Scuola Internazionale Superiore di Studi Avanzati

Geometry and Mathematical Physics

Introduction to linear ODEs in the Complex Domain and Isomonodromy Deformations

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1 Introduction

solve integrable systems, physics problems such as Schrödinger equation.

Bessel equation

$$z^2y'' + zy' + (z^2 - \nu^2)y = 0$$

where $\nu \in \mathbb{C}$.

Airy

$$y'' = zy$$

Gauss equation

$$z(1-z)y'' + (\gamma - (\alpha + \beta + 1)z)y' - \alpha\beta y = 0$$

where $\alpha, \beta, \gamma \in \mathbb{C}$.

The Bessel equation occurs in solving the Laplace equation $\nabla\varphi = 0$, and the Gauss equation occurs when solving the Schrödinger equation.

More generally, using the notation $y^{(m)} = \frac{d^m y}{dz^m}$, a linear ODE of order m has the form

$$y^{(m)} + a_1(z)y^{(m-1)} + \cdots + a_m(z)y = b(z).$$

As we will see, this is a special case of a system of ODEs

$$\frac{dy}{dz} = \underbrace{\begin{pmatrix} a_{11(z)} & \cdots & a_{1m}(z) \\ \vdots & & \vdots \\ a_{n1}(z) & \cdots & a_{nm}(z) \end{pmatrix}}_{A(z)} y$$

and y can be either a vector or a matrix. This is a *linear system of order 1*.

What if the matrix is not analytic? That is, if the a_{ij}'s have poles in the complex plane? The question is

- locally, how to represent the solutions closed to singularities?
- globally, how to connect the solutions around different singularities?

Another theme is the study of *monodromy propertie*: if we take a solution and take a loop around a singularity, the solution will transform under $y(z) \mapsto y(z)M_\gamma$, where M_γ is the *monodromy matrix*. We will see the result depends only on the homotopy class of the path, and can be described by the monodromy group.

If we have an ODE where the matrix depends on parameters

$$\frac{dY}{dz} = A(z, t_1, \ldots, t_p)Y,$$

then not only will the solution $Y(z,t)$ depends on the parameters t, but also will the singularities and the monomromy matrix $M_\gamma(t)$. We will in particular study monodromy "data" that do not depend on t. This is called *monodromy preserving deformation*.

We may also require

$$\frac{\partial Y}{\partial t_j} = \Omega_j(z,t)Y, j = 1, \ldots, p.$$

subject to the compatibility conditions

$$\frac{\partial^2 Y}{\partial t_j \partial t_k} = \frac{\partial^2 Y}{\partial t_k \partial t_j}$$
$$\frac{\partial^2 Y}{\partial z \partial t_j} = \frac{\partial^2 Y}{\partial t_j \partial z}$$

These compatibilities are nonlinear differential equations for A and Ω_j's.

These compatibilies arise naturally in physics problems, such as

- the Painlevé equations,
- structure of manifolds (Dubrovin-Frobenius manifolds),
- random matrices
- nonlinear PDEs such as KdV and KP

1.1 Preliminaries

If f is holomorphic on a domain B then f is smooth. The reason is Cauchy integral formula

$$f(z) = \frac{1}{2\pi i} \oint_\gamma \frac{f(z')}{z'-z} dz'$$

for γ a simple counterclockwise path around z, and if $w(z')$ is continuous on γ, let

$$F_m(z) = \int \frac{w(z')}{(z'-z)^m} \frac{dz'}{2\pi i}$$

for $m \geq 1$, then $F_m(z)$ is homolomorphic inside (resp. outside) γ if z is inside (resp. outside), and

$$\frac{dF_m}{dz} = mF_{m+1}(z).$$

Together we have given f holomorphic then

$$\frac{d^m f}{dz^m} = \frac{m!}{2\pi i} \int \frac{f(z')}{(z'-z)^{m+1}} dz$$

and if f is holomorphic on the ball of radius R centred at z_0 then $f(z) = \sum c_k(z-z_0)^k$ absolutely convergent in any ball $|z-z_0| \leq r \leq R$, a Taylor series with radius of convergence R. Because of this we say f is analytic, meaning there exists a Taylor series with finite radius, and holomorphic is the same as analytic. The real counterpart is obviously false by considering the smooth function $f(x) = e^{-1/x^2}, x \in \mathbb{R}$ which does not have a Taylor series with positive radius of convergence at the origin.

1.2 Convergence in the space of holomorphic functions

Let $H(B)$ be the space of functions $f : B \to \mathbb{C}$ analytic. Given a sequence of functions $\{f_n\}_{n\in\mathbb{N}}$ where $f_n \in H(B)$, what does it mean that $f_n \to f \in H(B)$? The correct notion is locally uniform convergence.

Recall that if $K \subseteq \mathbb{C}$ is compact then $(C(K), \|\cdot\|_\infty)$ is complete.

Lemma 1.1. *Let $B \subseteq \mathbb{C}$ be a domain. There exists a sequence $\{K_n\}_{n\in\mathbb{N}}$ of compact sets contained in B such that*

1. *$B = \bigcup_{n=0}^{\infty} K_n$,*
2. *K_n is contained in the interior of K_{n+1},*
3. *for all $K \subseteq B$ compact, exists m such that $K \subseteq K_m$.*

Lemma 1.2. *Let (X, d) be a metric space. Define*

$$\begin{aligned} \mu : X \times X &\to [0,1) \\ (x,y) &\mapsto \frac{d(x,y)}{1+d(x,y)} \end{aligned}$$

then μ is a distance and d, μ define the same topology on X.

For $f, g \in C(B)$, define $d_m(f,g) = \|f-g\|_{K_m,\infty}$, then

$$\rho(f,g) = \sum_{n=0}^{\infty} 2^{-n} \frac{d_n(f,g)}{1+d_n(f,g)}.$$

Theorem 1.3. *$(C(B), \rho)$ is a complete metric space. Given $\{f_m\} \subseteq C(B)$, $f_m \to f$ in metric ρ if and only if $f_m \to f$ uniformly on every compact subset of B, or equivalently on every K_n.*

Theorem 1.4 (Weierstrass)**.** *$H(B) \subseteq C(B)$ is closed in $C(B)$. In particular it is complete.*

1.3 Systems of ODEs

Let $D \subseteq \mathbb{C}^{n+1}$ be a domain. We denote a point in D by $(z, y_1, \ldots, y_n)$ and write an analytic function $f_j : D \to \mathbb{C}$ as $f_j(z, y_1, \ldots, y_n)$ or simply $f_j(z, y)$.

Definition (system of ODEs)**.** A *system of ODEs* of the first order and dimension n

$$y' = f(z,y),$$

meaning

$$\begin{aligned} y_1 &= f_1(z, y_1, \ldots, y_n) \\ &\vdots \\ y_n &= f_n(z, y_1, \ldots, y_n) \end{aligned}$$

is the problem of finding a domain $B \subseteq \mathbb{C}$ and $y : B \to \mathbb{C}^n$ analytic such that

1. $(z, y(z)) \in D$ and

2. $y'(z) = f(z, y(z))$.

Definition (initial value problem)**.** Let $(z_0, y_0) \in D$. An *initial value problem* (ivp) is the problem of finding $B \ni z_0$ and a solution $y(z)$ such that $y(z_0) = y_0$. It is usually written as

$$y' = f(z, y), y(z_0) = y_0.$$

Remark. Suppose $y = (y_1, \ldots, y_n)$. Then

$$|y| = \sum_{j=1}^{n} |y_j|$$

is a norm.

1.4 Existence and uniqueness theorem

Theorem 1.5 (existence and uniqueness theorem)**.** *Suppose $f : D \to \mathbb{C}^n$ is a vector valued function on a domain that is analytic and bounded, where*

$$D = \{(z, y) \in \mathbb{C}^{n+1} : |z - z_0| < a, |y - y_0| < b\}$$

is a polydisc. Let $M = \sup_{z \in D} |f(z)|$. The the ivp

$$y' = f(z, y), y(z_0) = y_0$$

has a unique solution $y(z)$ analytic on $U_\alpha(z_0) = \{z \in \mathbb{C} : |z - z_0| < \alpha\}$ where $\alpha = \min\{a, b/M\}$.

Exercise. Let $y(z) = y_0 + \int_{z_0}^{z} f(\zeta, y(\zeta))d\zeta$.

1. If $y(z)$ is an anlytic solution of the ivp then it also solves I.
2. Conversely if $y(z)$ is an analytic solution of I then it also solves the ivp.

