned} ({}_AM)_f &\to M_g \\ \frac{m}{f^n} &\mapsto \frac{m}{g^n} \end{aligned}$$

is an isomorphism of abelian groups. □

One might expect the same to work with coherent sheaves, but this fails in general. Consider the following example. If $X = \operatorname{Spec} k$ for k is a field then a sheaf of $\mathcal{O}_X$-module is exactly a k-vector space. Every sheaf of $\mathcal{O}_X$-module is quasicoherent, and is coherent if and only if it is finite dimesional. However consider $\varphi : \mathbb{A}^n_k \to \operatorname{Spec} k$. Then $\varphi_*(\mathcal{O}_{\mathbb{A}^n_k})$ is not coherent because $k[x_1, \dots, x_n]$ is not a finite dimensional k-vector space if $n \geq 1$.

5.1 Ideal sheaf and closed subscheme

We have defined closed embedding but have not define closed subscheme. We will do so via quasicohernet sheaves.

Definition (ideal sheaf)**.** Let X be a scheme and $\varphi : Y \to X$ a closed embedding. We define the *ideal sheaf* to be the kernel of $\varphi^\# : \mathcal{O}_X \to \varphi_*\mathcal{O}_Y$.

It is denoted by $\mathcal{I}_\varphi, \mathcal{I}_{Y/X}$ or $\mathcal{I}_Y$.

Lemma 5.16. *If X is locally noetherian then so is Y and $\mathcal{I}_Y$ is coherent.*

Proof. Cover X by $U_i = \operatorname{Spec} A_i$ where A_i is noetherian. Let $V_i = \varphi^{-1}(U_i)$ where $V_i = \operatorname{Spec} B_i$. As $\varphi^\# : A_i \twoheadrightarrow B_i$, B_i is also noetherian. $\mathcal{I}_Y(U_i)$ is an ideal in a noetherian ring so finitely generated. ☐

Proposition 5.17. *Let X be a scheme and $\mathcal{I} \subseteq \mathcal{O}_X$ a quasicoherent subsheaf. Then exists a closed embedding $\varphi : Y \to X$ such that $\mathcal{I}_Y = \mathcal{I}$.*

Sketch proof. As a set let

$$Y = \{x \in X : \mathcal{I}_x \to \mathcal{O}_{X,x} \text{ not an isomorphism}\}$$

with induced topology. Let $\varphi : Y \to X$ be inclusion. Let $\mathcal{O}_Y = \varphi^{-1}(\mathcal{O}_X/\mathcal{I})$. ☐

Lemma 5.18. *Let A, B, C be rings, $\varphi : A \to B, \psi : A \to C$ surjective ring maps. Then $\alpha : B \to C$ such that $\psi = \alpha \circ \varphi$ is unique if it exists. It exists if and only if* $\ker \varphi \subseteq \ker \psi$.

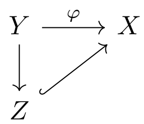
Corollary 5.19. *Let X, Y, Z be closed schemes, $\varphi : Y \to X, \psi : Z \to X$ closed embeddings. Then $\alpha : Z \to Y$ such taht $\psi = \varphi \circ \alpha$ is unique if it exists, it exists if and only if $\mathcal{I}_Y \subseteq \mathcal{I}_Z$.*

Corollary 5.20. *Two closed embedding are isomorphic if and only if $\mathcal{I}_Y \cong \mathcal{I}_Z$.*

Definition (closed subscheme)**.** A *closed subscheme* of X is an isomorphism class of closed embeddings. Equivalently it is a quasicoherent ideal sheaf on X.

Definition (closed subscheme image)**.** Let $\varphi : Y \to X$ be a morphism of schemes. The *closed subscheme image* of φ is the subscheme defined by $\mathcal{I}_\varphi = \ker(\varphi^\# : \mathcal{O}_X \to \varphi_* \mathcal{O}_Y)$.

Exercise. It is the smallest closed subscheme $Z \hookrightarrow X$ such that there exists a commutative diagram (necessarily unique)



i.e. if φ factors through another closed embedding $W \hookrightarrow X$ then there is a unique morphism $Z \to W$.

(? Y is dense in Z and contains a dense open subset. Constructible set)

Proposition 5.21. *Let $\varphi : Y \to X$ be a locally closed embedding, i.e. it is of the form $Y \xrightarrow{closed} U \xrightarrow{open} X$. Let Z be the closed subscheme image of φ. Then the diagram*

$$\begin{array}{ccc} Y & \longrightarrow & U \\ \downarrow & & \downarrow \\ Z & \longrightarrow & X \end{array}$$

is cartesian. In particular $Y \hookrightarrow Z$ is open and φ can be written as $Y \xrightarrow{open} Z \xrightarrow{closed} X$.

Proof. Let $W = U \cap Z = U \times_X Z$ so we have a commutative diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[rr, bend left, "\text{closed}"] & W \arrow[r] \arrow[d] & U \arrow[d] \\ & Z \arrow[r] & X \end{tikzcd}$$

For all $V \subseteq U$ open we have

$$\begin{array}{ccccccc} & & \mathcal{I}_{W/U}(V) & & & & \\ & & \| & & & & \\ 0 & \longrightarrow & \mathcal{I}_{Z/X}(V) & \longrightarrow & \mathcal{O}_X(V) & \longrightarrow & \mathcal{O}_Y(\varphi^{-1}(V)) \\ & & \downarrow & & \| & & \| \\ 0 & \longrightarrow & \mathcal{I}_{Y/U}(V) & \longrightarrow & \mathcal{O}_U(V) & \longrightarrow & \mathcal{O}_Y(\varphi^{-1}(V)) \end{array}$$

where one can verify that $\mathcal{I}_{Z/X}|_U = \mathcal{I}_{W/U}$. It follows that $\mathcal{I}_{W/U} \cong \mathcal{I}_{Y/U}$ so $Y \cong W$. $\square$

Exercise. Conversely, let $Z \hookrightarrow X$ be a closed embedding and $Y \hookrightarrow Z$ be an open embedding. Then $Y \hookrightarrow Z \hookrightarrow X$ is locally closed.

Corollary 5.22. *Let $\varphi : Y \to X$ be a locally closed embedding. Then it is a closed embedding if and only if $\varphi(Y)$ is a closed subet of X.*

Lemma 5.23. *Let X be a scheme, S an affine scheme and $\varphi : X \to S$ a separated morphism. If $U, V \subseteq X$ are open affines then so is $U \cap V$.*

"separated scheme over affine schemes are analogous to Hausdorff spaces"

Proof. Have

$$U \cap V = U \times_X V = (U \times_S V) \times_{X \times_S X} X$$

Δ is closed so $U \times_X V \to U \times_S V$ is a closed embedding. $U \times_S V$ is affine and hence so is $U \times_X V$. $\square$

Example. For an example of a scheme X with open affines U, V whose intersection is not affine, do the following: glue two copies of $\mathbb{A}_k^2$ along $\mathbb{A}_k^2 \setminus \{0\}$ using identity to obtain the plane with two origins. Take U and V to be the affine plane containing each origin. Show their intersection $\mathbb{A}_k^2 \setminus \{0\}$ is not affine by showing $\mathcal{O}(\mathbb{A}^2) \to \mathcal{O}(\mathbb{A}^2 - \{0\})$ is an isomorphism (cover $\mathbb{A}^2 - \{0\}$ by $D(x)$ and $D(y)$ and use sheaf axioms).

5.2 Global Spec

Recall that if $\varphi : X \to Y$ is a morphism of schemes then $\mathcal{A} = \varphi_* \mathcal{O}_X$ is a quasicoherent sheaf of algebras on Y. If $Y = \operatorname{Spec} B$ is affine then $\mathcal{A} \cong \widetilde{A}$ where $A = \mathcal{A}(Y) = \mathcal{O}_X(X)$. Let $Z = \operatorname{Spec} A$ and $\pi : Z \to Y$ induced by $B \to A$.

Exercise. Show exists a unique $\psi : X \to Z$ such that $\varphi = \pi \circ \psi$ and $\psi^\#$ is id_A. Show that if φ factors as $X \xrightarrow{\psi_1} Z_1 \xrightarrow{\pi_1} Y$ with π_1 affine then exists a unique $\alpha : Z \to Z_1$ such that $\pi = \pi_1 \circ \alpha, \psi_1 = \alpha \circ \psi$.

Theorem 5.24. *Let Y be a scheme, $\mathcal{A}$ a quasicoherent sheaf of algebras on Y. Then exists a unique scheme $X = \operatorname{Spec} \mathcal{A}$ and an affine morphism $\pi : X \to Y$ together with an isomorphism $\gamma : \pi_* \mathcal{O}_X \to \mathcal{A}$.*

Proof. Uniqueness is local on Y so we may reduce to the case Y affine, which is the exercise. For existence, cover Y by affines $\{Y_i\}$. Let $X_i = \operatorname{Spec} \mathcal{A}(Y_i)$ and $\pi_i : X_i \to Y_i$ the map induced by $\mathcal{O}_Y \to \mathcal{A}$. Let $Y_{ij} = Y_i \cap Y_j$ and $X_{ij} = \pi_i^{-1}(Y_{ij}) \subseteq X_i$. Then by uniqueness there exists a canonical isomorphism $\psi_{ij} : X_{ij} \to X_{ji}$ inducing the isomorphism

$$\mathcal{O}_{Y_i}(Y_{ij}) \to \mathcal{A}(Y_{ij}) \leftarrow \mathcal{O}_{Y_j}(Y_{ij}).$$

They satisfy the cocycle conditions and glue to a scheme X and $\pi : X \to Y$. □

5.3 Operations on sheaves of modules

Morally every operation we can do with modules, such as direct sum, tensor product, taking kernels etc have an analogue for sheaves of modules.

Suppose $\mathcal{F}, \mathcal{G}$ are sheaves of $\mathcal{O}_X$-modules. We define a sheaf $\mathcal{H}om(\mathcal{F}, \mathcal{G})$ of $\mathcal{O}_X$-modules by

$$U \mapsto \operatorname{Hom}_{\mathcal{O}_U}(\mathcal{F}|_U, \mathcal{G}|_U).$$

Exercise. Show that it is a sheaf of $\mathcal{O}_X$-modules. Show that if $\mathcal{F}, \mathcal{G}$ are both quasicoherent then so is $\mathcal{H}om(\mathcal{F}, \mathcal{G})$. Show that if X is locally noetherian and $\mathcal{F}, \mathcal{G}$ are coherent then so is $\mathcal{H}om(\mathcal{F}, \mathcal{G})$.

Given a sheaf of modules $\mathcal{F}$, we define its dual to be

$$\mathcal{F}^\vee = \mathcal{H}om(\mathcal{F}, \mathcal{O}_X).$$

Exercise. If $\mathcal{E}, \mathcal{F}$ are locally free (of finite rank), show that there is a natural isomorphism of locally free sheaves

$$\mathcal{H}om(\mathcal{E}, \mathcal{F}) \to \mathcal{H}om(\mathcal{F}^\vee, \mathcal{E}^\vee).$$

We define $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ to be the sheaf associated to the presheaf

$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U).$$

If $\mathcal{F}, \mathcal{G}$ are quasicoherent then so is $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ and for all $U \subseteq X$ affine

$$\mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{G}(U) \to (\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G})(U)$$

is an isomorphism.

Exercise. Show that if X is locally noetherian, $\mathcal{F}, \mathcal{G}$ are coherent then $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ is coherent.

Exercise. Let X be a ringed space, $\mathcal{E}, \mathcal{F}, \mathcal{G}$ sheaves of $\mathcal{O}_X$-modules. Show that there is a natural homomorphism

$$\mathcal{H}om(\mathcal{E}, \mathcal{F}) \otimes_{\mathcal{O}_X} \mathcal{H}om(\mathcal{F}, \mathcal{G}) \to \mathcal{H}om(\mathcal{E}, \mathcal{G})$$

given by composition.

Exercise.

1. Let X be a ringed space, $r \geq 1$. Show that for every sheaf of $\mathcal{O}_X$-module $\mathcal{F}$, the presheaf

$$U \mapsto \mathcal{F}(U) \otimes_{\mathcal{O}_X(U)} \mathcal{O}_X^{\oplus r}(U) \cong \mathcal{F}(U)^{\oplus r}$$

   is a sheaf.

2. Let $\mathcal{E}$ be a locally free sheaf of rank r. Show its dual $\mathcal{E}^\vee$ is locally free of rank r. Show that there exists a natural homomorphism $\mathcal{E}^\vee \otimes \mathcal{E} \to \mathcal{O}_X$, which is an isomorphism if $r = 1$.

Definition (invertible sheaf)**.** Let X be a ringed space. An *invertible sheaf* on X is a locally free sheaf of rank 1.

Definition (Picard group)**.** The *Picard group* $\mathrm{Pic}(X)$ of X is the set of isomorphism classes of invertible sheaves on X.

Exercise. $\mathrm{Pic}(X)$ is an abelian group with multiplication given by tensor product and identity $\mathcal{O}_X$.

Definition (effective Cartier divisor)**.** Let X be a scheme. An *effective Cartier divisor* on X is a closed subscheme D such that $\mathcal{I}_D$ is an invertible sheaf. We denote $\mathcal{O}_X(-D) = \mathcal{I}_D$ and $\mathcal{O}_X(D) = \mathcal{I}_D^\vee$.

Exercise. If D is an effective Cartier divisor on a scheme X, show that the inclusion $\mathcal{I}_D \to \mathcal{O}_X$ induces a morphism $\mathcal{O}_X \to \mathcal{O}_X(D)$ and hence a section $s_D \in \Gamma(X, \mathcal{O}_X(D))$.

Definition (simple cyclic cover)**.** Let X be a scheme, $\mathcal{L}$ an invertible sheaf, $n \geq 1$, $s \in \Gamma(X, (\mathcal{L}^\vee)^{\otimes n})$ corresponds to $\varphi : \mathcal{L}^{\otimes n} \to \mathcal{O}_X$, which is assumed to be injective. Let $\mathcal{A}_i = \mathcal{L}^{\otimes i}$ and $\mathcal{A} = \bigoplus_{i=0}^{n-1} \mathcal{A}_i$. Make $\mathcal{A}$ into a $\mathbb{Z}/n\mathbb{Z}$-graded algebra with product

$$\mathcal{L}^{\otimes i} \otimes \mathcal{L}^{\otimes j} \to \begin{cases} \mathcal{L}^{\otimes(i+j)} & i+j<n \\ \mathcal{L}^{\otimes(i+j)} \xrightarrow{\varphi\otimes\mathrm{id}} \mathcal{O}_X \otimes \mathcal{L}^{\otimes i+j-n} \cong \mathcal{L}^{\otimes(i+j-n)} & i+j \geq n \end{cases}$$

Let $D \subseteq X$ be the effective Cartier divisor such that $\mathcal{I}_D = \varphi(\mathcal{L}^{\otimes n})$. The scheme $Y = \operatorname{Spec} \mathcal{A}$ is called the *(simple) cyclic cover* of order n of X with *branch divisor* D.

5.4 Line bundles and divisors

Definition (Krull dimension)**.** The *Krull dimension* of a topological space X is the supremum of n such that there is a chain of irreducible closed subsets

$$Z_0 \subsetneq Z_1 \subsetneq \cdots Z_n \subseteq X.$$

Example. For $\mathbb{A}_k^n$ we have

$$\{0\} \subsetneq \mathbb{A}_k^1 \subsetneq \mathbb{A}_k^2 \subsetneq \cdots \subsetneq \mathbb{A}_k^n$$

so $\dim \mathbb{A}_k^n \geq n$. Using Noether normalisation one can show $\dim \mathbb{A}_k^n = n$. By going up/down every chain of irreducible closed subsets can be extended to one of length n.

Example.

1. Let $X = \mathbb{A}_k^n$ where $k = \overline{k}$. Then a prime divisor is precisely $\operatorname{Spec} k[x_1, \dots, x_n]/(f)$ where $f \neq 0$ is irreducible. To see this since $k[x_1, \dots, x_n]$ is a UFD, (f) is prime if and only if f is irreducible. If $f \neq 0$ is irreducible then it has height one: if $0 \subseteq \mathfrak{p} \subseteq (f)$ then for all $g \in \mathfrak{p}$, $(g) \subseteq (f)$ so either $g = 0$ or equality. Conversely if $\mathfrak{p}$ has height 1 then choose $g \in \mathfrak{p}$ nonzero. Then exists an irreducible factor f of g such that $f \subseteq \mathfrak{p}$, so equality.

2. Let $X = \mathbb{P}_k^n$ where $k = \overline{k}$. Then a prime divisor is precisely $\operatorname{Proj} k[x_0, \dots, x_n]/(f)$ (consider moving this elsewhere) where $f \in k[x_0, \dots, x_n]$ irreducible homogeneous.

   Proof. Every closed subset of X can be described as $\bigcap Z(g_i)$ for some $g_1, \dots, g_r$ homogeneous. f irreducible if and only if all dehomogenisation is either irreducible or constant. Being reduced, irreducible and dimension are local properties. Conversely, suppose Z is an effective prime divisor and wlog $D(x_0) \cap Z \neq \emptyset$. Then $Z \cap D(x_0) = \operatorname{Spec} k[x_1^0, \dots, x_n^0]/(g)$ where g is irreducible. Let f be the homogenisation of g and $W = \operatorname{Proj} k[x_0, \dots, x_n]/(f)$. W is reduced and irreducible and $W \cap D(x_0) = Z \cap D(x_0)$. Thus $W = \overline{W \cap D(x_0)} = \overline{Z \cap D(x_0)} = Z$ where both have the reduced closed subscheme structure (why does this hold topologically?) $\square$

Let X be a reduced irreducible separated noetherian scheme. Let $n = \dim X$. Let $\eta \in X$ be the generic point and $K(X) = \mathcal{O}_{X,\eta}$ be its function field.

Definition (effective prime divisor)**.** An *(effective) prime divisor* is a closed reduced irreducible subscheme of X of dimension $n-1$.

Assumption: every effective prime divisor on X is Cartier. For example from above it is true for $\mathbb{A}^n_k$ and so it is true for every X with a cover by $\mathbb{A}^n_k$.

Definition (divisor, effective divisor)**.** Let X be a scheme. The *group of divisors* on X, $\operatorname{Div}(X)$, is the free abelian group generated by prime divisors. An element $D = \sum a_i D_i \in \operatorname{Div}(X)$ is called a *divisor*, and is *effective* if $a_i \geq 0$ for all i.

The assignment on prime divisors

$$\begin{aligned} \operatorname{Div}(X) &\to \operatorname{Pic}(X) \\ D &\mapsto \mathcal{O}_X(D) \end{aligned}$$

extends uniquely to a group homomorphism.

Assumption: for all $D \subseteq X$ prime divisor with generic point η_D, we require $\mathcal{O}_{X,\eta_D}$ to be a DVR. Then we have a valuation $v_D : K(X)^* \to \mathbb{Z}$ and

$$\begin{aligned} \mathcal{O}_{X,\eta_D} &= \{f \in K(X)^* : v_D(f) \geq 0\} \cup \{0\} \\ \mathfrak{m}_{\eta_D} &= \{f \in K(X)^* : v_D(f) > 0\} \cup \{0\} \end{aligned}$$

Lemma 5.25. *Let $f \in K(X)^*$. Then there are only finitely many prime divisors such that $v_D(f) \neq 0$ is finite.*

Proof. As X is noetherian it has a finite cover by open affines. If $U \cap X$ is open then there is a bijection

$$\begin{aligned} \{D \subseteq X \text{ prime divisor}, D \cap U \neq \emptyset\} &\leftrightarrow \{D_U \subseteq U \text{ prime divisor}\} \\ D &\mapsto D \cap U \\ \overline{D}_U &\leftarrow\!\shortmid D_U \end{aligned}$$

so it is enough to prove in case X affine. Let $X = \operatorname{Spec} A$ so $K(X) = K(A)$. As if $f = \frac{g}{h} \in K(A)^*$ then $D(f) = v_D(g) - v_D(h)$, it is enough to assume $f \in A$. Note $v_D(f) \neq 0$ if and only if $f(\eta_D) = 0$ if and only if $D \subseteq Z(f)$. If $f \neq 0$ then $Z(f) \subsetneq X$ so $D \subseteq Z(f) \subsetneq X$ implies D is an irreducible component of $Z(f)$. A is noetherian implies that $Z(f)$ has finitely many irreducible components. $\square$

We can thus define a group homomorphism

$$\begin{aligned} \operatorname{div} : K(X)^* &\to \operatorname{Div}(X) \\ f &\mapsto \sum_{D \text{ prime}} v_D(f) \cdot [D] \end{aligned}$$

Definition. Let $\mathcal{O}_X^*$ be the sheaf

$$U \mapsto \{f \in \mathcal{O}_X(U) : f(p) \neq 0 \text{ for all } p \in U\}.$$

Example. $\mathcal{O}^*_{\mathbb{A}^n_k}(\mathbb{A}^n_k) = k^*$.

Theorem 5.26. *Let X be a scheme satisfying all assumptions so far. Then we have an exact sequence of abelian groups*

$$0 \longrightarrow \mathcal{O}_X^*(X) \longrightarrow K(X)^* \xrightarrow{\text{div}} \text{Div}(X) \xrightarrow{\mathcal{O}_X(-)} \text{Pic}(X) \longrightarrow 0$$

Proof. We introduce another group

$$\widetilde{\text{Pic}}(X) = \{(\mathcal{L}, s) : \mathcal{L} \text{ line bundle}, s \in \mathcal{L}_\eta \setminus \{0\}\}/\sim$$

where $(\mathcal{L}, s) \sim (\mathcal{L}', s')$ if and only if exists an isomorphism $\varphi : \mathcal{L} \to \mathcal{L}'$ such that $\varphi(s) = s'$. It is a group via tensor product. Claim the natural map $\pi : \widetilde{\text{Pic}}(X) \to \text{Pic}(X)$ is surjective: take a nonempty open $U \subseteq X$ with a trivialisation $\alpha : \mathcal{L}|_U \cong \mathcal{O}_U$. Then α induces $\mathcal{L}_\eta \cong \mathcal{O}_\eta$ and $\alpha^{-1}(1) \in \mathcal{L} \setminus \{0\}$. Note that α is unique up to $\mathcal{O}_\eta^* = K(X)^*$.

On the other hand

$$\begin{aligned} \ker p &= \{(\mathcal{O}_X, s) : \text{exists } \varphi : \mathcal{O}_X \to \mathcal{O}_X \text{ such that } \varphi(1) = s\}/\sim \\ &= \mathcal{O}_\eta^*/\{s \in \mathcal{O}_{X,\eta}^* : (\mathcal{O}_X, s) \cong (\mathcal{O}_X, 1)\}. \end{aligned}$$

A morphism $\varphi : \mathcal{O}_X \to \mathcal{O}_X$ is the same as (multiplication by) a global section f, and is an isomorphism when it induces isomorphisms on all stalks, i.e. $f(x) \neq 0$ for all $x \in X$. Thus $f \in \mathcal{O}_X^*$ and $\ker \pi = K(X)^*/\mathcal{O}_X^*(X)$.

It is then left to show $\text{Div}(X) \cong \widetilde{\text{Pic}}(X)$. If D is a prime divisor then by assumption we have a nonzero section $s_D \in \Gamma(X, \mathcal{O}_X(D))$. Then we extend the map $D \mapsto (\mathcal{O}_X(D), (s_D)_\eta)$ to a homomorphism $\text{Div}(X) \to \widetilde{\text{Pic}}(X)$. Conversely, suppose $(\mathcal{L}, s) \in \widetilde{\text{Pic}}(X)$. Fix D a divisor. Choose an trivialising open subset U such that $U \cap D \neq \emptyset$. Let $\varphi|_U : \mathcal{L}|_U \to \mathcal{O}_U$ be the trivialisation. We define $v_D(s) = v_D(\varphi_U(s))$. Check this is well-defined and we define

$$\begin{aligned} \widetilde{\text{Pic}}(X) &\to \text{Div}(X) \\ (\mathcal{L}, s) &\mapsto \sum_{D \text{ prime}} v_D(s) \cdot [D] \end{aligned}$$

As before this is a finite sum.

We show the composition gives identity on $\widetilde{\text{Pic}}(X)$. Note first that given $(\mathcal{L}, s) \in \widetilde{\text{Pic}}(X)$, the argument above shows that it has precisely one automorphism, given by $1 \in K(X)^*$ (while an invertible sheaf has non trivial automorphism, if we fix a stalk then it has none). Thus given $(\mathcal{L}, s), (\mathcal{L}', s') \in \widetilde{\text{Pic}}(X)$ an isomorphism between them is unique. Thus it suffcies to give local isomorphism $(\mathcal{L}, s) \to (\bigotimes_{i=1}^r \mathcal{O}_X(D_i)^{\otimes a_i}, \bigotimes_{i=1}^r t_i^{\otimes a_i})$, and they glue.

We may thus assume $X = \text{Spec}\, A$ and $\mathcal{L}|_X$ is trivial, i.e. $\mathcal{L}|_X \cong \mathcal{O}_X$ and $s = \frac{f}{g} \in K(X)^*$. By Krulls' principal ideal theorem $Z(f), Z(g)$ are unions of prime divisors. Let $D_1, \dots, D_r$ be all such divisors. wlog $\mathcal{I}_{D_i}$ is freely generated by $f_i \in A \setminus \{0\}$. Let $b_i = v_{D_i}(f), c_i = v_{D_i}(g)$. Claim $f = u \cdot \prod_{i=1}^r f_i^{b_i}$ with u invertible (by Krull), $g = v \prod_{i=1}^r f_i^{c_i}$. Up to isomorphism we may assume $u = v = 1$. Finally check. □

Corollary 5.27.

1. $\operatorname{Pic}(\mathbb{A}_K^n) = 0$ *so a prime divisor is exactly a prime ideal.*
2. $\operatorname{Pic}(\mathbb{P}_k^n) = \mathbb{Z}$.

5.5 Split sequence

Definition ((locally) split sequence of quasicoherent sheaves)**.** Let X be a scheme. An exact sequence of quasicoherent sheaves

$$0 \longrightarrow \mathcal{F}_1 \xrightarrow{f} \mathcal{F}_2 \xrightarrow{g} \mathcal{F}_3 \longrightarrow 0$$

splits if exists an isomorphism $\varphi : \mathcal{F}_2 \to \mathcal{F}_1 \oplus \mathcal{F}_3$ such that $\varphi \circ f$ is the inclusion of the first factor and g is the composition of φ and projection to the second factor. It *splits locally* if exists an open cover of X on which the sequence split.

Lemma 5.28. *If $\mathcal{F}_3$ is locally free then the sequence is splits locally*

Lemma 5.29. *Let A be a ring, $g : M_2 \to M_3$ a surjection between free modules of rank n and r respectively, then $M_1 = \ker g$ is locally free of rank $n - r$.*

Sketch proof. Fix isomorphisms $M_2 \cong A^{\oplus n}, M_3 \cong A^{\oplus r}$. Then g is given by an $(n \times r)$ matrix (g_{ij}). For each $I = (1 \le i_1 < \cdots < i_r \le n)$, let $a_I \in A$ be the determinant of the corresponding minor. g is surjective so for all $\mathfrak{p} \in \operatorname{Spec} A$ the localisation is surjection, and remains so after tensoring with $\kappa(\mathfrak{p})$. Thus eixsts I such that $a_I \in \kappa(\mathfrak{p})$ is nonzero. On $D(a_I)$, g splits so its kernel is free. □

Corollary 5.30. *Let X be a scheme. Suppose in the short exact sequence of quasicoherent sheaves*

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

$\mathcal{F}_2$ and $\mathcal{F}_3$ are locally free of rank n and r respectively. Then the sequence splits locally and $\mathcal{F}_1$ is locally free of rank $n - r$.

Lemma 5.31. *Let X be a scheme. Suppose*

$$0 \longrightarrow \mathcal{F}_1 \longrightarrow \mathcal{F}_2 \longrightarrow \mathcal{F}_3 \longrightarrow 0$$

is an exact sequence of locally free sheaves. Then

1. *for every* $\mathcal{G}$ *quasicoherent*

$$0 \longrightarrow \mathcal{F}_1 \otimes \mathcal{G} \longrightarrow \mathcal{F}_2 \otimes \mathcal{G} \longrightarrow \mathcal{F}_3 \otimes \mathcal{G} \longrightarrow 0$$

   is exact and splits locally.