Proof. The proof uses Picard's method of successive approximation: let

$$y_0(z) = y_0$$
$$y_{m+1}(z) = y_0 + \int_{z_0}^{z} f(\zeta, y_n(\zeta))d\zeta.$$

We first show $y_n(z)$ is analytic on $U_\alpha(z_0)$. y_1 is analytic because the integrand is analytic (does not depend on the path) and

$$|y_1(z) - y_0| \leq \int_{z_0}^{z} |f(\zeta, y_0)|d|\zeta| \leq M \cdot |z - z_0| < b.$$

Inductively y_n is analytic and $|y_n(z) - y_0(z)| \leq M|z - z_0| < b$.

Next for every $K \subseteq D$ compact, the partial derivatives $\frac{\partial f_i(z,y)}{\partial y_j}$ are bounded on K. Then f is Lipschitz continuous with respect to y, i.e. exists $k > 0$ such that

$$|f(z, y) - f(z, \tilde{y})| \leq k|y - \tilde{y}|.$$

Let $K(\varepsilon) = \{|z - z_0| \leq \alpha - \varepsilon\}$. Then

$$|y_n(z) - y_0| \leq M|z - z_0| \leq b - \varepsilon M.$$

Now using telescopic sum

$$y_m(z) = y_0 + \sum_{j=0}^{m-1} (y_{j+1}(z) - y_j(z))$$

We can estimate the sum by

$$\begin{aligned} |y_2 - y_1| &\leq \int_{z_0}^{z} |f(\zeta, y_1(\zeta) - f(\zeta, y_0)|d|\zeta| \\ &\leq k \int_{z_0}^{z} |y_1(\zeta) - y_0)d|\zeta| \\ &\leq kM \int_{z_0}^{z} |\zeta - z_0|d|\zeta| \\ &= \frac{M}{k} \frac{k^2|z - z_0|^2}{2} \end{aligned}$$

using Lipschitz continuity. By induction we can show the sum coverges. Thus on $K(\varepsilon)$, $y_m(z)$ converges uniformly to some $y(z)$.

Next we show $y(z)$ on K is an analytic solution of the ivp, or equivalently the corresponding integral equation

$$y(z) = y_0 + \int_{z_0}^{z} f(\zeta, y(\zeta))d\zeta.$$

For reality check, note that $|y(z) - y_0| = \lim_{n\to\infty} |y_n(z) - y_0| < b$ so the expression $f(z, y(z))$ makes sense. By uniform convergence

$$\lim_{n\to\infty} f(z, y_n(z)) = f(z, y(z))$$

so

$$\begin{aligned} y(z) &= \lim_{n\to\infty} y_{n+1}(z) \\ &= y_0 + \lim_{n\to\infty} \int_{z_0}^{z} f(\zeta, y_n(\zeta))d\zeta \\ &= y_0 + \int_{z_0}^{z} \lim_{n\to\infty} f(\zeta, y_n(\zeta))d\zeta \text{ uniform convergence} \\ &= y_0 + \int_{z_0}^{z} f(\zeta, y(\zeta))d\zeta \end{aligned}$$

For unqiueness, suppose $Y(z)$ is another solution defined on $\}|z - z_0| \leq \alpha - \varepsilon - \varepsilon'\}$ such that $(z, Y(z)) \in D$. We show that on this domain $Y(z) = \lim_{n\to\infty} y_n(z)$. This can be done using the integral equation to estimate $|Y(z) - y_n(z)|$ inductively and show the limit is zero. Thus $y(z)$ and $Y(z)$ conincides on $|z - z_0| \leq \alpha - \varepsilon - \varepsilon'$, so $y(z)$ is the analytic continuation of $Y(z)$.

To move from compact subsets to the entire domain, let $\{\varepsilon_m\}$ be a decreasing sequence converging to 0 and let $K_m = \{|z - z_0| \leq \alpha - \varepsilon_m\}$. Then $\{K_m\}$ is a candidate for the sets in lemma 1. By construction

$$y_n^{K_m}(z) = y_n^{K_{m+1}}(z)|_{K_m}$$

so as $n \to \infty$,

$$y^{K_m}(z) = y^{K_{m+1}}|_{K_m}$$

so $y^{K_{m+1}}$ is the analytic continuation of y^{K_m} to K_{m+1}. Thus

$$y_{n+z}(z) = y_0 + \int_{z_0}^{z} f(\zeta, y_n(\zeta))d\zeta$$

converges on each K_m uniformly, so by lemma 3 it converges to an analytic solution $y(z)$. Thus $y(z)$ is the unique solution on $U_\alpha(z_0)$. □

2 Linear systems of ODEs

Consider a matrix-valued functions

$$A : U_a(z_0) = \{z \in \mathbb{C} : |z - z_0| < a\} \to \mathrm{Mat}(n, \mathbb{C})$$
$$b : U_a(z_0) \to \mathbb{C}^n$$

with each entry analytic. We are interested in a *linear system of ODEs*

$$y'(z) = \underbrace{A(z)y + b(z)}_{f(z,y)}.$$

We use the matrix norm $|A| = \sum |A_{ij}|$ which can be easily checked to satisfy $|Ay| \leq |A||y|$.

Lemma 2.1. *Suppose $A, b : B \to \mathbb{C}$ be analytic and $K \subseteq B$ compact. Then $f(z, y)$ is bounded on K and Lipschitz continuous with respect to y.*

Proof.

$$|f(z, y(z)) - f(z, \tilde{y}(z))| = |A(z)(y - \tilde{y})| \leq |A(z)||y - \tilde{y}| \leq k|y - \tilde{y}|.$$

□

Theorem 2.2. *Suppose A, b are analytic on $U_a(z_0)$. Then the ivp*

$$y' = A(z)y + b(z), y(z_0) = y_0$$

where y_0 is any complex number, has a unique analytic solution in $U_a(z_0)$.

The proof is essentially the same, but note that there is no bound on y_0, and the domain of the solution has the same radius as the domain of A and b. This is because f is analytic on $D = \{z : |z - z_0| < \alpha\} \times \mathbb{C}$ so there is no bound on the second variable.

Theorem 2.3. *Let A, b be analytic on B, a simply connected domain. Then the ivp*

$$y' = A(z)y + b(z), y(z_0) = y_0$$

has unique solution $y(z)$ analytic on B.

Suppose $f_1 : U_1 \to \mathbb{C}, f_2 : U_2 \to \mathbb{C}$ are analytic, $U_1 \cap U_2 \neq \emptyset$ and $f_1(z) = f_2(z)$ for $z \in U_1 \cap U_2$ then f_2 is the analytic continuation of f_1 on U_2.

Suppose $f : U \to \mathbb{C}$ is anlytic and γ is a path from $z_0 \in U$ to $\tilde{z}$. Then we say f has an *analytic continuation along* γ if we can cover γ with a finite number of open balls such that f has analytic continuation on the union of the balls.

Theorem 2.4 (monodromy theorem)**.** *Suppose B is a connected domain. Suppose f is analytic on an open ball $U \subseteq B$. Assume that f has analytic continuation along any curve in B. Then if γ_1 and γ_2 are two homotopic path from $z_0 \in U$ to $\tilde{z}$ then the analytic continuations $f_{\gamma_1}, f_{\gamma_2}$ agree on a neighbourhood of $\tilde{z}$.*

In particular if in addition B is simply connected then f has a unique analytic continuation on the whole B.

Proof of Theorem 2.3. Suppose $\tilde{z} \in B$ and take a path γ from z_0 to $\tilde{z}$. Note that as B is open there exists $\varepsilon > 0$ such that $B(z, \varepsilon) \subseteq B$ for all $z \in \operatorname{im}\gamma$. Cover the compact path γ with finitely many open balls and for sufficiently small ε' we can solve the ivp on $B(z', \varepsilon')$. The solution is unique □

2.1 Homogeneous system

A *homogeneous system* is one in which $b = 0$, i.e. $y' = A(z)y$. Let us asume that A is analytic on B simply connected. Then the system defines a linear operator

$$\begin{aligned} E : H(B) &\to H(B) \\ y &\mapsto y' - A(z)y \end{aligned}$$

Then the space of solutions to the ODEs is $\ker E$.

Theorem 2.5. *If the system has dimension n then $\dim\ker E = n$.*

Proof. Let $v_1, \dots, v_n$ be a basis of $\mathbb{C}^n$. Define $y_i(z)$ to be the unique analytic solution on B of $y_i(z_0) = v_i$. $\ker E$ is spanned by y_i's. Linear independence can be checked at z_0. □

$y_1(z), \dots, y_n(z)$ are called a *fundamental system*. We define $Y(z)$ to be the matrix whose columnes are $y_i(z)$. Then

$$\frac{dY}{dz} = [\frac{dy_1}{dz}|\cdots|\frac{dy_n}{dz}] = [Ay_1|\cdots|Ay_n] = A[y_1|\cdots|y_n] = A(z)Y(z).$$

$Y(z)$ is called a *fundamental matrix solution.*

Corollary 2.6. *Analytic continuation along any curve preservse linear independence.*

In other words, $\det Y(z_0) \neq 0$ implies $\det Y(z) \neq 0$ for all $z \in B$. Suppose we have another fundamental matrix solution

$$\tilde{Y}(z) = [\tilde{y}_1(z)|\cdots|\tilde{y}_n(z)] = [\sum_{k=1}^n c_{k1}y_k(z)|\cdots|\sum_{k=1}^n c_{kn}y_k(z)] = Y(z)C$$

where $C = (c_{ij})$ nonsingular.