2. *for every morphism* $\varphi : Y \to X$

$$0 \longrightarrow \varphi^* \mathcal{F}_1 \longrightarrow \varphi^* \mathcal{F}_2 \longrightarrow \varphi^* \mathcal{F}_3 \longrightarrow 0$$

   is exact and splits locally.

Proof. Cover X by affine opens on which the original sequence splits and is free. □

6 Differentials

Definition (Kähler differential)**.** Let $\varphi : A \to B$ be a morphism of rings. We define $\Omega_{B/A}$, the module of *Kähler differentials* to be the free B-module generated by $\mathrm{d}b, b \in B$ subject to the relations

$$\begin{aligned} \mathrm{d}(fg) &= f\mathrm{d}g + g\mathrm{d}f \\ \mathrm{d}(f+g) &= \mathrm{d}f + \mathrm{d}g \\ \mathrm{d}(\varphi(a)) &= 0 \text{ for } a \in A \end{aligned}$$

Definition (derivation)**.** Let M be a B-module. An A-*derivation* $\delta : B \to M$ is an A-linear map such that

$$\delta(fg) = f\delta(g) + g\delta(f).$$

We denote the space of such derivations by $\mathrm{Der}_A(B, M)$.

Exercise.

1. $\mathrm{Der}_A(B, M)$ is a B-submodule of $\mathrm{Hom}_A(B, M)$.
2. For any M there is a natural bijection

$$\begin{aligned} \mathrm{Der}_A(B, M) &\to \mathrm{Hom}_B(\Omega_{B/A}, M) \\ \delta &\mapsto (\mathrm{d}b \mapsto \delta(b)) \end{aligned}$$

3. Let $P_n = A[x_1, \dots, x_n]$. Define $\delta_i = \frac{\partial}{\partial x_i}$. Show $P_n \in \mathrm{Der}_A(P_n, P_n)$.
4. Show that $\Omega_{P_n/A}$ is generated by $\mathrm{d}x_1, \dots, \mathrm{d}x_n$.

Lemma 6.1. *$\Omega_{P_n/A}$ is the free P_n-module generated by $\mathrm{d}x_1, \dots, \mathrm{d}x_n$.*

Proof. They are generators by exercise 4. If there is a relation $\sum f_i \mathrm{d}x_i = 0$ then applying δ_i shows $f_i = 0$. □

Proposition 6.2. *Suppose $A \to B \xrightarrow{\pi} C$ are ring homomorphisms with π surjective with kernel I. Then there is an exact sequence of C-modules*

$$I \otimes_B C \xrightarrow{\alpha} \Omega_{B/A} \otimes_B C \xrightarrow{\beta} \Omega_{C/A} \longrightarrow 0.$$

Proof. Define

$$\begin{aligned} \alpha &: f \otimes g \mapsto \mathrm{d}f \otimes g \\ \beta &: \mathrm{d}f \otimes g \mapsto g\mathrm{d}\pi(g) \end{aligned}$$

Check these are well-defined. β is surjective since π is. Exactness at the second term is left as an exercise. □

Note as $C = B/I$, we can write the C-module $I \otimes_P C$ as I/I^2. In this notation $\alpha(f) = \mathrm{d}f \otimes 1$. One can check $\alpha(f_1 f_2) = 0$ for $f_1, f_2 \in I$ and $\alpha(gf_1) = g\alpha(f_1)$.

Corollary 6.3. *If $R = A[x_1, \ldots, x_n]/I$ then we have an exact sequence*

$$I/I^2 \longrightarrow \bigoplus_{i=1}^n R\mathrm{d}x_i \longrightarrow \Omega_{R/A} \longrightarrow 0.$$

Proposition 6.4. *Let R be a finitely generated A-algebra. Write $R \cong P_1/I_1 \cong P_2/I_2$ with P_1, P_2 free polynomial A-algebra. Let K_1, K_2 be defined by the exact sequence*

$$0 \longrightarrow K_i \longrightarrow I_i/I_i^2 \longrightarrow \Omega_{P_i/A} \otimes_{P_i} R \longrightarrow \Omega_{R/A} \longrightarrow 0$$

Then K_1 is canonically isomorphic to K_2.

Proof. Suppose $P_1 = A[x_1, \ldots, x_n], P_2 = A[y_1, \ldots, y_m]$, then there is a natural surjection $P_3 = A[x_1, \ldots, x_n, y_1, \ldots, y_m] \to R$ and we will show both K_1 and K_2 are isomorphic to K_3. To show this it suffices to show it for polynomial ring with one more variable. Namely we want to prove the following: if $\pi : P = A[x_1, \ldots, x_n] \to R$ surjective with kernel I, choose $r \in R$ and define

$$\begin{aligned} \pi' : P' = A[x_1, \ldots, x_n, y] &\to R \\ x_i &\mapsto \pi(x_i) \\ y &\mapsto r \end{aligned}$$

Let $I' = \ker \pi'$. Then we want to show $K \cong K'$.

As π is surjective can find $g \in P$ such that $\pi(g) = r$, and then $g - y \in \ker \pi'$. As a P-module $I' = (g - y)P' \oplus I$ so $I'/(I')^2 = (g - y)P'/I' \oplus I/I^2$. Then we have a commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & K & \longrightarrow & I/I^2 & \xrightarrow{\alpha} & \bigoplus_{i=1}^n R\mathrm{d}x_i \\ & & \downarrow & & \downarrow{\scriptstyle (\mathrm{id},0)} & & \downarrow{\scriptstyle (\mathrm{id},0)} \\ 0 & \longrightarrow & K' & \longrightarrow & I/I^2 \oplus R \cdot (g-y) & \xrightarrow{\begin{pmatrix} \alpha & * \\ 0 & \mathrm{id} \end{pmatrix}} & \bigoplus_{i=1}^n R\mathrm{d}x_i \oplus R\mathrm{d}y \end{array}$$

and it follows that the kernels are naturally isomorphic. □

In fancier language, we have proved that $I/I^2 \to \Omega_{P/A} \otimes_A R$ is well-defined up to a canonical quasi-isomorphism, so defines an object in $\mathbf{D}(\mathbf{Mod}_R)$. This is the *naïve cotangent complex*.

Definition (smooth homomorphism)**.** A ring homomorphism $A \to R$ is *smooth* if it is finitely presented, $K = 0$ and $\widetilde{\Omega}_{R/A}$ is locally free (i.e. $\Omega_{R/A}$ is finite projective).

Example.

1. $A \to A[x_1, \ldots, x_n]$ is smooth.
2. $\mathbb{C} \to \mathbb{C}[x, y]/(xy - 1) = R$ is smooth:

$$R \cdot (xy - 1) \xrightarrow{\alpha} R\mathrm{d}x \oplus R\mathrm{d}y \longrightarrow \Omega_{R/\mathbb{C}} \longrightarrow 0$$

As a vector space $R \cong \mathbb{C} \oplus x\mathbb{C}[x] \oplus y\mathbb{C}[y]$. Suppose $f = a + xb(x) + yc(y) \in R$ is such that $f(xy-1) \in \ker \alpha$. Then $fx\mathrm{d}y = fy\mathrm{d}x = 0$. This is the same as

$$\begin{aligned} ax + x^2 b(x) + c(y) &= 0 \\ ay + b(x) + y^2 c(y) &= 0 \end{aligned}$$

so $b(x) = c(y) = 0$ and it follows that $f = 0$. Thus α is an isomorphism onto its image, which is $\{f\mathrm{d}x + f\mathrm{d}y : f \in R\}$ so $\Omega_{R/\mathbb{C}}$ is free of rank 1.

Exercise. Determine if $\mathbb{C} \to \mathbb{C}[x,y]/(xy)$ is smooth.

Proposition 6.5. *Let A be a ring, B a finitely presented A-algebra. Then*

1. *If $S \subseteq A$ is multiplicative and B is smooth over A then $S^{-1}B$ is smooth over $S^{-1}A$.*
2. *Let $f_1, \dots, f_n \in A$ such that $\{D(f_i)\}$ covers $\operatorname{Spec} A$. Then if for all i, B_{f_i} is smooth over A_{f_i} then B is smooth over A.*
3. *Let $g \in B$. If B is smooth over A then so is B_g.*
4. *Let $g_1, \dots, g_m \in B$ such that $\{D(g_i)\}$ covers $\operatorname{Spec} B$. Then if for all j, B_{g_j} is smooth over A then B is smooth over A.*

Proof. Write $B = P/I$ where $P = A[x_1, \dots, x_N]$.

1. Localisation is exact and preserves local freeness. In fact this holds for any flat base change.
2. $(\ker \alpha)_{f_i} = 0$ for all i implies $\ker \alpha = 0$.
3. $B_g \cong B[y]/(yg-1) \cong A[x_1, \dots, x_N, g]/J$ where $J = (I, yg-1)$. Note

   $$J/J^2 = (I/I^2) \otimes_B B_g \oplus B_g(yg-1).$$

   Then we have a commutative diagram

   $$\begin{array}{ccccc} I/I^2 \otimes_B B_g & \xrightarrow{\alpha_g} & \bigoplus_{i=1}^N B_g \mathrm{d}x_i & \longrightarrow & \Omega_{B/A} \otimes_B B_g \\ \downarrow{\scriptstyle (\mathrm{id},0)} & & \downarrow{\scriptstyle (\mathrm{id},0)} & & \downarrow \\ (I/I^2) \otimes_B B_g \oplus B_g \cdot (g-y) & \xrightarrow{\beta = \begin{pmatrix} \alpha_g & * \\ 0 & g \end{pmatrix}} & \bigoplus_{i=1}^N B_g \mathrm{d}x_i \oplus B_g \mathrm{d}y & \longrightarrow & \Omega_{B_g/A} \end{array}$$

   As g is a unit in B_g,

   $$\ker \beta = \ker \alpha_g = 0, \Omega_{B_g/A} = \operatorname{coker} \beta = \operatorname{coker} \alpha_g = \Omega_{B/A} \otimes_B B_g.$$

4. $(\ker \alpha)_{g_i} = 0$ for all i by 3.

□

Definition (relative dimension, étale morphism)**.** Let A be a ring, B an A-algebra. We say B is *smooth A of relative dimension n* if it is smooth over A and $\widetilde{\Omega}_{B/A}$ is locally free of rank n.

As a special case, B is *étale* over A if it is smooth of relative dimension zero.

Definition (smooth morphism)**.** A morphism $\varphi : X \to Y$ of schemes is *smooth* (resp. *smooth of relative dimension n*, *étale*) if we can cover Y by open affines V_i such that for each i we can cover $\varphi^{-1}(V_i)$ by open affines U_j such that $\mathcal{O}_X(U_j)$ is smooth (resp. smooth of relative dimension n, étale) over $\mathcal{O}_Y(V_i)$.

Corollary 6.6. *$\varphi : X \to Y$ is smooth (resp. smooth of relative dimension n, étale) if and only if for all $V \subseteq Y$ open affine, for all $U \subseteq \varphi^{-1}(V)$ affine such that $\mathcal{O}_X(U)$ is smooth (resp. smooth of relative dimension n, étale) over $\mathcal{O}_Y(V)$.*

Proof. For the nontrivial direction, cover V by principal open affines which are also principal open in some V_i. Same for U. Now apply Proposition 6.5. □

Exercise. Open embeddings are étale.

Fact: any smooth morphism is open.

Definition (standard smooth)**.** Let A be a ring, B an A-algebra. Then B is *standard smooth* over A if exists a presentation $B \cong A[x_1, \dots, x_n]/(f_1, \dots, f_r)$ such that $\det \left(\frac{\partial f_i}{\partial x_j}\right)_{1 \le i,j \le r}$ maps to a unit in B.

Claim (Stacks 10.136)

1. If B is standard smooth over A, then B is smooth over A and $\Omega_{B/A}$ is freely generated by $\mathrm{d}x_{r+1}, \dots, \mathrm{d}x_n$.
2. If B is smooth over A we can cover $\operatorname{Spec} B$ by $D(h)$ such that B_h is standard smooth over A.

Example. $\mathbb{A}^n_X$ and $\mathbb{P}^n_X$ are smooth of relative dimension n over X. Suffice to check this for $X = \operatorname{Spec} A$ affine, in whice case $\mathbb{A}^n_X = \operatorname{Spec} A[x_1, \dots, x_n]$ and one easily checks it is smooth of relative dimension X. $\mathbb{P}^n_A$ has an affine cover by $\mathbb{A}^n_A$.

Fact: let X be a scheme smooth over $\operatorname{Spec} k$ where k is algebraically closed. Then X is reduced. If X is irreducible then every prime divisor D is Cartier. (smooth over a field implies locally factorial, implying every Weil divisor is Cartier)

Idea of proof: enough to show that for all $p \in D$ such that $\kappa(p) = k$ (because our scheme is locally finite type over a field, such points are dense) , $\mathcal{I}_{D,p}$ is a free module of rank 1. Let $\hat{\mathcal{O}}_{X,p} = \varprojlim \mathcal{O}_{X,p}/\mathfrak{m}_p^N \cong k[[x_1, \dots, x_n]]$. Every minimal nonzero prime ideal in $\hat{\mathcal{O}}_{X,p}$ is principal, $\hat{\mathcal{O}}_{X,p}$ is faithfully flat over $\mathcal{O}_{X,p}$.

so enough to show $\mathcal{I}_{D,p}$ is principal.

$\hat{\mathcal{I}}_{D,p} \subseteq \hat{\mathcal{O}}_{X,p}$ has a generator u in $\mathcal{O}_{X,p}$, i.e. $u : \mathcal{O}_{X,p} \to \mathcal{I}_{D,p}$ such that passing to formal completion is surjective.

Corollary 6.7. *Suppose $f \in S = k[x_0, \dots, x_n]$ is homogenous of degree e. Let $X = \operatorname{Proj} S/(f)$. Let $Y = \bigcap_{i=0}^n Z(\frac{\partial f}{\partial x_i})$. Then $X \setminus Y$ is a quasiprojective variety over k smooth of relative dimension over* $\operatorname{Spec} k$.

Proof. Before we start the proof note the identity

$$ef = \sum_{i=0}^n x_i \frac{\partial f}{\partial x_i}$$

which can be easily checked using linearity and induction. Take $p \in U_0 \cap X = \operatorname{Spec} k[y_1, \dots, y_n]/g$ where $g = f(1, y_1, \dots, y_n)$. Then $\frac{\partial g}{\partial y_j} = \frac{\partial f}{\partial x_j}(1, y_1, \dots, y_n)$. Suppose $p \notin Y$, so exists some j such that $\frac{\partial f}{\partial x_i}(p) \neq 0$. Claim that we can take $j \neq 0$, as otherwise

$$0 = ef(p) = x_0(p) \frac{\partial f}{\partial x_0}(p) \neq 0.$$

Let $h = \frac{\partial g}{\partial y_j}$. Then $p \in W = X \cap U_0 \cap \{h \neq 0\}$ and W has presentation

$$\operatorname{Spec} k[y_1, \dots, y_n, y_{n+1}]/(g, y_{n+1}h - 1)$$

so by rearranging the variables, we see

$$\begin{pmatrix} \frac{\partial g}{\partial y_j} & \frac{\partial g}{\partial y_{n+1}} \\ \frac{\partial (y_{n+1}h - 1)}{\partial y_j} & \frac{\partial (y_{n+1}h-1)}{\partial y_{n+1}} \end{pmatrix} = \begin{pmatrix} h & 0 \\ * & h \end{pmatrix}$$

has unit determinant in W. □

Theorem 6.8. *Let $\varphi : X \to Y$ be a smooth morphism (resp. smooth of relative dimension n, étale) of schemes. Then any base change of φ is also smooth (resp. smooth of relative dimension n, étale).*

Proof. Being smooth is local on both source and target so we can assume $Y = \operatorname{Spec} A$ and $X = \operatorname{Spec} B$ where B is a smooth A-algebra. After localising we can further assume B is standard smooth over A, say $B = A[x_1, \dots, x_n]/(f_1, \dots, f_r)$.

Let A' be any A-algebra and define $B' = B \otimes_A A'$. Then $B' = A'[x_1, \dots, x_n]/(\tilde{f}_1, \dots, \tilde{f}_r)$ where $\tilde{f}_i$ is the image of f_i in $A'[x_1, \dots, x_n]$. $\frac{\partial f_i}{\partial x_j}$ is mapped to $\frac{\partial \tilde{f}_i}{\partial x_j}$. It is then enough to note that $B \to B'$, as any ring map, maps invertible elements to invertible elements. □

Exercise. Give an alternative proof using the definition of smoothness. Hint:

$$0 \longrightarrow I/I^2 \longrightarrow \Omega_{P/A} \otimes_P B \longrightarrow \Omega_{B/A} \longrightarrow 0$$

splits locally and I/I^2 is locally free.

To summarise, let $\varphi : X \to Y$ be a morphism of schemes. For any $U = \operatorname{Spec} B \subseteq X, V = \operatorname{Spec} A \subseteq Y$ such that $\varphi(U) \subseteq V$, we have an induced quasicoherent sheaf on U, namely $\widetilde{\Omega}_{B/A}$. We showed that $\Omega_{B/A}$ commutes with localisation in both A and B. We want to prove that there is a (unique up to isomorphism) way to define a quasicoherent sheaf Ω_φ or $\Omega_{X/Y}$ together with a morphism $d : \mathcal{O}_X \to \Omega_{X/Y}$ of sheaves of $\varphi^{-1}(\mathcal{O}_Y)$-modules such that

1. (local on source) for every $U \subseteq X$, let $\tilde{\varphi} = \varphi|_U$. Then there is an isomorphism $\Omega_\varphi|_U \cong \Omega_{\tilde{\varphi}}$ such that

$$\begin{array}{ccc} \mathcal{O}_X|_U & \xrightarrow{d|_U} & \Omega_\varphi|_U \\ \| & & \downarrow \cong \\ \mathcal{O}_U & \xrightarrow{d} & \Omega_{\tilde{\varphi}} \end{array}$$

   commutes.

2. (local on target) if $V \subseteq Y$ is open and $\varphi(X) \subseteq V$, let $\psi : X \to V$ be the induced morphism. Then there is an isomorphism $\Omega_\varphi \cong \Omega_\psi$ such that

$$\begin{array}{ccc} \mathcal{O}_X & \xrightarrow{d} & \Omega_\varphi \\ & \searrow^{d} & \downarrow \cong \\ & & \Omega_\psi \end{array}$$

   commutes.

3. (agree with Kähler differentials for affines) if $X = \operatorname{Spec} B, Y = \operatorname{Spec} A$, then there is an isomorphism $\Omega_{X/Y} \cong \widetilde{\Omega}_{B/A}$ such that

$$\begin{array}{ccc} \mathcal{O}_X(X) & \xrightarrow{d} & \Omega_{X/Y}(X) \\ \| & & \downarrow \cong \\ B & \xrightarrow{d} & \Omega_{B/A} \end{array}$$

   commutes.

Note that the isomorphisms are unique if they exist, since the image of B under d generates $\Omega_{B/A}$.

If such an object exists then we immediately know that

Corollary 6.9. *Suppose $f : X \to Y, g : Y \to Z$ are morphisms of schemes. Then there is a natural exact sequence*

$$f^*\Omega_{Y/Z} \longrightarrow \Omega_{X/Z} \longrightarrow \Omega_{X/Y} \longrightarrow 0$$

If f is a closed embedding then exists a natural exact sequence

$$f^*\mathcal{I}_{X/Y} \xrightarrow{d} f^*\Omega_{Y/Z} \longrightarrow \Omega_{X/Z} \longrightarrow 0$$

There are two ways to prove this:

1. mimic the definition of $\widetilde{M}$ on Spec A. Afterall we know that $\Omega_\varphi(U)$ should be for every $U \subseteq X$ open affine such that $\varphi(U)$ is contained in an open affine in Y.

2. via normal bundle of diagonal.

Lemma 6.10. *Let $f : Z \to W$ be a locally closed embedding. Write $f = h \circ g$ where $g : Z \to U$ closed embedding, $h : U \to W$ open embedding. Then $g^*\mathcal{I}_{Z/U}$ is well-defined. We write it as $f^*\mathcal{I}_{Z/W}$.*

Proof. Use closed subscheme image... $g^*\mathcal{I}_{Z/U} = g^*h^*\mathcal{I}_{\overline{Z}/\overline{W}} = f^*I_{\overline{Z}/\overline{W}}$. □

Definition (sheaf of Kähler differentials)**.** Let $\varphi : X \to Y$ be a morphism of schemes. Define

$$\Omega_{X/Y} = \Delta_\varphi^*\mathcal{I}_{X/X\times_Y X}.$$

For $f \in \mathcal{O}_X(U)$, define

$$\mathrm{d}f = \Delta_\varphi^*(\pi_1^* f - \pi_2^* f)$$

where $\pi_1, \pi_2 : X \times_Y X \to X$ are the natural projections.

Exercise. This satisfies the desired properties.

How to think about this: for smooth maps, dualising the second exact sequence in the corollary gives interpretation in terms of normal bundle.

6.1 Another interpretation for smoothness

Let $(A, \mathfrak{m})$ be a local ring with maximal ideal. Assume $\bigcap_{n\geq 0} \mathfrak{m}^n = 0$. Define the *completion* of A to be

$$\hat{A} = \varprojlim_n A/\mathfrak{m}^n.$$

Exercise.

1. Suppose $A = k[x_1, \dots, x_n]_{(x_1,\dots,x_n)}$. Then $\hat{A} = k[[x_1, \dots, x_n]]$.
2. If $f_1, \dots, f_r \in (x_1, \dots, x_n)$ and $B = A/(f_1, \dots, f_r)A$ then $\hat{B} = k[[x_1, \dots, x_n]]/(f_1, \dots, f_r)$.
3. Implicit function theorem holds for $k[[x_1, \dots, x_n]]$.
4. $k[x_1, \dots, x_n]/(f_1, \dots, f_r)$ is smooth of relative dimension $(n-r)$ near the origin if and only if $k[[x_1, \dots, x_n]]/(f_1, \dots, f_r) \cong k[[t_1, \dots, t_{n-r}]]$.
5. $\hat{A}$ is a faithfully flat A-algebra.

Let $\varphi : X \to Y$ be a morphism of schemes locally of finite type over an algebraically closed field k. Let $p \in X$ be a closed point. Then φ is smooth of relative dimension r at p ant $t_1, \dots, t_r \in \mathcal{O}_{X,p}$ are such that $\Omega_{\varphi,p}$ is the free module generated by $\mathrm{d}t_i$ if and only if $\varphi^\#$ induces an isomorphism

$$\begin{aligned} \hat{\mathcal{O}}_{Y,\varphi(p)}[[z_1, \dots, z_r]] &\to \hat{\mathcal{O}}_{X,p} \\ z_i &\mapsto t_i \end{aligned}$$

Fact: if X is a scheme smooth over a field k then every prime divisor is Cartier.

Definition (tangent space)**.** Suppose X is a scheme locally of finite type over an algebraically closed field k and $p \in X$ a closed point. The *tangent space* of X at p is

$$\begin{aligned} T_pX &= \mathrm{Der}_k(\mathcal{O}_{X,p}, \kappa(p)) \\ &= \mathrm{Hom}_{\mathcal{O}_{X,p}}(\Omega_{X/k,p}, \kappa(p)) \\ &= \mathrm{Hom}_k(\Omega_{X/k,p} \otimes_{\mathcal{O}_{X,p}} \kappa(p), \kappa(p)) \end{aligned}$$

Proposition 6.11. *There is a natural bijection*

$$T_pX \longleftrightarrow \{\varphi : \operatorname{Spec} k[\varepsilon]/\varepsilon^2 \to X : \varphi_{t=0} = p\}.$$

Proof. The statement is local on X so suppose $X = \operatorname{Spec} R$. A closed point p gives an R-algebra structure to $k = \kappa(p)$. Note $k[\varepsilon]/\varepsilon^2 = k \oplus k\varepsilon$ as a vector space map. One can check a k-linear map

$$\begin{aligned} \varphi : R &\to k[\varepsilon]/\varepsilon^2 \\ f &\mapsto f(p) + \lambda(f)\varepsilon \end{aligned}$$

is a homomorphism if and only if λ is a derivation. □

Corollary 6.12.

$$\dim T_pX = \dim \Omega_{X,p} \otimes \kappa(p).$$

Proposition 6.13. *Let X be a scheme locally of finite type over k, $Y \subseteq X$ a closed subscheme such that $\mathcal{I}_{Y/X}$ is principal. Then for all $p \in Y$ closed,*

$$\dim T_pX \geq \dim T_pY \geq \dim T_pX - 1.$$

Proof. wlog $X = \operatorname{Spec} R, Y = \operatorname{Spec} R/(f)$. Let $i : Y \to X$ be the closed embedding. In the short exact sequence

$$i^*\mathcal{I}_{Y/X} \longrightarrow i^*\Omega_{X/k} \longrightarrow \Omega_{Y/k} \longrightarrow 0$$

taking stalk at p and tensoring $\kappa(p)$ we get

$$i^*\mathcal{I}_{Y/X,p} \otimes \kappa(p) \longrightarrow (T_pX)^\vee \longrightarrow (T_pY)^\vee \longrightarrow 0$$

Since the leftmost vector space has dimension 1 the result follows. □

Example. We give an example of a prime divisor which is not Cartier. Let $X = \operatorname{Spec} \mathbb{C}[x,y,z]/(xy - z^2), Y = \operatorname{Spec} \mathbb{C}[x,y,z]/(x,z)$. Let p be the origin. $T_pX = \ker(T_p\mathbb{A}^3 \xrightarrow{\mathrm{d}(xy-z^2)} \mathbb{C}) = T_p\mathbb{A}^3$. $T_pY = \langle \frac{\partial}{\partial y} \rangle$ which is one-dimensional.

cf 2-Cartier

Another way to see X is integral: action of $\mathbb{Z}/2$ on $\mathbb{C}[U,V]$ via $U \mapsto -U, V \mapsto -V$. The invariants are $\mathbb{C}[U^2, V^2, UV]$. Can check the only relation is the obvious one. Being a subring of a domain, it is a domain. In terms of functions, this

Exercise: do for nth root of unity acting on $\mathbb{A}^2$. $a(u,v) = (au, av)$. Invariant polynomial ring generated by homogeneous monomials of degree n. Projective this is Veronese embdding of $\mathbb{P}^1$ in $\mathbb{P}^n$.

Another action is $a(u,v) = (au, a^{-1}v)$. Invariants are generated by u^n, v^n, uv. $xy = z^n$. This action is in $\mathrm{SL}(2,\mathbb{C})$. Related to Gorenstein singularity.

Note μ_n is a group scheme. However if $p = \operatorname{char} k$ divides n. μ_n is not reduced.

fact: in characteristic zero, group schemes are reduced, and in fact smooth over the base field.

In general if a group G acts on a ring R, we would like to define $\operatorname{Spec} R/G$ to be $\operatorname{Spec} R^G$, the invariant subring. In general the question of quotiening schemes by groups is very difficult. cf GIT

7 Projective schemes and morphisms

Recall that a morphism $Y \to X$ of schemes induces a cartesian diagram.

$$\begin{array}{ccc} \mathbb{P}^n_Y & \longrightarrow & \mathbb{P}^n_Y \\ \downarrow & & \downarrow \\ Y & \longrightarrow & X \end{array}$$

Definition (strictly projective and locally projective morphism)**.** A morphism of schemes $\varphi : X \to Y$ is called *strictly projective* if it factors as $X \xrightarrow{i} \mathbb{P}^n_Y \xrightarrow{\pi} Y$ with i a closed embedding. It is *locally projective* if exists an open cover $\{V_i\}$ of Y such that $\varphi|_{\varphi^{-1}(V_i)} : \varphi^{-1}(V_i) \to V_i$ is strictly projective for all i.

Exercise. Show that both properties are stable under base change.

Definition. A morphism $\varphi : X \to Y$ is called *strictly quasiprojective* if it factors as $X \xrightarrow{i} \mathbb{P}^n_Y \xrightarrow{\pi} Y$ with i a locally closed embedding.

Exercise. Define locally quasiprojective morphism and show both are stable under base change.