One application: suppose we have a homogeneous ODE of order n

$$u^{(n)} + p_1(z)u^{(n-1)} + \cdots + p_n(z)u = 0$$

where u is a scalar and $p_i : B \to \mathbb{C}$ analytic. Then this can be recast into a linear ODE by setting $y_1 = u, y_2 = y'$ etc:

$$y' = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ \vdots & & & \\ -p_n & -p_{n-1} & \cdots & -p_1 \end{pmatrix} y.$$

This is called the *companion matrix.* $u_1(z), \dots, u_n(z)$ is a fundamental system.

2.2 Inhomogeneous system

We will see that it suffices to study homogeneous case when studying linear systems. Suppose we have an inhomogeneous problem $y' = A(z)y + b(z)$, where A, b are analytic on a simply connected domain B. Then in matrix form it is

$$\frac{dW}{dz} = A(z)W + [b(z)| \cdots |b(z)]$$

where W is an (unknown) $n \times n$ matrix. Of course it is a special case of

$$\frac{dW}{dz} = A(z)W + [b(z)| \cdots |b(z)] \tag{NH}$$

where F is analytic and matrix valued. We call $\frac{dY}{dz} = A(z)Y$ the associated homogeneous system, and let its fundamental matrix system be $Y(z)$ with $\det Y(z) \neq 0$.

Theorem 2.7. *All solutions of (NH) are*

$$W(z) = Y(z)[C + \int_{z_0}^{z} Y^{-1}(\zeta)F(S)d\zeta]$$

where $Y(z)$ is a fundamental solution of the homogeneous system.
The ivp $W(z_0) = W_0$ has solution obtained with $C = Y(z_0)^{-1}W_0$.

Proof. Differentiate the purported solution,

$$\begin{aligned} W' &= Y'[C + \int_{z_0}^{z} Y^{-1}(\zeta)F(\zeta)d\zeta] + Y(z)Y^{-1}(z)F(z) \\ &= A(z)Y(z)[C + \int_{z_0}^{z} Y^{-1}(\zeta)F(\zeta)d\zeta] + F(z) \\ &= A(z)W(z) + F(z) \end{aligned}$$

Every solution of (NH) is obtained from some ivp W_0, from which we derive $C = Y(z_0)^{-1}W_0$. Therefore we obtain every solution. ☐

The formula is obtained by “variation of parameter”. In fact it is instructive to do the following easy exercise: we seek a solution of the form $W(z) = Y(z)C(z)$. Substitute into (NH), we will get $C'(z) = Y^{-1}(z)F(z)$. Integrate to get the desired solution in the theorem.

Now back to the homogeneous problem $\frac{dY}{dz} = A(z)Y$. Suppose A is holomorphic on B. Take $z_0 \in B, r > 0$ maximal such that $U_r(z_0) \subseteq B$. In particular A is holomorphic in the ball $U_r(z_0)$. A has a power series representation $A(z) = \sum A_j(z - z_o)^j$ convergent uniformly in every compact subset of $U_r(z_0)$ and the radius of convergence is r. What we know is that

1. there exists a unique solution to the ivp $Y(z_0) = Y_0$.
2. $Y(z)$ is analytic in $U_r(a)$.

Thus $Y(z) = \sum Y_k(z-z_0)^k$ for $|z-z_0| < r$. Substitute into the equation,

$$\sum_{k=1}^{\infty} kY_k(z-z_0)^{k-1} = \sum_{j=0}^{\infty}\sum_{k=0} A_j Y_k (z-z_0)^{j+k}$$

i.e.

$$\sum_{m=0}^{\infty}(m+1)Y_{m+1}(z-z_0^m = \sum_{m=0}^{\infty}(\sum_{k=0}^{m} A_{m-k}Y_k)(z-z_0)^m.$$

We get the recurrence relation

$$(m+1)Y_m = \sum_{k=0}^{m} A_{m-k}Y_k$$

with (arbitrary) initial condition Y_0. Up to order M, the solution has local representation

$$Y(z) = \sum_{k=0}^{M} Y_k(z-z_0^k + O((z-z_0)^{M+1}).$$

3 Singularities and monodromies

3.1 Classification of isolated singularities

Suppose $f : B \setminus \{a\} \to \mathbb{C}$ is holomorphic where B is a domain. Then a is called an *isolated singularity*. By basic complex analysis a is one of the below

1. a is a *removable singularity* if $|f(z)|$ is bounded in some $U_r(a) \setminus \{a\}$, in which case the limit $\lim_{z\to a} f(z)$ exists, and $\tilde{f}(z) = \begin{cases} f(z) & z \neq a \\ \lim_{z\to a} f(z) & z = a \end{cases}$ is holomorphic on B.

2. a is a *pole* if $\lim_{z\to\infty} |f(z)| = \infty$, in which case exists r such that $f(z) = \frac{g(z)}{(z-a)^m}$, where g is holomorphic on $U_r(a)$ and $g(a) \neq 0$ and $m \geq 1$. m is called the order of the pole. $f(z)$ has a Laurent exptension at $z = a$.

3. a is an *essential singularity* if $\lim_{z\to a} |f(z)|$ does not exist, in which case $f(z)$ takes all possible complex values except possibly one (called the lacunary value) in a neighbourhood of a.

3.2 Singularities of linear systems

Now consider $y' = A(z)y + b(z)$ where A, b are analytic on $B \setminus \{a_0, \ldots, a_m\}$. Let $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. Then for a function $f(z)$ defined on $\mathbb{C}$, we can consider ∞ as an isolated singularity and use the classification before.

We call $a_1, \ldots$ singularities of the linear differential system.

> **Theorem 3.1.** *Given a solution $y(z)$ of a linear system, the set of singularities of y is a subset of the singularities of the equation.*

Proof. If A, b are analytic in a ball (which is simply connected) then all $y(z)$ are analytic there too. If $z = a$ is a singularity of $y(z)$, then necessarily $z = a$ is a singularity of either $A(z)$ or $b(z)$. □

Remark.

1. The converse to the theorem is false. Consider $y' = \frac{\mu}{z-a} y$ where $\mu \in \mathbb{C}$. The solution $y = c(z-a)^\mu$ is
   (a) regular at $z = a$ if $\mu \in \mathbb{N}$,
   (b) a pole if $\mu \in \{-1, -2, \ldots\}$,
   (c) a branch point if $\mu \in \mathbb{C} \setminus \mathbb{Z}$.

2. Note that if we consider A, b as functions defined on $\overline{\mathbb{C}}$, then $z = \infty$ can be a singularity of the solution even if it is not a singularity of the equation (i.e. A, b have a removable singularity at ∞). For example consider $y' = y$: solution $y(z) = ce^z$. $z = \infty$ is a singularity.

3. The theorem only applies to linear ODEs. Consider for example $y' = -y^2$. Solution $y(z) = \frac{1}{z-a}$. $z = a$ is a pole of $y(z)$ but not a singularity of the equation.

3.3 Monodromy for homogeneous systems

Consider the homogeneous problem $\frac{dY}{dz} = A(z)Y$ where $A : B \setminus \{a\} \to \mathbb{C}$ analytic. We have seen that analytic continuation along ℓ from z_0 to z' defines $y(z)$ analytic in a neighbourhood of z'. Take another path $\tilde{\ell}$ which is not homotopic to ℓ and analytic continuation along ℓ' defines $\tilde{y}(z)$.

Note that if B is a disc, we can take a *branch cut* L from a to the boundary of B, and then $B \setminus L$ is simply connected. $y, \tilde{y}$ are analytic on $B \setminus L$. $y, \tilde{y}$ are called *branches* of each other. Let $\gamma = \ell^{-1} \cdot \tilde{\ell}$, then travelling along γ transforms y to $\tilde{y}$. $\tilde{y}$ is an *analytic continuation* of y.

If there are more than one singularity $\{a_1, \ldots, a_m\}$ in B, we can do a branch cut L_i from each a_i, and $y, \tilde{y}$ are analytic on $B \setminus \{L_1, \ldots, L_m\}$.

For a homogeneous system $\frac{dY}{dz} = A(z)Y$, we proved that there exsits a fundamental solution $Y(z) = [y_1(z)|\cdots|y_n(z)]$ analytic on $\mathbb{C} \setminus \{L_1, \ldots, L_n\}$. If we take another path we get $\tilde{Y}(z)$. But columns of Y form a basis, we can express

$$\tilde{Y}(z) = [\sum m_{i1} y_i|\cdots|\sum_i m_{ni} y_i] = Y(z) M_\gamma,$$

where $M_\gamma = (m_{ij})$ is called the *monodromy matrix* of $Y(z)$ associated to (the homotopy class) of γ.