Theorem 7.1. *Given a scheme X and $n \geq 1$, the morphism $\pi : \mathbb{P}^n_X \to X$ is proper.*

Note that as properness is preserved under base change, we only have to prove the result for affine schemes. If time permits we will discuss the proof.

Corollary 7.2.

1. *Every locally projective morphism is proper.*
2. *Every locally quasiprojective morphism is separated.*

Chow's lemma gives a sort of converse to the first statement, which says that under some favourable circumstances a proper morphism can be turned into a locally projective morphism by modifying the source slightly. Thus the intuition for properness is projectiveness.

Proof. We prove 2 and 1 follows from the same argument. Locally closed embedding is separated (one way to see is that open embedding is separated and closed embedding is affine). Separatedness is local on base so we may assume $\varphi : X \to Y$ is strictly quasiprojective, which is a composition of two separated morphisms. $\square$

Classical algebraic geometry is concerned with quasiprojective schemes over algebraically closed field k. Note that for k a field $\varphi : X \to \operatorname{Spec} k$ is strictly (quasi)projective if and only if it is locally (quasi)projective. Also X is (locally) of finite type (since closed embedding is affine so of finite type, open embedding is locally of finite type and $\mathbb{P}^n_k \to \operatorname{Spec} k$ is of finite type) so (locally) noetherian.

Proposition 7.3. *Let $\varphi : X \to Y$ be locally (resp. strictly) quasiprojective. Then it is locally (resp. strictly) projective if and only if it is proper.*

In particular a quasiprojective variety is proper if an only if it is projective.

Proof. The composition $X \to \mathbb{P}^n_Y \to Y$ is proper and $\mathbb{P}^n_Y \to Y$ is separated so $X \to \mathbb{P}^n_Y$ is universally closed so closed. Thus it is a closed embedding. □

Corollary 7.4. *If X and Y are quasiprojective over $\operatorname{Spec} k$ the any morphism $X \to Y$ over k is separated.*

Corollary 7.5. *Let $\varphi : X \to Y$ be a morphism of projective schemes over k. Let Z be the closed subscheme image of $\varphi(X)$ in Y. Then $\varphi : X \to Z$ is surjective.*

Corollary 7.6. *If X is projective over k, Y quasiprojective over k then any k-morphism $X \to Y$ is proper.*

Corollary 7.7. *Let X be a reduced proper scheme over an algebraically closed field k. If X is connected then the natural map $k \to \mathcal{O}_X(X)$ is an isomorphism.*

Proof. There is a natural bijection $\mathcal{O}_X(X) \cong \operatorname{Hom}_k(X, \mathbb{A}^1_k)$. Let $i : \mathbb{A}^1_k \hookrightarrow \mathbb{P}^1_k$. Let $\varphi : X \to \mathbb{A}^1_k$ be a morphism induced by $f \in \mathcal{O}_X(X)$. Then the image of $i \circ \varphi$ is closed and connected in $\mathbb{P}^1_k$. It is not $\mathbb{P}^1_k$ so must be a k-point (here we used $k = \overline{k}$). As X is reduced $i \circ \varphi$ factors through the reduced structure $\operatorname{Spec} k \hookrightarrow \mathbb{A}^1$. □

Example. We verify a special case of valuative criterion. Let R be a DVR, say with uniformiser t. Let K be its field of fractions. Then we would like to show that there exists exactly one dotted arrow making the following diagram commute

$$\begin{array}{ccc} \operatorname{Spec} K & \longrightarrow & \mathbb{P}^n_Y \\ \big\downarrow & \nearrow & \big\downarrow \\ \operatorname{Spec} R & \longrightarrow & Y \end{array}$$

wlog suppose the image of $\operatorname{Spec} K$ is in U_0. Then since $R \to K$ is a monomorphism there is at most one such morphism.

$$\begin{array}{ccc} K & \longleftarrow & A[x_i^0]/(x_0^0 - 1) \\ \big\downarrow & \swarrow & \\ R & & \end{array}$$

Let f_i be the image of x_i^0 in K. Then either $f_i = 0$ or $f_i = t^{m_i} g_i$ where m_i is the order of vanishing of f_i. If all f_i's are zero then of course we can extend so suppose not, and wlog $m_1 \le m_i$ for all i such that $f_i \ne 0$. Looking at the chart

U_1, the morphism $\operatorname{Spec} K \to U_{01} \to U_{10}$ (where $U_{01} \subseteq U_0$ and $U_{10} \subseteq U_1$ are isomorphic) gives

$$(x_0^1)^{-1} \mapsto x_1^0 \mapsto f_1$$

so $f_1 \in K$ is invertible and hence is nonzero. The we can define a morphism $U_1 \to \operatorname{Spec} R$ by

$$x_i^1 \mapsto f_i \cdot f_1^{-1} = t^{m_i - m_1} g_i g_1^{-1} \in R$$

since $m_i - m_1 \geq 0$.

7.1 Proj construction

Before proving (a special case of) the theorem, we give a different construction of $\mathbb{P}_A^n$ for A a ring, similar to $\operatorname{Spec} A$. Let $R = \bigoplus_{n \geq 0} R_n$ be a graded ring. We will define a ringed space $\operatorname{Proj} R$, prove it is a scheme, and give a canonical isomorphism $\operatorname{Proj} A[x_0, \dots, x_n] \cong \mathbb{P}_A^n$ where $\deg x_i = 1$ and $\deg a = 0$ for $a \in A$.

As a set

$$\operatorname{Proj} R = \{\mathfrak{p} \subseteq R : \mathfrak{p} \text{ homogeneous prime}, \mathfrak{p} \not\supseteq \bigoplus_{n \geq 1} R_n\}.$$

Recall that an ideal $I \subseteq R$ is *homogeneous* if it is generated by homogeneous elements, or equivalently for every $f \in I$, each homogeneous component of f is in I. $\bigoplus_{n \geq 1} R_n$ is called the *irrelevant ideal.*

(over $\overline{k}[x_0, \dots, x_n]$ where k is infinite, $t \cdot \mathfrak{p} = \mathfrak{p}$ for all $t \in k^*$ if and only if $\mathfrak{p}$ is homogeneous).

$\operatorname{Proj} R$ is given the Zariski topology, namely the closed subsets are

$$Z(f) = \{\mathfrak{p} \in \operatorname{Proj} R : f \in \mathfrak{p}\} = \bigcap_{i=0}^{\deg f} Z(f_i).$$

In other words a basis of the toplogy is $\{D(f) = \{\mathfrak{p} : f \notin \mathfrak{p}\}\}$ for $f \in R$ homogeneous.

For the structure sheaf, those pullback to regular functions under $\pi : \mathbb{A}_k^{n+1} \setminus \{0\} \to \mathbb{P}_k^n$. We define $\mathcal{O}_{\operatorname{Proj} R}$ to be the sheaf

$$U \mapsto \textit{invariant}$$

For $f \in R$ homogeneous of degree a, let $R_f = \bigoplus_{i \in \mathbb{Z}} (R_f)_i$ where the $\frac{g}{f^r}$ has degree $m - ra$ if g has degree m. Then

$$\mathcal{O}_{\operatorname{Proj} R}(D(f)) = (R_f)_0.$$

Proposition 7.8. *This gives a unique defined structure sheaf* $\mathcal{O}_{\operatorname{Proj} R}$.

The proof is a tedious check which we omit.

Theorem 7.9. *If* $R = A[x_0, \dots, x_n]$ *then* $\operatorname{Proj} R \cong \mathbb{P}_A^n$.

Sketch proof. The isomorphism is given its restriction to $D(x_i)$ and U_i. We take as example $i = 0$. On the level of rings there are two maps which are inverses to each other

$$(A[x_0, \dots, x_n]_{x_0})_0 \longleftrightarrow A[x_0^0, \dots, x_n^0]/(x_0^0 - 1)$$

The map goes from left to right is "dehomogenisation": an element of LHS can be written as a sum of $\frac{f}{x_0^{\deg f}}$ where f is homogeneous, so can be written as a sum of monomial. Thus

$$\frac{f}{x_0^{\deg f}} = F(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$$

and we send it to $F(x_1^0, \dots, x_n^0)$. Conversely given G of RHS we send it to its homogenisation

$$x_0^{\deg G} G(\frac{x_1}{x_0}, \dots, \frac{x_n}{x_0})$$

of degree 0.

One then checks that for f homogeneous $D(f) \cap U_0 = D(F)$ and satisfies cocycle conditions so glue to an isomorphism. □

Lemma 7.10. *Let $\varphi^\# : R \to S$ be a homomorphism of graded rings surjective in degree ≥ 1. Then it induces a morphism $\varphi : \operatorname{Proj} S \to \operatorname{Proj} R$.*

The condition on surjectivity is to ensure that the pullback of an ideal does not contain the irrelevant ideal. For example the inclusion $k[x_0, x_1] \to k[x_0, x_1, x_2]$ fails to define a morphism $\mathbb{P}_k^2 \to \mathbb{P}_k^1$ since (x_0, x_1) pulls back to the irrelevant ideal. Geometrically, the point $[0, 0, 1]$ has nowhere to go since $[0, 0]$ is not a point on $\mathbb{P}_k^1$.

Proof. On points we define $\varphi(\mathfrak{p}) = \varphi^{-1}(\mathfrak{p})$. Surjectivity in degree ≥ 1 implies $\varphi(\mathfrak{p}) \in \operatorname{Proj} R$. It is continuous because $\varphi^{-1}(D(f)) = D(\varphi^\#(f))$. It induces a homomorphism $\varphi^\# : \mathcal{O}_{\operatorname{Proj} R} \to \varphi_* \mathcal{O}_{\operatorname{Proj} S}$ because for $f \in R_d$, $\varphi^\# : R \to S$ induces $R_f \to S_{\varphi^\#(f)}$, which we take to be $\varphi^\#(D(f))$. □

Exercise. Show that there is a group homomorphism from $\mathrm{GL}(n+1, A)$ to the automorphism group of $\mathbb{P}_A^n$ with kernel the multiples of the identity matrix (hint: the corresponding graded ring homomorphism is exactly what one would expect). (check: We will later see these are the only morphisms commuting with projection)

7.2 Line bundles on $\mathbb{P}^n$

It is not difficult to deduce from the proof of Theorem 5.24 that for $\varphi : X \to Y$ a morphism, $\psi : Z \to Y$ an affine morphism, there is a bijection

$$\{\alpha : X \to Y \text{ such that } \psi \circ \alpha = \varphi\} \longleftrightarrow \{\psi_* \mathcal{O}_Z \to \varphi_* \mathcal{O}_X \text{ morphism of } \mathcal{O}_Y\text{-algebras}\}.$$

Goal: do something similar with Proj. Key missing step: we need an invertible sheaf. (?check this statement)

Consider the inclusion $H_n = \mathbb{P}^{n-1}_A \hookrightarrow \mathbb{P}^n_A$ induced by killing off x_n. One can check that

$$\mathcal{I}_{H_n}|_{U_i} = \begin{cases} x^i \mathcal{O}_{U_i} & i \neq n \\ \mathcal{O}_{U_i} & i = n \end{cases}$$

so a rank 1 locally free sheaf and an effective Cartier divisor. Similary we may define H_j for $0 \leq j \leq n$.

Lemma 7.11. *There exists an isomorphism* $\mathcal{O}_{\mathbb{P}^n_A}(-H_i) \cong \mathcal{O}_{\mathbb{P}^n_A}(-H_n)$ *induced by* $\frac{x_i}{x_n}$.

Definition. We let $\mathcal{O}_{\mathbb{P}^n_A}(1) = \mathcal{O}_{\mathbb{P}^n_A}(H_n)$.

The isomorphism in lemma implies that each H_i defines a section $s_i \in \Gamma(\mathbb{P}^n_A, \mathcal{O}(1))$.

Theorem 7.12. *There is a natural isomorphism of graded* A*-algebras*

$$\begin{aligned} A[t_0, \ldots, t_n] &\to \bigoplus_{d \geq 0} \Gamma(\mathbb{P}^n_A, \mathcal{O}(d)) \\ t_i &\mapsto s_i \end{aligned}$$

8 Group schemes

Definition (group scheme)**.** A *group scheme* G over X is a scheme G over X together with

- $m : G \times_X G \to G$ multiplication morphism,
- $e : X \to G$ identity morphism,
- $i : G \to G$ inverse morphism

such that all these are morphisms over X and

- m is associative

$$\begin{array}{ccc} G \times_X G \times_X G & \xrightarrow{(m, \mathrm{id}_G)} & G \times_X G \\ \big\downarrow{\scriptstyle (\mathrm{id}_G, m)} & & \big\downarrow{\scriptstyle m} \\ G \times_X G & \xrightarrow{\quad m \quad} & G \end{array}$$

- e is left and right identity, i.e. both compositions

$$\begin{array}{ccccccc} & & G \times_X X & & & & \\ & \overset{\cong}{\nearrow} & & \overset{\mathrm{id}_G \times e}{\searrow} & & & \\ G & & & & G \times_X G & \xrightarrow{m} & G \\ & \underset{\cong}{\searrow} & & \underset{e \times \mathrm{id}_G}{\nearrow} & & & \\ & & X \times_X G & & & & \end{array}$$

  are id_G.

- i is left and right inverse, i.e. both compositions

$$\begin{array}{ccccccc} & & & & G \times_X G & & \\ & & & \overset{i \times \mathrm{id}_G}{\nearrow} & & \overset{m}{\searrow} & \\ G & \xrightarrow{\Delta} & G \times_X G & & & & G \\ & & & \underset{\mathrm{id}_G \times i}{\searrow} & & \underset{m}{\nearrow} & \\ & & & & G \times_X G & & \end{array}$$

  are $G \to X \xrightarrow{e} G$.

Example.

1. Let A be a ring. Define

$$\mathrm{GL}(n, A) = D(\det) \subseteq \operatorname{Spec} A[a_{ij}] := M$$

where $1 \leq i, j \leq n$. Let $G = \mathrm{GL}(n, A)$. Then G is a group scheme over

$X = \operatorname{Spec} A$. m is given by matrix multiplication, i.e.

$$\begin{aligned} A[a_{ij}] &\to A[b_{ij}, c_{ij}] \\ a_{ij} &\mapsto \sum_{k=1}^{n} b_{ik} c_{kj} \end{aligned}$$

which restricts to a morphism $m : G \times G \to G$. e is induced by $a_{ij} \mapsto \delta_{ij}$ and i is induced by mapping a_{ij} to the (i, j)th entry of the inverse of a matrix, which we know can be expressed as a rational function.