Recall from algebraic topology that $\pi_1(\overline{\mathbb{C}}\{a_1, \ldots, a_m, \infty\}, z_0)$ is the free group generated by $\gamma_1, \ldots, \gamma_m$. Also there is an automorphism given by conjugations

$$\begin{aligned} \pi(\overline{\mathbb{C}} \setminus \{a_1, \ldots, a_m, \infty\}, z_0) &\to \pi(\overline{\mathbb{C}} \setminus \{a_1, \ldots, a_m, \infty\}, z_1) \\ \gamma &\mapsto \lambda\gamma\lambda^{-1} \end{aligned}$$

where λ is a path from z_1 to z_0. Now suppose $Y(z_1)$ is the analytic continuation of $Y(z_0)$ along λ^{-1}. Then obviously the analytic continuations of $Y(z_0)$ along $\gamma \cdot \lambda^{-1}$ and along $\lambda^{-1} \cdot (\lambda \cdot \gamma \cdot \gamma^{-1})$ yields the same result

$$\begin{array}{ccc} Y(z_0) & \xrightarrow{\gamma} & Y(z_0)M_\gamma \\ \downarrow{\scriptstyle \lambda^{-1}} & & \downarrow{\scriptstyle \lambda^{-1}} \\ Y(z_1) & \xrightarrow{\lambda\gamma\lambda^{-1}} & Y(z_1)M_\gamma \end{array}$$

But the bottom row is precisely the monodromy transformation associated to $\lambda\gamma\lambda^{-1}$ so we have $Y(z)M_\gamma = Y(z)M_{\lambda\gamma\lambda^{-1}}$. Since $Y(z)$ is invertible, $M_{\lambda\gamma\lambda^{-1}} = M_\gamma$.

Theorem 3.2. *Let z_0, Y_0 be given ($\det Y_0 \neq 0$) and let $Y(z)$ be the unique fundamental matrix solution analytic in $\mathbb{C} \setminus \{L_1, \ldots, L_m\}$ solving the ivp.*

$$\begin{aligned} \phi : \pi_1(\overline{\mathbb{C}} \setminus \{a_1, \ldots, a_m, \infty), z_0) &\to \mathrm{GL}(n, \mathbb{C}) \\ \gamma &\mapsto M_\gamma \end{aligned}$$

this is called the monodromy representation of π_1. *This is an anti-representation.*

Proof. Exercise. □

$\gamma_1, \dots, \gamma_m$ generates the group $\mathcal{M} = \langle M_{\gamma_1}, \dots, M_{\gamma_n} \rangle$, the *monodromy group* associated with the ivp $Y(z_0) = Y_0$.

If we have another ivp $Y(z_0) = \hat{Y}_0$, ther is a unique solution $\hat{Y}(z)$ defined on $\mathbb{C} \setminus \{L_1 \cup \cdots L_m\}$. The two solutions are related by $\hat{Y}(z) = Y(z)C$ where $C = Y_0^{-1}\hat{Y}_0 \in \mathrm{GL}(n, \mathbb{C})$. Thus along the path γ,

$$\hat{Y}(z) = Y(z)C \mapsto Y(z)M_\gamma \cdot C = Y(z)C \cdot \underbrace{C^{-1}M_\gamma C}_{\hat{M}_\gamma}.$$

Thus from the data of $A(z)$, we get $\langle C^{-1}M_{\gamma_1}C, \dots, C^{-1}M_{\gamma_n}C \rangle$ for all $C \in \mathrm{GL}(n, \mathbb{C})$. This is an equivalence relation on monodromy groups.

Definition (direct monodromy map)**.** Fix points $a_1, \dots, a_m \in \mathbb{C}$. The *direct monodromy map* is the map

$$\{A(z) \text{ isolated singularities } a_1, \dots, a_m, \infty\} \to \{\text{monodromy group}\}/\text{conjugation}.$$

It is worth remarking here the opposite problem, that is, given monodromy groups up to conjugation, can we determine the differential system? This is called the *inverse monodromy problem* or *Riemann-Hilbert problem*.

3.4 Solutions on universal covers

Fix a base point z_0. Analytic continuation gives a function $Y(z) =$ "$y(z, \ell)$" that depends on (the homotopy class of) the path ℓ. Define an equivalence relation $(z, \ell) \sim (z', \ell')$ if and only if $z = z', \ell \simeq \ell'$. The the set of equivalence classes is in bijection with the universal cover. The fundamental group acts on the universal cover by deck transformation. If there is no chance of confusion we denote a point $[(z, \ell)]$ by $\tilde{z}$.

The analytic continuation of $Y(z)$ along any path ℓ gives a function $\mathbb{Y}$ on the universal convering so, defines a function $\mathbb{Y}(\tilde{z})$ is a function of $\tilde{z}$ on the universal cover. $Y(z)$ is a *branch* of $\mathbb{Y}(\tilde{z})$.

We have seen that $\pi_1(\overline{\mathbb{C}} \setminus \{0, \infty\}; z_0) \cong \mathbb{Z}$. A point $[(z, \ell)] \in \widetilde{\overline{\mathbb{C}} \setminus \{0, \infty\}}$ can be represented by $(z, \ell_m = \gamma^m \cdot \ell)$ where $m \in \mathbb{Z}$ and ℓ is a path from z_0 to z. We give the another presentation. Take a branch cut $(-\infty, 0]$ in the complex domain and consider the principal branch of logarithm, defined so that $\ln z \in \mathbb{R}$ for $z > 0$. Then $z \mapsto \ln z$ represents this: $\ln z = \ln |z| + i \arg z$. If a path crosses the branch cut m times, the $\arg z$ is transformed to $\arg z + 2\pi m$. In this case the universal covering is the exponential map

$$\begin{aligned} \exp : \widetilde{\overline{\mathbb{C}} \setminus \{0, \infty\}} &\to \overline{\mathbb{C}} \setminus \{0, \infty\} \\ \ln z + 2\pi i m &\mapsto z \end{aligned}$$

For this reason $\widetilde{\overline{\mathbb{C}} \setminus \{0, \infty\}}$ is called the *Riemann surface of the logarithm*. By a slightly confusing change of notation, we can represent a point in the universal cover by $\tilde{z} = |z|e^{i \arg z}$, where the exponential is a formal symbol (so $|z|e^{i \arg z}$ and $|z|e^{i \arg z + 2\pi i m}$ are different points). The advantage of this notation is that monodromy can be computed easily just by symbolic manipulation. For example

$$\tilde{z}^c = e^{c \ln \tilde{z}} = |z|^c e^{ic \arg z}$$

whose meaning is that if the locally defined function $z \mapsto z^c$ transforms with a multiple of $e^{2\pi i c}$ when taken along the path that loops around the origin once.

4 Classification of isolated singularities

4.1 Matrix exponential and logarithm

We endow $\mathrm{Mat}(n, \mathbb{C})$ with the matrix norm $|A| = \sum |A_{ij}|$. It is a complete metric space. Then

$$e^A = \sum_{k=0}^{\infty} \frac{A^k}{k!}$$

is convergent. Similarly for f analytic, we can define

$$e^{A \cdot f(z)} = \sum_{k=0}^{\infty} \frac{A^k f(z)^k}{k!}$$

which is locally uniformly convergent so analtyic. As a special case, take the principal branch of logarithm and we can define

$$z^A = e^{A \log z}.$$

Example. If going along a path transforms $z \mapsto z \cdot e^{2\pi i}$ then

$$z^A = e^{A \ln z} \mapsto e^{2\pi i A} z^A$$

and indeed $e^{2\pi i A}$ is the monodromy matrix. We can think of this as defined on the universal covers by $\tilde{z}^A = |z|^A e^{Ai \arg z}$, where the formal exponential helps us keep track of monodromy actions.

Exercise. Matrix exponential has the following properties:

1. if $A = GBG^{-1}$ for some $G \in \mathrm{GL}(n, \mathbb{C})$ then $e^A = Ge^B G^{-1}$. It follows that $z^A = Gz^B G^{-1}$.
2. If $[A, B] = 0$ then $e^{A+B} = e^A e^B$.
3. $\det e^A = e^{\operatorname{tr} A}$. In particular $\det e^A \neq 0$ (hint: put into Jordan form).
4. $\frac{d}{dz} e^{Af(z)} = Af'(z) e^{Af(z)}$ so $\frac{d}{dz} z^A = \frac{A}{z} \cdot z^A$. Thus $\frac{dY}{dz} = \frac{A}{z} Y$ has a fundamental solution $Y(z) = z^A$. As usual we write $\mathbb{Y}(\tilde{z}) = \tilde{z}^A$ for the function defined on the universal cover.