2. In the above example if $n = 1$ then we write $\mathbb{G}_{m,X} = \mathrm{GL}(1, A) = \operatorname{Spec} \mathcal{O}_X[t, t^{-1}]$.

3. $\mathrm{SL}(n, A) = \operatorname{Spec} A[a_{ij}]/(\det -1) \subseteq M$ as a closed subscheme is a group scheme.

4. $\mathbb{A}^n_A$ is a group scheme. If $n = 1$ then we write $\mathbb{G}_{a,X} = \mathbb{A}^1_A = \operatorname{Spec} \mathcal{O}_X[t]$.

Definition (group action)**.** Let G be a group scheme over X and Y a scheme over X. A *(left) action* of G on Y over X is a morphism $a : G \times_X Y \to Y$ over X such that

1. commutes with multiplication:

$$\begin{array}{ccc} G \times_X G \times_X Y & \xrightarrow{m \times \mathrm{id}_Y} & G \times_X Y \\ \downarrow{\scriptstyle \mathrm{id}_G \times a} & & \downarrow{\scriptstyle a} \\ G \times_X Y & \xrightarrow{a} & Y \end{array}$$

2. identity acts trivially, i.e. the following composition is id_Y

$$Y \xrightarrow{\cong} X \times_X Y \xrightarrow{e \times \mathrm{id}_Y} G \times_X Y \xrightarrow{a} Y$$

Example.

1. Projection to Y gives the trivial action for any group scheme G.

2. G acts on itself via m.

Example. Let $A = \bigoplus_{d \geq 0} A_d$ be a graded algebra, $X = \operatorname{Spec} A_0, Y = \operatorname{Spec} A$. Then $G = \mathbb{G}_{m,X}$ acts on Y via

$$\begin{aligned} A &\to A \otimes_{A_0} A_0[t, t^{-1}] = A[t, t^{-1}] \\ f &\mapsto t^d f \end{aligned}$$

if $f \in A_d$.

Exercise. Set $A_0 = k, A = k[x_1, \dots, x_n]$ with $\deg x_i = 1$. Then the action is just the diagonal action by the torus $t(x_1, \dots, x_n) = (tx_1, \dots, tx_n)$.

Exercise. Show $\mathrm{GL}(n, A)$ acts on $\mathbb{A}^n_A$.

Definition (equivariant morphism)**.** Let G be a group scheme over X acting on Y_1 and Y_2 via a_1, a_2. A morphism $f : Y_1 \to Y_2$ over X is *G-equivariant* if the following diagram commutes:

$$\begin{array}{ccc} G \times_X Y_1 & \xrightarrow{\mathrm{id}_G \times f} & G \times_X Y_2 \\ \big\downarrow a_1 & & \big\downarrow a_2 \\ Y_1 & \xrightarrow{\quad f \quad} & Y_2 \end{array}$$

Lemma 8.1. *Suppose $G = \mathbb{G}_{m,A}$ acts on $Y_1 = \operatorname{Spec} B, Y_2 = \operatorname{Spec} C$ where B and C are graded A-algebras (the action induced by grading). Then $\varphi : Y_1 \to Y_2$ is $\mathbb{G}_m$-equivariant if and only if $\varphi^\# : C \to B$ is a homomorphism of graded A-algebra.*

Proof. On the level of rings

$$\begin{array}{ccc} C & \longrightarrow & C[t, t^{-1}] \\ \big\downarrow & & \big\downarrow \\ B & \longrightarrow & B[t, t^{-1}] \end{array}$$

For $f \in C_d$, let $g = \varphi^\#(f) = \sum_{i=0}^r g_i$ where g_i's are homogeneous. Then

$$\begin{array}{ccc} f & \longmapsto & t^d f \\ \big\downarrow & & \big\downarrow \\ \sum g_i & \longmapsto & \sum t^i g_i \end{array}$$

so $t^d \sum g_i = \sum t^i g_i$ so g is homogeneous of degree d. □

Corollary 8.2. *An endormorphism $\varphi : \mathbb{A}^n_A \to \mathbb{A}^n_A$ is $\mathbb{G}_m$-equivariant if and only if it is linear, i.e. $x_i \mapsto \sum_j a_{ij} x_j$ for a unique choice of $a_{ij} \in A$.*

Definition (vector bundle)**.** Let X be a scheme. A *trivial vector bundle* on X is a scheme $\pi : E \to X$ with a $\mathbb{G}_{m,X}$ action which is $\mathbb{G}_{m,X}$-isomorphic to $\mathbb{A}^r_X$. A *vector bundle* of rank r on X is a scheme $E \to X$ with a $\mathbb{G}_{m,X}$ such that there exists an open cover $\{U_i\}$ such that $E|_{U_i} \to U_i$ is trivial.

What is implied in this definition is that group actions are stable under base change.

Remark. It is a theorem that a vector bundle is the same as one that is locally trivial over the étale site or other flat sites.

More generally

Definition (principal bundle/torsor)**.** Let X be a scheme and G a group scheme over X. A *trivial principal G-bundle* (or *trivial G-tosor*) is a scheme Y over X with a G-action which is G-isomorphic to G. A *principal G-bundle* (or *G-torsor*) is a scheme that is locally a trivial principal G-bundle.

Proposition 8.3. *There is an equivalence of categories*

$$\{\text{rank } r \text{ vector bundle on } X\}/\cong \longleftrightarrow \{\text{principal } \mathrm{GL}(r)\text{-bundle}\}$$

Remark. If $\mathcal{E}$ is locally free of rank r then $\bigwedge^r \mathcal{E}$ is locally free of rank 1, a line bundle which we call $\det \mathcal{E}$. Then by the correspondence between bector bundles and locally free sheaves in the next section, there is an equivalence of categories between

$$\{\text{rank } r \text{ vector bundle with trivial det}\} \longleftrightarrow \{\text{principal } \mathrm{SL}(r)\text{-bundle}\}.$$

8.1 Correspondence between vector bundles and locally free sheaves

Let $\pi : E \to X$ be a rank r vector bundle, $\mathcal{A} = \pi_* \mathcal{O}_E$. Then on a trivialising neighbourhood U, $\mathcal{A}|_U \cong \mathcal{O}_U[x_1, \dots, x_r]$ so $\mathcal{A}$ is a quasicoherent sheaf of $\mathcal{O}_X$-modules. On intersection $U_i \cap U_j$, the lemma in the previous section states that different trivialisations induce the same grading. Thus there is a gobal grading $\mathcal{A} = \bigoplus_{d \geq 0} \mathcal{A}_d$ on X.

For any graded algebra there is a map $\operatorname{Sym}^* \mathcal{A}_1 \to \mathcal{A}$. In general this map is neither injective nor surjective, but in our case it is (locally so globally) an isomorphism and $\mathcal{A}_1$ is locally free of rank r.

Thus we have constructed an equivalence of categories

$$\begin{aligned} \{\text{vector bundle of rank } r\} &\longleftrightarrow \{\text{locally free sheaves of rank } r\} \\ E &\mapsto (\pi_* \mathcal{O}_E)_1 \\ \operatorname{Spec} \operatorname{Sym}^* \mathcal{F} &\leftarrow\!\shortmid \mathcal{F} \end{aligned}$$

Definition (abelian cone)**.** Let X be a locally noetherian scheme. An *abelian cone* over X is a scheme $A \to X$ with a $\mathbb{G}_m$-action such that $\pi_* \mathcal{O}_A = \bigoplus_{d \geq 0} \mathcal{A}_d$ with

1. $\mathcal{O}_X \to \mathcal{A}_0$ is an isomorphism,
2. $\mathcal{A}_1$ is coherent,
3. $\operatorname{Sym}^* \mathcal{A}_1 \to \mathcal{A}$ is an isomorphism.

Exercise. Show that this is equivalent to saying that A is the kernel of a morphism of vector bundles on X, i.e. the fibre product

$$\begin{array}{ccc} A & \longrightarrow & X \\ \downarrow & & \downarrow{\scriptstyle 0} \\ E & \longrightarrow & F \end{array}$$

for some $\mathbb{G}_m$-equivariant morphism $E \to F$.

Remark. A *cone* is one where one replaces isomorphism in 3 by surjective. They define global Proj.

What is Proj $\mathcal{A}$ in this language? Assume $\mathcal{A} = \bigoplus_{d \geq 0} \mathcal{A}_d$ is a quasicoherent sheaf of graded $\mathcal{O}_X$-algebra satisfying the definition of a cone. Then we have a closed embbedding $\operatorname{Spec} \mathcal{A} \to \operatorname{Spec} \operatorname{Sym}^* \mathcal{A}_1$. $C = \operatorname{Spec} \mathcal{A}$ is called a cone over X, and it is a subscheme of $\mathcal{A}(C)$, called the associated abelian cone, that is invariant under $\mathbb{G}_m$. Conversely every cone is a $\mathbb{G}_m$-equivariant closed subscheme of an abelian cone.

In this language, Proj $\mathcal{A}$ is the quotient of Spec $\mathcal{A}$ minus the zero section by $\mathbb{G}_m$-action. The zero section is a closed subscheme $X_0 \hookrightarrow C$ (isomorphic to X) obtained by ...

Quotient: $\operatorname{Spec} \mathcal{A} \setminus X_0 \to \operatorname{Proj} \mathcal{A}$ is a principal $\mathbb{G}_m$-bundle, where the principal opens are $D(f)$ where f homomogeneous for $f \in \mathcal{A}_1(U)$.

Exists an equivalence of categories

$$\{\text{abelian cones over } X \text{ with } \mathbb{G}_m\text{-equivariant morphism}\} \longleftrightarrow \{\text{coherent sheaf on } X\}.$$

In particular if $X \to Y$ a morphism locally of finite type of locally noetherian schemes, then $\Omega_{X/Y}$ is coherent. There are two natural candidates for the "tangent bundle" of X over Y: one may take the dual sheaf

$$\Theta_{X/Y} = \mathcal{H}om(\Omega_{X/Y}, \mathcal{O}_X).$$

In general we lose torsion information in this process. A more refined approach is to use the cone over X

$$\mathcal{T}_{X/Y} = \operatorname{Spec} \operatorname{Sym}^* \Omega_{X/Y}.$$

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If the cone has a zero section $s_0 : X \to C$ induced by

$$0 \longrightarrow \bigoplus_{d \geq 1} \mathcal{A}_1 \longrightarrow \mathcal{A} \xrightarrow{s_0^\#} \mathcal{O}_X \longrightarrow 0$$

then $C \setminus s_0(X) \to \operatorname{Proj} \mathcal{A}$ is a principal $\mathbb{G}_m$-bundle.

Example (projective bundle)**.** Let $\mathcal{A} = \operatorname{Sym}^* \mathcal{E}$ where $\mathcal{E}$ is a locally free sheaf. Then we define $\mathbb{P}(\mathcal{E}) = \operatorname{Proj}(\mathcal{A})$. Let $E = \operatorname{Spec} \operatorname{Sym}^* \mathcal{E}$, then we have a principal $\mathbb{G}_m$-bundle $E \setminus s_0(X) \to \mathbb{P}(\mathcal{E})$. Thus $\mathbb{P}(\mathcal{E})$ parameterises lines in fibres of E. Sometimes we also denote it by $P(E)$ (note $\mathcal{E}$ is a locally free sheaf while E is a vector bundle).

Theorem 8.4 (Bertini)**.** *Let $X \subseteq \mathbb{P}^n_k$ be a closed subscheme smooth of dimension d over* Spec k*, an algebraically closed field. Let $\check{\mathbb{P}}^n$ be the scheme parameterising hyperplanes in $\mathbb{P}^n$. Then there exists a nonempty open $U \subseteq \check{\mathbb{P}}^n$ such that for every $a \in U(k)$, we have $X \cap H_a$ is smooth of dimension $(d-1)$.*

Analogue of Sard's theorem: take incidence correspondence

$$\Gamma = \{(p, \ell) \in \mathbb{P}^n \times \check{\mathbb{P}}^n : p \in \ell\}.$$

Then Bertini's theorem says that the generic fibre of $\Gamma \cap (X \times \check{\mathbb{P}}^n) \to \check{\mathbb{P}}^n$ is smooth.

Sketch proof. Let $p \in X(k)$. Then locally near p, we can find $A = k[t_1, \dots, t_{d+r}]/(f_1, \dots, f_r)$ standard smooth of dimension d. $\Omega_X(p)$ has dimension d with basis $\mathrm{d}t_{d+1}, \dots, \mathrm{d}t_{d+r}$. Let $\ell \in S_1$. We want to know when is A/ℓ nonsinglar at p. Restruct to U, ℓ is a regular function. Then $\ell|_U = g \in k[t_1, \dots, t_{d+r}]$. Then $X \cap Z(\ell)$ is smooth at p of dimension $d-1$ if and only if $\mathrm{d}\ell(p) \neq 0 \in \Omega_X(p)$.

To take care of all points at the same time, we consider

$$\Gamma = \{(p, \ell) : \mathrm{d}\ell(p) = 0 \in \Omega_X(p), \ell(p) = 0\} \subseteq X \times \check{\mathbb{P}}^n.$$

Then

$$\Gamma = \{(p, \ell) : X \cap Z(\ell) \text{ not smooth of dimension } d-1 \text{ at } p\}.$$

We have projections $\alpha : \Gamma \to X, \beta : \Gamma \to \check{\mathbb{P}}^n$. Then Bertini's theorem asserts that $\check{\mathbb{P}}^n \setminus \beta(\Gamma)$ contains nonempty open subset in $\check{P}^n$. To do so we show $\beta(\Gamma)$ is closed in $\check{\mathbb{P}}^n$ of dimension $< n$.

Claim $\alpha : \Gamma \to X$ is a projective bundle associated to a bundle of rank $n-d$. This implies Γ is proper so $\beta(\Gamma)$ is closed. To find the dimension, locally Γ is the product of X and $\mathbb{P}^{n-d-1}$. Since $\Gamma \to X$ is smooth of relative dimension $n-d-1$, Γ is smooth of dimension $n-1$. Thus $\dim \beta(\Gamma) \leq n-1$.

Since the two vector bundles have rank n and rank d respectively,

$$0 \longrightarrow N^\vee_{X/\mathbb{P}^n} \longrightarrow \Omega_{\mathbb{P}^n}|_X \longrightarrow \Omega_X \to \qquad 0$$

$N^\vee_{X/\mathbb{P}^n}$ is locally free of rank $n-d$. Check $\Gamma = P(N_{X/\mathbb{P}^n}(1))$. For example on $U_0 = \{x_0 \neq 0\}$,

$$S_1 \to \mathcal{O}_{U_0}$$
$$\ell \mapsto \frac{\ell}{x_0}$$

□

Corollary 8.5. *Let $X \subseteq \mathbb{P}^n$ as before. Fix e a positive integer and let S_e be the set of homogeneous polynomials of degree e in $S = k[x_0, \dots, x_n]$. Let $P(S_e) = S_e \setminus \{0\}/k^*$. Then exists $U \subseteq P(S_e)$ nonempty open such that for all $f \in U$, $X \cap Z(f)$ is nonsingular of dimension $dim X - 1$.*

Sketch proof. Consider the subring $S' = \bigoplus_{a \geq 0} S_{ad}$ with grading $S'_a = S_{ad}$. One way to think about S' are the invariant subring under $\mu_d = \operatorname{Spec} k[t]/(t^d - 1)$ (μ_d is smooth if and only if $\operatorname{char} k$ does not divide d). S' is generated as a k-algebra by monomials of degree d, i.e. there is a surjection of graded algebra $R = k[y_0, \dots, Y_N] \twoheadrightarrow S'$ where $N = \binom{n+d+1}{d} - 1$. Then we have an embedding $\operatorname{Proj} S' \cong \mathbb{P}^n \hookrightarrow \operatorname{Proj} R = \mathbb{P}^N$, which is the *Veronese embedding of $\mathbb{P}^n$ of degree d* sending $(x_0, \dots, x_n)$ to the tuple of all monomials of degree d (with a chosen ordering).

A hyperplane in $\mathbb{P}^N$ is the same as a linear combination of degree d monomials in x_i. Thus if $v(\mathbb{P}^n) \cap H = Z(f)$ where f is a polynomial in $x_0, \dots, x_n$, then $X \cap Z(f) \cong v(X) \cap H$ via v. □

Remark. We can apply this repeatedly, starting with $X = \mathbb{P}^n$. Then for generic $f_i \in S_{e_i}$, $Y = \bigcap_{i=1}^r Z(f_i)$ is smooth of dimension $n-r$. Such a Y is called a *complete intersection of degree* $(e_1, \dots, e_r)$. In particular if $r = 1$, Y is called a *smooth hypersurface* of degree e_1.

Fact: In Bertini's theorem, if we assume X is irreducible and $\dim X > 1$ then $X \cap Z(\ell)$ is also irreducible.

More examples can be obtained by generalising the result to other spaces. For example complete intersection in products of projective spaces, in weighted projective spaces.

8.2 Another way to construct new smooth varieties

Simple cyclic covering

Let X be a scheme, $\mathcal{L} \in \operatorname{Pic} X$ and $s \in \Gamma(X, \mathcal{L}^{\otimes n})$ for some $n \geq 2$. Let $A = \mathcal{O}_X \oplus \mathcal{L}^\vee \oplus \cdots \oplus (\mathcal{L}^\vee)^{\otimes(n-1)}$ which is locally free sheaf of rank n. It can be made into a $\mathbb{Z}/n\mathbb{Z}$-graded algebra. In local coordinates where $\mathcal{L} \cong \mathcal{O}_X$ generated by t, $\mathcal{A} = \mathcal{O}_X[t]/(t^n - s)$. Then we have a surjection $\operatorname{Sym}^*(\mathcal{L}^\vee) \to \mathcal{A}$ with kernel generated by $u - s(u) \in (\mathcal{L}^\vee)^{\otimes n} \oplus \mathcal{O}_X$. Then we have a closed embedding $\widetilde{X} = \operatorname{Spec} \mathcal{A} \hookrightarrow L = \operatorname{Spec} \operatorname{Sym}^* \mathcal{L}^\vee$. We ask when is $\widetilde{X}$ the simple cyclic cover branched on S also smooth?

(taking nth root of unity)

Exercise. We get isomorphic $\widetilde{X}$ if we replace s by us where $u \in \Gamma(X, \mathcal{O}_X^*)$, so the construct descends to D, the effective Cartier divisor associated to s.

D is called the *branched divisor*, and $\widetilde{D} = (\pi^{-1}(D))_{\text{red}}$ is called the *ramification divisor*.

Claim: If X is smooth and n is coprime to $\operatorname{char} k$ then $\widetilde{X}$ is smooth if and only if D is smooth: smoothness is local so we can assume $X = \operatorname{Spec} A, \mathcal{L} \cong \mathcal{O}_X$ where $A = k[x_1, \dots, x_{d+r}]/(f_1, \dots, f_r)$ is standard smooth. Then hte cover $\widetilde{X}$ is given by

$$\operatorname{Spec} A[t]/(t^n - s) = \operatorname{Spec} \widetilde{A}$$

where $\widetilde{A} = A[x_1, \dots, x_{d+r}, t]/(f_1, \dots, f_r, t^n - s)$. s is nonzero implies that $t \neq 0$ so $\frac{\partial(t^n - s)}{\partial t} = nt^{n-1} \neq 0$.

What happens on the branch divisor, i.e. $s = 0$? This is the same as $t = 0$. Our only hope is to use $\frac{\partial s}{\partial x_j}$. We seek $(r+1)$ variables among x_i such that...

In conclusion: $\widetilde{X}$ is smooth of dimensionequals to $\dim X$ if and only if $\operatorname{char} k$ is coprime to n and D is smooth of dimension $\dim X - 1$.

Typical application: construct a smooth projective genus 3 curve. Tkae a degree 4 hypersurface in $\mathbb{P}^2$, or double cover ($n = 1$) of $\mathbb{P}^1$ branched on 8 points. In fact these are all the possibilities for genus 3!

c.f. unirationality of moduli space

9 Sheaf cohomology

We recall some vanishing theorems in algebraic geometry:

1. (Grothendieck) If X is affine and $\mathcal{F} \in \mathbf{Qcoh}(X)$ then $H^i(X, \mathcal{F}) = 0$ for all $i > 0$. Intuition: $\Gamma(X, -) : \mathbf{Qcoh}(X) \to \mathbf{Mod}_A$ is exact.

2. Let X be a smooth manifold. Then for each $\mathcal{F}$ sheaf of C^∞_X-modules, $H^i(X, \mathcal{F}) = 0$ for $i > 0$.

3. (Serre vanishing) Let A be a noetherian ring, $N \geq 0, X \subseteq \mathbb{P}^N_A$ a closed subscheme and $\mathcal{F} \in \mathbf{Coh}(X)$. Then exists n_0 such that for all $n \geq n_0$, for all $i > 0$, $H^i(X, \mathcal{F}(n)) = 0$ where $\mathcal{F}(n) = \mathcal{F} \otimes (\mathcal{O}_{\mathbb{P}^N_A}(n)|_X)$. The analogous result for complex manifolds is Kodaira vanishing.

Computationally we use Cech cohomology. The comparison theorem for us is

> **Theorem 9.1** (Leray). *Let X be a topological space, $\mathcal{U} = \{U_i\}_{i \in I}$ an open cover and $\mathcal{F}$ a sheaf such that for all $p \geq 0$, for all $i_0 < \cdots < i_p$, for all $n > 0$,*
>
> $$H^n(U_{i_0} \cap \cdots \cap U_{i_p}, \mathcal{F}|_{U_{i_0} \cap \cdots \cap U_{i_p}}) = 0.$$
>
> *Then $\check{H}^i(\mathcal{U}, \mathcal{F})$ is canonically isomorphic to $H^i(X, \mathcal{F})$ for all $i \geq 0$.*

Example. Let $X = \mathbb{P}^1_k, U_0 = \operatorname{Spec} k[t], U_1 = \operatorname{Spec} k[s], U_{01} = \operatorname{Spec} k[t,s]/(ts - 1)$. Let $\mathcal{F} = \mathcal{O}_{\mathbb{P}^1}(-m)$ be the ideal sheaf of regular functions vanishing at origin to order m. $\mathcal{F}(U_0) = t^m k[t]$ with basis $\{t^m, t^{m+1}, \dots\}$. $\mathcal{F}(U_1) = k[s]$ with basis $\{1, s, s^2, \dots\}$ and $\mathcal{F}(U_{01}) = k[t,s]/(ts-1)$ with basis $\{1, t, s, t^2, s^2, \dots\}$. Then $\check{H}^1(\mathcal{U}, \mathcal{O}(-m))$ has basis $\{t, t^2, \dots, t^{m-1}\}$. Thus $\dim H^1(\mathbb{P}^1, \mathcal{O}(-m)) = m - 1$.

This shows that Serre vanishing for $\mathcal{O}_{\mathbb{P}^1}$ on $\mathbb{P}^1$, where $n_0 = -1$.

By the same argument for all $m \geq 0$, $H^1(\mathbb{P}^1, \mathcal{O}(m)) = 0$.

Remark. In general, if a scheme X is separated over an affine scheme and has an open affine by N affines then $H^i(X, \mathcal{F}) = 0$ for all $i \geq N$ for all $\mathcal{F}$ quasicoherent. This implies that if X is projective then $H^i(X, \mathcal{F})$ for all $i > \dim X$ for all $\mathcal{F}$ quasicoherent.

Fact: If we denote $h^i(X, \mathcal{F}) = \dim_k H^i(X, \mathcal{F})$, then we have the following results: for any $N \geq 0, m \in \mathbb{Z}, i \geq 0$, we have $h^i(\mathbb{P}^N, \mathcal{O}(m)) = 0$ except

$$h^0(\mathbb{P}^N, \mathcal{O}(m)) = \binom{N+1}{m}$$

for $m \geq 0$ and

$$h^N(\mathbb{P}^N, \mathcal{O}(m)) = \dim h^0(\mathbb{P}^N, \mathcal{O}(-m-N-1)$$

for $m \leq -(n+1)$. This is a special case of *Serre duality*:

Theorem 9.2. *Let X be a smooth connected projective scheme of dimension d over an algebraically closed field k. Then there is a natural isomorphism*

$$H^d(X, \Omega_X^d) \cong k$$

where $\Omega_X^d = \bigwedge \Omega_{X/k} = \det \Omega_{X/k}$, the canonical sheaf *or the* dualising sheaf.

Moreover for all $\mathcal{F}$ locally free of rank r, there is a perfect pairing for all $0 \leq i \leq d$

$$H^i(X, \mathcal{F}) \times H^{d-i}(X, \mathcal{F}^\vee \otimes \Omega_X^d) \to H^d(X, \Omega_X^d) \cong k.$$

It admits a generalisation to "slightly singular" schemes. Let $i : X \hookrightarrow \mathbb{P}_k^n$ be a closed embedding. We assume i is a regular embedding of codimension $n-d$, i.e. $i^*\mathcal{I}_{X/\mathbb{P}_k^n}$ is locally free of rank $n-d$, X is d-dimensional and

$$0 \longrightarrow i^*\mathcal{I}_{X/\mathbb{P}_k^n} \longrightarrow \Omega_{\mathbb{P}_k^n/k}|_X \longrightarrow \Omega_{X/k} \longrightarrow 0$$

exact. Then Serre vanishing holds with ω_X in place of Ω_X^d, where ω_X is the *dualising sheaf* defined by

$$\omega_X = \det(\Omega_{\mathbb{P}_k^n/k}|_X) \otimes \det(i^*\mathcal{I}_{X/\mathbb{P}_k^n})^\vee.$$

9.1 Cohomology of complex manifolds

Let X be an n-dimensional complex manifold, $\mathcal{O}_X$ the sheaf of holomorphic functions and $\mathcal{A}^{p,q}$ the sheaf of (p,q)-forms. We have two differentials $\partial : \mathcal{A}^{p,q} \to \mathcal{A}^{p+1,q}, \overline{\partial} : \mathcal{A}^{p,q} \to \mathcal{A}^{p,q+1}$.

We have a complex

$$0 \longrightarrow \mathcal{O}_X \longrightarrow \mathcal{A}^{0,0} \xrightarrow{\overline{\partial}} \mathcal{A}^{0,1} \longrightarrow \cdots \longrightarrow \mathcal{A}^{0,n} \longrightarrow 0$$

which is a resolution of $\mathcal{O}_X$ since $\mathcal{A}^{0,0}$ is the sheaf of $\mathbb{C}$-valued smooth functions, and those f such that $\overline{\partial} f = 0$ are precisely those satisfying the Cauchy-Riemann equation so is holomorphic. Note that in fact this is a sheaf of $\mathcal{O}_X$-modules as

$$\overline{\partial}(f\alpha) = f\overline{\partial}\alpha + \overline{\partial} f \wedge \alpha = f\overline{\partial}\alpha$$

for f holomorphic.

Let $\mathcal{E}$ be a locally free sheaf of $\mathcal{O}_X$-modules of rank r, i.e. the holomorphic sections of a rank r holomorphic bundle E. Tensoring with $\mathcal{E}$ is exact so we get an exact sequence

$$0 \longrightarrow \mathcal{E} \longrightarrow \mathcal{A}^{0,0}(E) \xrightarrow{\overline{\partial}} \mathcal{A}^{0,1}(E) \longrightarrow \cdots \longrightarrow \mathcal{A}^{0,n}(E) \longrightarrow 0$$

where $\mathcal{A}^{p,q}(E) = \mathcal{A}^{p,q} \otimes_{\mathcal{O}_X} \mathcal{E}$.

Note also that $\mathcal{A}^{0,0}$ admits a partition of unity, so all $\mathcal{A}^{0,0}$-modules are acyclic. Thus $H^q(X, \mathcal{E})$ can be calculated as $\overline{\partial}$-closed $(0,q)$-forms with valued in E modulo exact forms. This is called *Dolbeault cohomology*.

We will now interpret Serre duality in the context of complex geometry. By discussion above $H^n(X, \Omega_X^n)$ is $\mathcal{A}^{0,n}$-forms (automatically $\overline{\partial}$-closed) with values in $\mathcal{A}^{n,0}$ module exact forms, so $\mathcal{A}^{n,n}$-forms modulo $\overline{\partial}\ \mathcal{A}^{n,n-1}$-forms. As X is compact, integration gives a map to $\mathbb{C}$.