   For $k \geq 2$, $\frac{dY}{dz} = \frac{A}{z^k} Y$ has solution $Y(z) = \exp(-\frac{A}{k-1} \frac{1}{z^{k-1}})$. On the other hand $\frac{dY}{dz} = z^k AY$ for $k \geq 0$ has solution $Y(z) = \exp(\frac{A}{k+1} z^{k+1})$.
5. If $[A_i, A_j] = 0$ then

   $$\frac{dY}{dz} = (\frac{A_1}{z - a_1} + \cdots + \frac{A_m}{z - a_m})Y$$

   has a solution just as if they are numbers: $Y(z) = (z-a_1)^{A_1} \cdots (z-a_m)^{A_m}$.
6. Suppose all matrices commute and $h \geq 0, k \geq 2$, then

   $$\frac{dY}{dz} = (\frac{1}{z^k}(A_0 + A_1 z + \cdots + A_{k-2} z^{k-2}) + \frac{B}{z} + z^h(C_0 + \frac{C_1}{z} + \cdots + \frac{C_h}{z^h}))Y$$

   has solution

   $$\mathbb{Y}(\tilde{z}) = \tilde{z}^B \exp(\frac{A_0}{1-k} \frac{1}{z^{k-1}} + \cdots + \frac{A_{k-2}}{-z}) \exp(\frac{C_0}{h+1} z^{h+1} + \cdots + C_h z) = \Phi(z) \tilde{z}^B.$$

Definition. A *logarithm of* A, where $\det A \neq 0$, written $L = \ln A$, is a matrix such that $e^L = A$.

Matrix logarithm is not unique: suppose there exists an L, then $L + 2\pi ikI$ is also a logarithm.

We can compute matrix logarithm by making a series of reductions. Assuming everything exists, note

$$e^{\ln A} = A = GJG^{-1} = Ge^{\ln J}G^{-1} = e^{G \ln JG^{-1}}$$

so we can let $\ln A = G \ln JG^{-1}$ (up to some issue of determinancy). Thus it is left to define and compute $\ln J$.

Exercise. Check that exp preserves block-diagonal form. Conversely suppose e^A is block-diagonal then A is also block diagonal.

Write $A = GJG^{-1}$ where J is the Jordan normal form with Jordan blocks $J_1, \dots, J_s$. Write $J_i = \lambda_i I_i + H_i$ where λ_i is the eigenvalue of J_i and $H = \begin{pmatrix} 0 & 1 & & \\ & 0 & 1 & \\ & & & \ddots \end{pmatrix}$. The size of Jordan block J_i is m_i. We define

$$\begin{aligned} \ln J_i &= \ln(\lambda_i I_i (I_i + \frac{H_i}{\lambda_i})) \\ &= \ln(\lambda_i I_i) + \ln(I_i + \frac{H_i}{\lambda_i}) \\ &= (\ln \lambda_i) I_i + \sum_{k=0}^{\infty} \frac{(-1)^{k+1}}{\lambda_i} (\frac{H_i}{\lambda_i})^k \\ &= (\ln \lambda_i) I_i + \sum_{k=0}^{m_i - 1} \frac{(-1)^{k+1}}{\lambda_i} (\frac{H_i}{\lambda_i})^k \end{aligned}$$

as H_i is nilpotent.

Again note that $\ln J_i$ is defined up to integer multiples of $2\pi i I_i$.

4.2 Local structure of fundamental matrices at isolated singularities

Consider $\frac{dY}{dz} = A(z)Y$ where $A(z)$ is analytic on $U(a) \setminus \{a\}$. The universal covering can be represented by logarithm

$$\ln(z - a) = \ln |z - a| + i \arg(z - a).$$

We do the following steps:

1. take a branch cut, define a fundamental solution $Y(z)$.
2. Take a loop so $(\tilde{z} - a) \mapsto (\tilde{z} - a)e^{2\pi i}$, and the branch $Y(z) \mapsto Y(z)M$.
3. $\mathbb{Y}(\tilde{z}) = Y(z)M^k$ (i.e. defined on all branches). Note that M is also the action of an element of the fundamental group on $\mathbb{Y}$ (in deck transformation).

Theorem 4.1. *Given the system* $\frac{dY}{dz} = A(z)Y$ *where* $A(z)$ *is analytic on* $U(a) \setminus \{a\}$, *for all* $\tilde{z} \in \widetilde{U(a) \setminus \{a\}}$, *we have the following representation*

$$\mathbb{Y}(\tilde{z}) = \Phi(z) \cdot (\tilde{z} - a)^L$$

where $L = \frac{1}{2\pi i} \ln M$. $\Phi(z)$ *is single valued on* $U(a) \setminus \{a\}$, $\det \Phi(z) \neq 0$ *for* $z \neq a$.

Proof. Define $\Phi(\tilde{z}) = \mathbb{Y}(\tilde{z})(\tilde{z} - a)^{-L}$. It is single valued as when taken along a path,

$$\Phi(\tilde{z}) \mapsto \mathbb{Y}(\tilde{z})M \cdot \underbrace{e^{-2\pi i L}}_{M^{-1}}(\tilde{z} - a)^{-L} = \mathbb{Y}(\tilde{z})(\tilde{z} - a)^{-L} = \Phi(\tilde{z}).$$

Also $\det \Phi(\tilde{z}) = \det \mathbb{Y}(\tilde{z}) \det(\tilde{z} - a)^{-L} \neq 0$. ☐

Example. The last exercise of the previous section is put in this form.

Now let's do some computation. Write $L = GJG^{-1}$ and so

$$\mathbb{Y}(\tilde{z}) = \Phi(z)(\tilde{z} - a)^L = \underbrace{\Phi(z)G}_{\Psi(z)}(\tilde{z} - a)^J G^{-1}$$

Then $\tilde{\mathbb{Y}}(\tilde{z}) = G\mathbb{Y}(\tilde{z})$ is another fundamental solution with structre

$$\tilde{\mathbb{Y}}(\tilde{z}) = \Psi(z)(\tilde{z} - a)^J.$$

For simplicity suppose $a = 0$, then

$$\tilde{z}^{J_i} = z_i^\lambda \begin{pmatrix} 1 & \ln z & \frac{(\ln z)^2}{2!} & \cdots & \frac{(\ln z)^{m_i - 1}}{(m_i - 1)!} & \\ & 1 & \ln z & & & \\ \vdots & & & & & \\ & & \ddots & & & 1 \end{pmatrix}$$

4.3 Regular singularities

Definition (regular singularity)**.** Given $f : \widetilde{U_r(a) \setminus \{a\}} \to \mathbb{C}$ analytic, $z = a$ is a *regular singularity* if exists $m \in \mathbb{R}_+$ such that

$$\lim_{\substack{z \to a \\ \alpha < \arg(\tilde{z} - a) < \beta}} (\tilde{z} - a)^m f(z) = 0$$

for all $\beta - \alpha < 2\pi$. Otherwise $z = a$ is an *irregular singularity*.

Easy to see that if f, g has regular singularity at $z = a$ then so do $f + g$ and fg.

Example.

1. $f(z) = \frac{\psi(\tilde{z})}{(\tilde{z} - a)^r}$ where $\psi(z)$ is analytic at $z = a$, $r \in \mathbb{R}$ has a regular singularity.

2. $f(z) = (\tilde{z}-a)^{\mu} \ln(\tilde{z}-a)^{\nu}$ where $\mu, \nu \in \mathbb{C}$ has a regular singularity.
3. $f(z) = \exp(\frac{1}{(\tilde{z}-a)^{\mu}})$ where $\mu \in \mathbb{C}$ has an irregular singularity.
4. The solution $\mathbb{Y}(z) = \Phi(z)(\tilde{z}-a)^L$, (suppose $(\tilde{z}-a)^L$ is regular) depends on $\Phi(z)$. For example if $z = a$ a removable singularity or a pole of $\Phi(z)$ then $z = a$ is regular for $\mathbb{Y}(\tilde{z})$.

Definition. $z = a$ is a *regular singularity* for $\frac{dY}{dz} = A(z)Y$ if exists a fundamental solution $\mathbb{Y}(\tilde{z})$ such that $z = a$ is regular for $\mathbb{Y}(\tilde{z})$.

Remark. Every other fundamental solution can be given by $\mathbb{Y}(\tilde{z})C$ for $\det C \neq 0$ so this is well-defined.