Given two locally free sheaves $\mathcal{E}, \mathcal{F}$, we obtain the map

$$H^p(X, \mathcal{E}) \otimes H^q(X, \mathcal{F}) \to H^{p+q}(X, \mathcal{E} \otimes \mathcal{F})$$

via wedge on forms and tensor product on locally free sheaves, i.e.

$$(\alpha \otimes e) \cdot (\beta \otimes f) \mapsto (\alpha \wedge \beta) \otimes (e \otimes f).$$

It is easy to check that this is well-defined and descends to cohomologies.

One can use analysis to prove holomorphic Serre duality where X is a closed complex submanifolds of $\mathbb{P}^N$. To connect this back to algebraic geometry, we use the GAGA principle. Let X be a smooth closed subscheme of $\mathbb{P}^N$. As X is smooth over $\mathbb{C}$, it can be locally expressed as the spectrum of $\mathbb{C}[x_1, \dots, x_{n+r}]/(f_1, \dots, f_r)$ so by implicity function theorem defines locally a complex manifold $X^{\text{an}} \subseteq \mathbb{C}^{n+r}$. The "transition functions" are rational functions so holomorphic. A theorem of Chow says that all ? closed submanifolds of projective spaces arise this way.

Given a rank r locally free bundle $E \to X$, we can define a vector bundle $E^{\text{an}} \to X^{\text{an}}$. Serre's GAGA says that there is a canonical isomorphism

$$H^i(X, \mathcal{E}) \to H^i(X^{\text{an}}, \mathcal{E}^{\text{an}})$$

for all $i \geq 0$.

Theorem 9.3 (Riemann-Roch)**.** *Let C be a smooth projective connected curve over $k = \overline{k}$. Let $\mathcal{L} \in \operatorname{Pic}(C)$ with degree d (i.e. the degree of the corresponding divisor). Let g be the genus of C, defined by*

$$h^0(C, \Omega_C) = h^0(C, K_C) = h^1(C, \mathcal{O}_C)$$

where the last equality comes from Serre duality.

$$1 = h^0(C, \mathcal{O}_C) = h^1(C, \Omega_C).$$

Then

$$\chi(C, L) = \deg(\mathcal{L}) + 1 - g.$$

In particular $\deg(K_C) = 2g - 2$.

To generalise a bit, if $\mathcal{E}$ is a locally free sheaf of rank r on C then

$$\chi(E) = \deg(\det \mathcal{E}) + r(1 - g).$$

10 Curves

For this section we work with k-schemes where $k = \overline{k}$. We mean by a curve a smooth projective connected scheme, a point a k-valued point, and morphisms morphisms of k-schemes.

Recall that if C is a cruve then a prime divisor on $\mathbb{C}$ is a point. Smoothness implies that every prime divisor is Cartier. We have a surjective map

$$\begin{aligned} \mathrm{Div}(C) &\to \mathrm{Pic}(C) \\ D &\mapsto \mathcal{O}_C(D) \end{aligned}$$

and a degree map $\deg : \mathrm{Div}(C) \to \mathbb{Z}$.

We quote the fact that if X is a projective scheme, $\mathcal{F} \in \mathbf{Coh}(X)$ then $\Gamma(X, \mathcal{F})$ is a finite-dimensional k-vector space. Moreover $H^i(X, \mathcal{F})$ is finite-dimensional for all $i \geq 0$, whose dimension we denote by $h^i(X, \mathcal{F})$.

Theorem 10.1 (Riemann-Roch)**.** *Let $D \in \mathrm{Div}(C)$. Then*

$$\chi(\mathcal{O}(D)) = \deg(D) + \chi(\mathcal{O}_C)$$

where $\chi(X, \mathcal{F}) = \sum_{i \geq 0} (-1)^i h^i(X, \mathcal{F})$.

Proof. Let $p \in C$ and let $\mathcal{O}_p$ be the skyscraper sheaf. Given $\mathcal{L} \in \mathrm{Pic}(C)$, tensoring with $\mathcal{I}_p = \mathcal{O}_C(-p) \hookrightarrow \mathcal{O}_C$ induces a SRS

$$0 \longrightarrow \mathcal{L}(-p) \longrightarrow \mathcal{L} \longrightarrow \mathcal{Q} \longrightarrow 0$$

where $\mathcal{Q}$ is supported at p and is (non-canonically) isomorphic to $\mathcal{O}_p$.

We now show $\chi(\mathcal{O}_p) = 1$. Cover C by two open affines (?) U, V. If $p \in U, p \notin V$. Then Cech cohomology shows that $h^0 = 1, h^1 = 0$.

Now for any divisor D there is a short exact sequence

$$0 \longrightarrow \mathcal{O}(D - p) \longrightarrow \mathcal{O}(D) \longrightarrow \mathcal{O}_p \longrightarrow 0$$

so

$$\chi(\mathcal{O}(D - p)) = \chi(\mathcal{O}(D)) - 1.$$

Finally we write $D = \sum_{i=1}^n a_i p_i$ where $a_i \neq 0$ for all i. Induction on $\sum |a_i|$: if $\sum |a_i| = 0$ then $n = 0$ and $D = 0$ so $\deg D = 0$ and $\mathcal{O}(D) = {}_C$. For induction step, if $a_1 > 0$ then set $E = D - p_1$. Then the induction hypothesis for E as well as the discussion above shows

$$\chi(D) = \chi(E) + 1 = \deg E + \chi(\mathcal{O}_C) + 1 = \deg(D) + \chi(\mathcal{O}_C).$$

Same if $a_1 < 0$. ☐

Lemma 10.2. *Let X be a projective n-dimensional scheme. Then exists an open affine cover of X by $(n + 1)$ opens.*

Proof. Suppose $X \subseteq \mathbb{P}^N$ is closed. Take $H \subseteq \mathbb{P}^N$ a hyperplane. Then $X - H \subseteq \mathbb{P}^N - H$ is affine. Then the problem is reduced to show there exist hyperplanes $H_1, \dots, H_{n+1}$ such that $H_1 \cap \cdots \cap H_{n+1} \cap X = \emptyset$.

Consider the dual projective space $\check{\mathbb{P}}^N$ of hyperplanes in $\mathbb{P}^N$. Then exists $U_X \subseteq \check{\mathbb{P}}^N$ nonempty open such that for all $H \in U_X$ such that $\dim H \cap X \leq n-1$ (then equality by principal ideal theorem).

Proof. Induction on irreducible components of X. If X is irreducible then $\dim X \cap H = \dim X$ if and only if $H \supseteq X$. The "wrong" hyperplanes are give by

$$\Gamma(\mathbb{P}^N, \mathcal{I}_X(1)) \subsetneq \Gamma(\mathbb{P}^N, \mathcal{O}(1))$$

so defines a closed subset in $\check{\mathbb{P}}^N$. Call this closed subset Z_X and its complement U_X.

Suppose $X = X_1 \cup \cdots \cup X_r$ as irreducibles. Then $U_X = U_{X_1} \cap \cdots \cap U_{X_r}$. Any finite interesection of nonempty open is nonempty open.

Now induction on $\dim X$. If $\dim X = 0$ we can find $H \cap X = \emptyset$. For induction step, choose H_1 such that $\dim X \cap H_1 = \dim X - 1$. □

□

Corollary 10.3. *If X is a projective scheme of dimension d and $\mathcal{F}$ is a quasicoherent sheaf then*

$$H^i(X, \mathcal{F}) = 0$$

for all $i > d$.

Remark. One can extend "easily" Riemann-Roch for curves to any coherent sheaf $\mathcal{F}$ as follows:

$$\chi(\mathcal{F}) = \deg \mathcal{F} + \operatorname{rk} \mathcal{F} \cdot \chi(\mathcal{O}_C)$$

where $\operatorname{rk} \mathcal{F} = \dim_{K(\eta)} \mathcal{F}_\eta$ where η is the generic point of C. There is a natural short exact sequence

$$0 \longrightarrow \mathcal{F}_{\text{tors}} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}^{\vee\vee} \longrightarrow 0$$

where $\mathcal{F}_{\text{tors}}$ is the sheaf

$$U \mapsto \{s \in \mathcal{F}(U) : s = 0 \text{ in } \mathcal{F}_\eta\}.$$

For a smooth curve C, $\mathcal{F}^{\vee\vee}$ is locally free and we define

$$\deg \mathcal{F} = \deg \mathcal{F}^{\vee\vee} + h^0(\mathcal{F}_{\text{tors}})$$

and show $h^1(\mathcal{F}_{\text{tors}}) = 0$ (because $\mathcal{F}_{\text{tors}}$ has a filtration with quotients $\mathcal{O}_{p_i}$).

Key idea: for a scheme X we can define the K-groop $K_0(X)$, the Grothendieck group of coherent sheaves. Then we have a homomorphism $\chi : K_0(X) \to \mathbb{Z}$ if X is projective.

Note that if $\deg \mathcal{L} < 0$ then $\Gamma(C, \mathcal{L}) = 0$ (exist $s \in \Gamma(\mathcal{O}_C(p))$. If we can find a section then tensoring gives a section $\Gamma(C, \mathcal{L}(dp))$ where $\deg \mathcal{L}(dp) = 0$. But the only degree 0 invertible sheaf with nonzero global section is the strcture sheaf. Thus $\mathcal{L}(dp)$ is trivial, so every nonzero section never vanishes so s vanishes at p.

Combining this with Serre duality and Riemann-Roch we get

Corollary 10.4. *If* $\deg \mathcal{L} > 2g - 2 = \deg K_C$ *then* $h^0(\mathcal{L}^\vee \otimes K_C)h^1(\mathcal{L}) = 0$ *so*

$$h^0(\mathcal{L}) = \deg \mathcal{L} + 1 - g.$$

Theorem 10.5. *Let* $\varphi : X \to Y$ *be a morphism of smooth projective schem. Assume*

1. φ *injective on* k*-points,*
2. φ *injective on tangent space, i.e. for all* $p \in X$*,* $d\varphi(p) : T_pX \to T_{\varphi(p)}Y$ *injective.*

Then $Z = \varphi(X)$ *(as closed subscheme image) then* $\varphi : X \to Z$ *is an isomorphism.*

Sketch proof. □

Theorem 10.6. *If* $g(C) = 1$ *then exists a closed embedding* $C \hookrightarrow \mathbb{P}^2$ *as a degree* 3 *curve.*

Proof. $\deg K_C = 2g - 2 = 0$. Thus if $\deg \mathcal{L} > 0$ then

$$h^0(\mathcal{L}) = \chi(\mathcal{L}) = \deg \mathcal{L}.$$

Let $\mathcal{L} = \mathcal{O}_C(3p)$. Let s_0, s_1, s_2 be a basis of $H^0(C, \mathcal{L})$. Claim there exists a unique morphism $\varphi : C \to \mathbb{P}^2$ and isomorphism $\mathcal{L} \cong \varphi^*\mathcal{O}_{\mathbb{P}^2}(1)$ such that $s_i = \varphi^*(x_i)$. To show that φ exists, enough to show that for all $q \in C$ exists i such that $s_i(q) \neq 0$, since then we can define

$$\varphi(q) = [s_0(q), s_1(q), s_2(q)].$$

To prove this we note the short exact sequence

$$0 \longrightarrow \mathcal{O}(3p - q) \longrightarrow \mathcal{O}(3p) \longrightarrow \mathcal{O}(3p) \otimes \kappa(q) \longrightarrow 0$$

As $\deg \mathcal{O}(3p - q) = 2 > 0$, we have $h^0(\mathcal{O}(3p - q)) = 2$, so the map

$$H^0(\mathcal{O}_C(3p)) \to H^0(\mathcal{O}(3p) \otimes \kappa(q))$$

is nonzero, i.e. we can find $s_i \in \Gamma(C, \mathcal{O}_C(3p))$ such that $s_i(q) \neq 0$. Show

1. injectivity: for all q_1, q_2, $\varphi(q_1) \neq \varphi(q_2)$. Exists $s = \sum \lambda_i s_i \in \Gamma(C, \mathcal{O}(3p))$ such that $s(q_1) = 0, s(q_2) = 0$...

   Thus we can find $s \in \Gamma(C, \mathcal{O}(3p - q_1))$ such that $s(q_2) \neq 0$.
2. injectivity on tangent space: note that tangent vector is a morphism $\operatorname{Spec} k[\varepsilon]/\varepsilon^2 \to C$, and it is nonzero precisely when it is a closed embedding....

   Note that a k-algebra map $A \to k[\varepsilon]/\varepsilon^2$ is either surjective or factors thorugh $A \to k \to k[\varepsilon]/\varepsilon^2$.

Let $\mathcal{I}_D$ be the ideal sheaf of D and we have a short exact sequence

$$0 \longrightarrow \mathcal{I}_D \longrightarrow \mathcal{I}_q \longrightarrow \mathcal{O}_q \longrightarrow 0$$

(since C is a smooth curve, $\mathcal{O}_{C,q}$ is a DVR and $\mathcal{I}_{D,q} = (t^2)$ so $\mathcal{I}_D = \mathcal{O}_C(-2q)$).

All we need to show is that there exists $s \in \Gamma(c, \mathcal{O}(3d))$ such that $s(q) = 0$ but $s|_D \neq 0$. But the proof is exactly the same as before:

$$0 \longrightarrow \mathcal{O}(3p - 2q) \longrightarrow \mathcal{O}(3p - q) \longrightarrow \mathcal{O}(3p)|_D \longrightarrow 0$$

This shows that there is an embedding $C \hookrightarrow \mathbb{P}^2$. To check that the image has degree 3, we can either show $\deg \varphi^*\mathcal{O}(1) = 3$, or use adjunction

$$\deg K_C = \deg K_{\mathbb{P}^2} + \deg Z.$$

□

By a similar argument we can show if $g(C) = 0$ then taking p gives an isomorphism $C \to \mathbb{P}^1$. If $g \geq 2$ then taking $3K_C$ gives $C \hookrightarrow \mathbb{P}^N$. In fact one can show that either $2K_C$ (in fact K_C) gives a closed embedding, or K_C gives a two-to-one cover to a rational normal curve.

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