Theorem 4.2. *Suppose $z = a$ is regular for $\frac{dY}{dz} = A(z)Y$. Then in $\mathbb{Y}(\tilde{z}) = \Phi(z)(\tilde{z}-a)^L$, $z = a$ is a removable singularity or a pole for $\Phi(z)$, or a pole for $A(z)$.*

Proof. $\Phi(z) = \mathbb{Y}(\tilde{z})(\tilde{z}-a)^{-L}$ so $z = a$ is regular for $\Phi(z)$. Unpack the definition, we can find $m \in \mathbb{N}$ such that

$$\lim_{z \to a} (z-a)^m \Phi(z) = 0$$

for $\alpha < \tilde{z} - a < \beta$. But both $(z-a)^m$ and $\Phi(z)$ are single valued so we can cover the entire disc by sectors and remove the condition $\alpha < \tilde{z} - a < \beta$. Thus $\Phi(z) = \frac{1}{(z-a)^m} \cdot (z-a)^m \Phi(z)$ has at most a pole at $z = a$.

For the other condition,

$$\begin{aligned} A(z) &= \frac{d\mathbb{Y}(\tilde{z})}{dz} \mathbb{Y}(\tilde{z})^{-1} \\ &= \frac{d}{dz}(\Phi(z)(\tilde{z}-a)^L) \cdot (\tilde{z}-a)^{-L} \Phi(z)^{-1} \\ &= \frac{d\Phi(z)}{dz} \Phi^{-1}(z) + \Phi(z) \frac{L}{z-a} \Phi(z)^{-1} \end{aligned}$$

so $z = a$ is a pole. ☐

Example. Consider

$$A(z) = \frac{1}{z^2} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{1}{z} \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}$$

Write the solution as $Y(z) = (y_1, y_2)^T$, then we have

$$y_1' = \frac{1}{z^2} y_2, y_2' = \frac{1}{z} y_2.$$

Solve to get $y_1(z) = C_2 \ln z + C_1, y_2(z) = C_2 z$. We choose $(c_1, c_2) = (1, 0), (0, 1)$ to get the fundamental solution

$$\mathbb{Y}(\tilde{z}) = \begin{pmatrix} 1 & \ln \tilde{z} \\ 0 & z \end{pmatrix}$$

The monodromy associated to $z \mapsto ze^{2\pi i}$ is

$$\mathbb{Y}(\tilde{z}) \mapsto \begin{pmatrix} 1 & \ln \tilde{z} + 2\pi i \\ 0 & z \end{pmatrix} = \mathbb{Y}(\tilde{z}) \cdot e^{\left(\begin{smallmatrix} 0 & 2\pi i \\ 0 & 0 \end{smallmatrix}\right)} = \Phi(z) z^{\left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)}$$

where $\Phi = \left(\begin{smallmatrix} 1 & 0 \\ 0 & z \end{smallmatrix}\right)$ and $\det \Phi(z) \neq 0$ for $z \neq 0$.

4.4 Singularities of the first and second kind

Given the system $\frac{dY}{dz} = A(z)Y$ where $A : U_r(a) \setminus \{a\} \to \mathrm{Mat}(n, \mathbb{C})$ analytic and $z = a$ is an isolated singularity of A that is not an essential singularity. This means we can write $A(z) = \frac{\tilde{A}(z)}{(z-a)^{r+1}}$ where $\tilde{A}$ is analytic on $U_R(a)$ and $\tilde{A}(z) \neq 0$.

Definition (Fuchsian singularity)**.** If $r = 0$ then $z = a$ is called a *singularity of the first kind* (name used by Coddington-Levinson/Balsen-Jurkat-Lutz) or *Fuchsian singularity.* If $r \geq 1$ then $z = a$ is called a *singularity of the second kind*, or occassionally (and confusingly) called a *irregular singularity.* If $r \leq -1$ then it is not a singularity.

Theorem 4.3. *If $z = a$ is a Fuchsian singularity then $z = a$ is a regular singularity.*

Note that the converse is not true, as for example in the example above there is a pole of second order.

We can similarly classify singularities at infinity. Suppose $A(z)$ is analytic for $|z| > R$. Make the change of variable $z = \frac{1}{t}$ so $\frac{d}{dz} = -t^2 \frac{d}{dt}$, we have

$$\frac{dY}{dt} = -\frac{A(1/t)}{t^2} Y.$$

Thus $A_*(t) = A(\frac{1}{t})$ is analytic for $0 < |t| < \frac{1}{R}$ and $t = 0$ is an isolated singularity. Write $-\frac{A_*(t)}{t^2} = \frac{\tilde{A}(t)}{t^{r+1}}$ so

$$A_*(t) = -t^{-r+1} \tilde{A}(t) = -z^{r-1} \tilde{A}'(z)$$

so for $r = 0$ we call it a Fuchsian singularity and similarly for $r \geq 1$ and $r \leq -1$.

4.5 Linear systems with rational coefficients

Now suppose $A(z)$ is defined on $\overline{\mathbb{C}} \setminus \{a_1, \ldots, a_m, \infty\}$ and suppose $a_1, \ldots, a_m, \infty$ are at most poles. The locally around $z = a_j$ we can write $A(z) = A^{(j)}(z) + \mathrm{reg}(z - a_j)$ where the regular part is analytic at $z = a_j$ and $\mathrm{reg}(z - a_j) = O(1)$ as $z \to a_j$. In detail

$$A^{(j)}(z) = \begin{cases} \frac{1}{(z-a_j)^{r_j+1}} (A_0^{(j)} + A_1^{(j)}(z - a_j) + \cdots + A_{r_j}^{(j)}(z - a_j)^{r_j}) & r_j \geq 0 \\ 0 & r_j \leq -1 \end{cases}$$

and for $z = \infty$ write $A(z) = A^{(\infty)}(z) + \mathrm{reg}(\frac{1}{z})$ where $\mathrm{reg}(\frac{1}{z}) = O(\frac{1}{z})$ and

$$A^{(\infty)} = \begin{cases} z^{r_\infty - 1}(A_0^{(\infty)} + \frac{A_1^{(\infty)}}{z} + \cdots + \frac{A_{r_\infty - 1}^{(\infty)}}{z^{r_\infty - 1}}) & r_\infty \geq 1 \\ 0 & r_\infty \leq 0 \end{cases}$$

Lemma 4.4. *$A(z)$ is rational.*

Proof. $A(z) - \sum A^{(j)}(z) - A^{(\infty)}(z)$ is analytic on $\overline{\mathbb{C}}$ and tends to 0 as $z \to \infty$, so by Liouville is constant. □

Definition (Poincaré rank)**.** r_j's and r_∞ are called *Poincaré ranks* of the singularities.

4.6 Holomorphic and meromorphic equivalence

Suppose $G : U(a)\backslash\{a\} \to \mathrm{GL}(n, \mathbb{C})$ analytic with $z = a$ at most a pole. Consider a system $\frac{dY}{dz} = A(z)Y$. Given a *gauge transformation* $Y = G(z)\tilde{Y}$,

$$\frac{d\tilde{Y}}{dz} = \underbrace{(G(z)^{-1}A(z)G(z) - G(z)^{-1}\frac{dG(z)}{dz})}_{\tilde{A}(z)}\tilde{Y}$$

(we assume $G(z)$ is invertible away from $z = a$).

Definition (holomorphic/meromorphic equivalence)**.** We say two systems $\frac{dY}{dz} = A(z)Y, \frac{d\tilde{Y}}{dz} = \tilde{A}(z)Y$, where $A, \tilde{A}$ are analytic on $U(a) \setminus \{a\}$ with at most a pole at $z = a$, are *holomorphic equivalent* (resp. *meromorphic equivalent*) at $z = a$ if exists $G(z)$ invertible in $U(a)$ (resp. $U(a) \setminus \{a\}$) which transforms one system into the other.

One checks that this is an equivalence relation.

Theorem 4.5. *If $z = a$ is a regular singularity of $\frac{dY}{dz} = A(z)Y$ then it is a meromorphic equivalent to a Fuchsian system $\frac{d\tilde{Y}}{dz} = \frac{L}{z-a}\tilde{Y}$.*

Proof. Relabel the solution

$$\mathbb{Y}(\tilde{z}) = \underbrace{\Phi(z)}_{G(z)}\underbrace{(\tilde{z} - a)^L}_{\tilde{\mathbb{Y}}(z)}$$

and $\tilde{\mathbb{Y}}(z)$ is a fundamental solution of the required system. □

We can also consider *formal equivalence.* Consider the formal series

$$A(z) = \frac{1}{(z-a)^{p+1}}\sum_{k=0}^{\infty} A_k(z-a)^k$$

$$\tilde{A}(z) = \frac{1}{(z-a)^{q+1}}\sum_{k=0}^{\infty} \tilde{A}_k(z-a)^k$$

$$G(z) = \frac{1}{(z-a)^m}\sum_{k=0}^{\infty} G_k(z-a)^k$$

where $\det G_0 \neq 0$. If formally A and $\tilde{A}$ are related by the equation for gauge transformation above then the systems are *formally meromorphic equivalent.*

Remark. G has a formal inverse

$$
\begin{aligned}
G(z)^{-1} &= (z-a)^m (G_0 + \sum_{k=1}^{\infty} G_k (z-k)^k)^{-1} \\
&= (z-a)^m (I + \sum_{k=1}^{\infty} G_0^{-1} G_k (z-a)^k)^{-1} G_0^{-1} \\
&= (z-a)^m (I + \sum_{n=1}^{\infty} (-1)^n (\sum_{k=1}^{\infty} G_0^{-1} G_k (z-a)^k)^n) G_0^{-1}
\end{aligned}
$$

5 Structure offundamental matrix at a Fuchsian singularity

In this chapter we study the local structure of fundamental solution by doing example computations. Throughout consider the system $\frac{dY}{dz} = \frac{A(z)}{z}Y$ where A holomorphic at $z = 0$ and $A(0) \neq 0$. Write $A(z) = A_0 + \sum_{i=1}^{\infty} A_i z^i$. Let $J = G_0^{-1} A_0 G_0$ be the Jordan normal form.

The constant gauge transformation $Y(z) = G_0 \hat{Y}(z)$ gives

$$\frac{d\hat{Y}}{dz} = \frac{G_0^{-1} A(z) G_0}{z} \hat{Y}$$

so

$$\hat{A}(z) = J + \sum_{i=1}^{\infty} (G_0^{-1} A_i G_0) z^i.$$

Suppose we have put A_0 into Jordan normal form, i.e. $\frac{dY}{dz} = \frac{A(z)}{z}Y$, $A(z) = J + \sum_{i=1}^{\infty} A_i z^i = \sum_{i=0}^{\infty} A_i z^i$. We seek a gauge transformation $G(z)$ such that $\frac{d\tilde{Y}}{dz} = \frac{R(z)}{z}\tilde{Y}$ where $R(z) = \sum_{i=0}^{\infty} R_i z^i$ is "as simple as possible" so that we can solve for $\tilde{Y}$.

The condition can be expressed as the expression

$$G'(z)\tilde{Y} + \underbrace{G(z)\tilde{Y}'}_{\frac{R(z)}{z}\tilde{Y}} = \frac{A(z)}{z} G(z)\tilde{Y}$$

so we aim to "solve"

$$zG' + GR = A(z)G, \tag{$*$}$$

a system for $(G(z), R(z))$. Formally this means we are looking for series $G(z) = \sum_{i=0}^{\infty} G_i z^i, A(z) = \sum_{i=0}^{\infty} A_i z^i, R(z) = \sum_{i=0}^{\infty} R_i z^i$ that satisfy $(*)$. After some manipulations we get

$$\sum_{\ell=0}^{\infty} (\ell G_\ell + \sum_{j=0}^{\ell} G_j R_{\ell-j}) z^\ell = \sum_{\ell=0}^{\infty} (\sum_{j=0}^{\ell} A_{\ell-j} G_j) z^\ell$$

so for $\ell = 0$ we have $G_0 R_0 = A_0 G_0$., i.e. $G_0 R_0 = J G_0$. We choose $G_0 = I, R_0 = J$. For $\ell \geq 1$ we have

$$G_\ell(\ell \cdot I + J) - J G_\ell = (\sum_{j=1}^{\ell-1} (A_{\ell-j} G_j - G_j R_{\ell-j}) + A_\ell) - R_\ell \tag{$\dagger$}$$

for (G_ℓ, R_ℓ).

5.1 Case of diagonalisable A_0

Theorem 5.1. *If $J = \Lambda$ then the system $\frac{dY}{dz} = \frac{A(z)}{z}Y$ is holomorphically equivalent through $G_0 G(z)$ to*

$$\frac{d\tilde{Y}}{dz} = \frac{1}{z}(\Lambda + R_1 z + \cdots + R_k z^k)\tilde{Y} \tag{$\ddagger$}$$

where $(R_\ell)_{ij} \neq 0$ only if $\lambda_i - \lambda_j = \ell \geq 1$ integer. This is called the normal form *of the system. In particular if no difference $\lambda_i - \lambda_j$ is a non-zero integer then $\frac{d\tilde{Y}}{dz} = \frac{\Lambda}{z}\tilde{Y}$.*

Proof. We show formal equivalence only and defer convergence to Theorem 5.5. Equation (†) reads for $\ell = 1$

$$G_1\Lambda - \Lambda G_1 + G_1 = A_1 - R_1.$$

Taking (i, j) component,

$$(G_1)_{ij}(\lambda_j - \lambda_i + 1) = (A_1)_{ij} - (R_1)_{ij}$$

so if $\lambda_i - \lambda_j \neq 1$ then choose $R_1 = 0$ and $(G_1)_{ij} = \frac{(A_1)_{ij}}{\lambda_j - \lambda_i + 1}$. If $\lambda_i - \lambda_j = 1$ then $(R_1)_{ij} = (A_1)_{ij}$ and $(G_1)_{ij}$ is arbitrary.

Similarly for $\ell \geq 1$ in general

$$G_\ell\Lambda - \Lambda G_\ell + \ell G_\ell = (\cdots) - R_\ell$$

so

$$(G_\ell)_{ij}(\lambda_j - \lambda_i + \ell) = (\cdots)_{ij} - (R_\ell)_{ij}.$$

If $\lambda_i - \lambda_j \neq \ell$ then choose $(R_\ell)_{ij} = 0$ and $(G_\ell)_{ij} = \frac{(\cdots)}{\lambda_j - \lambda_i + \ell}$. If $\lambda_i - \lambda_j = \ell$ then we choose $(R_\ell)_{ij} = (\cdots)_{ij}$ and $(G_\ell)_{ij}$ is arbitrary.

Since there are only finitely many $\lambda_i - \lambda_j$, the system is of the form required. □

Corollary 5.2. *The system, if A_0 is diagonalisable, has a fundamental solution*

$$Y(z) = G_0 G(z) z^\Lambda z^R$$

where $R = R_1 + \cdots + R_k$ (which is nilpotent). In particular $z = 0$ is a Fuchsian singularity so regular (because $\det G(z) \neq 0$*).*

Proof. We know $Y(z) = G_0 G(z)\tilde{Y}(z)$ where $\tilde{Y}(z)$ satisfies (‡). Claim that $\tilde{Y}(z) = z^\Lambda z^R$ is a fundamental solution:

$$\frac{d}{dz}(z^\Lambda z^R) = \frac{\Lambda}{z} z^\Lambda z^R + z^\Lambda \frac{R}{z} z^R = \frac{1}{z}(\Lambda + z^\Lambda R z^{-\Lambda}) z^\Lambda z^R$$

and

$$(z^\Lambda R z^{-\Lambda})_{ij} = z^{\lambda_i - \lambda_j} R_{ij} = \begin{cases} z^\ell (R_\ell)_{ij} & \lambda_i - \lambda_j = \ell \geq 1 \\ 0 & \lambda_i - \lambda_j \neq \ell \geq 1 \end{cases}$$

so indeed it is a fundamental solution. □

Definition (resonance)**.** If $\lambda_i - \lambda_j \in \mathbb{Z} \setminus \{0\}$ we say the system is *resonant* at $z = 0$.

Exercise. Use the same strategy to prove

$$e^{2\pi i\Lambda} R = R e^{2\pi i\Lambda},$$

which implies that

$$e^{2\pi i\Lambda} z^R = z^R e^{2\pi i\Lambda}.$$

The monodromy of the solution is given by

$$Y(z) = G_0 G(z) z^\Lambda z^R \mapsto G_0 G(z) z^\Lambda e^{2\pi i\Lambda} z^R e^{2\pi i R} = \underbrace{G_0 G(z) z^\Lambda z^R}_{Y(z)} \underbrace{e^{2\pi i\Lambda} e^{2\pi i R}}_{M}.$$

We can reduce this further: write $\lambda_j = d_j + \rho_j$ where $d_j \in \mathbb{Z}, 0 \le \operatorname{Re} \rho_j < 1$ so $\Lambda = D + S$ where both D and S and diagonal and D has integer entries. Then $[D, S] = 0$. Also $[R, S] = 0$ since

$$[R, S]_{ij} = (\rho_j - \rho_i) R_{ij}$$

so either $\lambda_i - \lambda_i \in \mathbb{Z}$ so $\rho_j - \rho_i = 0$, or $\lambda_i - \lambda_j \notin \mathbb{Z}$ so $R_{ij} = 0$. Thus

$$Y(z) = G_0 G(z) z^{D+S} z^R = G_0 G(z) z^D z^S z^R = G_0 G(z) z^D z^{R+S} = \underbrace{G_0 G(z) z^D}_{\Phi(z)} z^L$$

where $L = R + S$. But this has exactly the form $Y(z) = \Phi(z) \cdot z^L$ where $\Phi = G_0 G(z) z^D$ is single valued with a pole at $z = 0$. In this representation the monodromy matrix is simply $M = e^{2\pi i L}$.

Let's discuss for a moment the freedom in the solution.

1. In $G_0^{-1} A_0 G_0 = J$, G_0 has freedom $G_0 \mapsto G_0 \Delta_0$ where $\Delta_0^{-1} J \Delta_0 = J$.
2. Fix G_0 and Λ. Recall that G_ℓ contains a finite number of arbitrary parameters.
3. Also R is fixed. Suppose

   $$Y(z) = G_0(I + \sum G_j z^j) z^D z^L.$$

   Suppose we can find
4. Freedom in R. The system may be put in two different normal forms with same Λ. Two such solutions $\tilde{Y}$ and $\tilde{\tilde{Y}}$ are related by a gauge transformation $\tilde{\tilde{Y}} = \Delta(z)\tilde{Y}$.

   $$\Delta(z) = \Delta_0(I + \Delta_1 z + \cdots + \Delta_k z^k)$$

   where $(\Delta_\ell)_{ij} \neq 0$ only if $\lambda_i - \lambda_j \ell \ge 1$ and i

5.2 Case of general A_0

In general, any A_0 can be written in Jordan normal form $J = \operatorname{diag}(J_1, \ldots, J_s)$ where $J_\ell - \lambda_\ell I_\ell + H_\ell$ where H_ℓ has only entries 1 right above the diagonal. We can then partition R_ℓ into blocks and the the problem effectively becomes solving the system of equations

$$XA - BX = C$$

where A is $m \times m$, X is $n \times m$, B is $n \times n$ and C is $m \times m$. Then

1. $XA - BX = 0$ has nontrivial solution if and only if A, B have a common eigenvalue.
2. $XA - BX = C$ with C given has a unique solution X if and only if A, B hasve non common eigenvalues.

3. $XA - BX = C$, with both X, C unknown and A, B with at least one common eigenvalue, always has a solution.

All these can be proven by regarding X and C as vectors and consider the linear system $Dx = c$. Then $\det D \neq 0$ if and only if A, B have no common eigenvalues. See Wasow.

Subsequently we can divide into cases according to if $\lambda_i - \lambda_j = \ell$. Thus we have the following more general theorem

Theorem 5.3. *The system* $\frac{dY}{dz} = \frac{A(z)}{z}Y$ *is holomorphically equivalent through* $G_0G(z)$ *to*

$$\frac{d\tilde{Y}}{dz} = \frac{1}{z}(J + R_1 z + \cdots + R_k z^k)\tilde{Y} \tag{‡}$$

where $(R_\ell)_{ij} \neq 0$ *only if* $\lambda_i - \lambda_j = \ell \geq 1$ *integer.*

Let $\lambda_j = d_j + r_j$ as before and write $S = J - D = \mathrm{diag}(S_1, \ldots, S_s)$ where $S_j = \rho_j I_j + H_j$. Let $L = R + S$ where $R = R_1 + \cdots + R_r$.

Corollary 5.4. *The system* $\frac{dY}{dz} = \frac{A(z)}{z}Y$ *has a fundamental solution*

$$Y(z) = G_0 G(z) z^D z^L$$

where $G(z) = I + \sum_{j=0}^{\infty} G_j z^j$.

5.3 Convergece of formal solution

We now prove covergence. Recall $(*)$

$$zG' + GR(z) = A(z)G,$$

assume $R(z)$ is a polynomial, $A(z)$ analytic at $z = 0$. Construct $y = (G_{11}, \ldots, G_{nn})^T$, $(*)$ becomes

$$zy' = F(z)y$$

where $F(z)$ is analytic in a ball $U_r(0)$ around $z = 0$ so can be locally written as a convergent power series.

Theorem 5.5. *Suppose* $zy' = F(z)y$ *has a formal solution* $y(z) = \sum c_j z^j$. *Then the series converges locally uniformly on* $U_r(0)$ *and thus defines an analytic function in* $U_r(0)$.

Note this is really a property of Fuchsian singularity as if we change LHS to $z^p y'$ where $p > 1$ then the conclusion does not hold.

Proof. Recall from complex analysis that if $f(z)$ is holomorphic on $U_r(0)$ then $f(z) = \sum c_k z^k$ where

$$c_k = \frac{f^{(k)}(0)}{k!} = \frac{1}{2\pi i}\oint_{C_{R_1}} \frac{f(\zeta)}{\zeta^{k+1}} d\zeta$$

so we can estimate

$$|c_k| \leq \frac{1}{2\pi} \oint_{C_{R_1}} \frac{|f(\zeta)|}{|f|^{k+1}} d|\zeta| \leq \frac{M(R_1)}{R_1^k}.$$

Conversely given $\sum c_k z^k$, assuming exists M such that $|c_k| \leq \frac{M}{R_1^k}$, then the series is uniformly convergence on $|z| \leq R_2$ for all $R_2 \leq R_1$, i.e. it defines an analytic function on $U_{R_1}(0)$.

Now given a formal solution $y = \sum c_j z^j$, substitute into the equation to get

$$\sum jC_j z^j = \sum_{ij} F_j C_j z^{i+j}$$

so

$$\sum \ell C_\ell z^\ell = \sum_\ell (\sum_{j=0}^{\ell} F_{\ell-j} C_j) z^\ell$$

so

$$(\ell I - F_0) C_\ell = F_\ell C_0 + F_{\ell-1} C_1 + \cdots + F_1 C_{\ell-1}.$$

For $\ell \geq N$ sufficiently large such that $\frac{|F_0|}{\ell} \leq 1$, then $(\ell I - F_0)$ has an inverse and we can solve for C_ℓ. We can esimate

$$\begin{aligned} |C_\ell| &\leq \frac{1}{\ell} \sum_{k=0}^{\infty} \frac{|F_0|^k}{\ell^k} (|F_\ell||C_0| + \cdots + |F_1||C_{\ell-1|}) \\ &= \frac{1}{\ell} \frac{1}{1 - |f_0|/\ell} (|F_\ell||C_0| + \cdots + |F_1||C_{\ell-1|}) \\ &\leq \frac{C}{\ell} (|F_\ell||C_0| + \cdots + |F_1||C_{\ell-1|}) \end{aligned}$$

We know that F is analytic in $U_r(0)$ so $|F_k| \leq \frac{M}{R_1^k}$. Thus

$$|C_\ell| \leq \frac{MC}{\ell} (\frac{|C_0|}{R_1^\ell} + \cdots + \frac{C_{\ell-1}}{R_1}) \leq MC(\frac{|C_0|}{R_1^\ell} + \cdots + \frac{C_{\ell-1}}{R_1})$$

If $\ell \leq N - 1$ (?) we can always choose $P > 0$ such that $|C_\ell| \leq (\frac{R}{R_1})^\ell$. Prove that for all ℓ, $|C_\ell \leq (P/R_1)^\ell$: assume this is true up to $\ell - 1$ and prove it for ℓ.

$$|\mathbb{C}_\ell| \leq MC(\frac{|C_0|}{R_1^\ell} + \cdots + \frac{C_{\ell-1}}{R_1}) \leq \frac{MC}{R_1^\ell}(P^{\ell-1} + P^{\ell-2} + \cdots + 1) = \frac{MC}{R_1^\ell} \frac{P^\ell - 1}{P - 1}$$

$P^{\ell-1}(P - 1 - MC) \geq 0$ this is possible, provided that P is sufficiently large.

Thus $|C_\ell| \leq \ldots$ so $\sum |C_\ell||z|^\ell < \infty$ provided $|z| \leq \frac{R_1}{P}$, so $y(z)$ is an analytic solution for $|z| \leq \frac{R_1}{P}$. From general theory we can analytically continue $y(z)$ as analytic solution in ball $|z| < r$. □

Thus every time we have a Fuchsian singularity, all formal computation are actual (meaning analytic) computation.

5.4 Completely Fuchsian system

We have studied the local theory of a Fuchsian system. In this section we study the behaviour if the a general system in which all the singularities in the complex plane are Fuchsian. Recall $A(z)$ is rational so necessarily such a system has the form

$$\frac{dY}{dz} = (\frac{A_1}{z - a_1} + \cdots + \frac{A_m}{z - a_m})Y$$

where A_j's are constant matrices. At $z = \infty$,

$$A(z) = \frac{1}{z}\sum_{j=1}^{m} A_j + O(\frac{1}{z^2})$$

which is convergent for $|z| > R$. Substitute $z = \frac{1}{t}$, the system becomes

$$\frac{dY}{dt} = -\frac{1}{t^2}A(\frac{1}{t})Y = \frac{1}{t}(\underbrace{-\sum A_j}_{A_\infty} + O(t))Y.$$

For each singularity a_j, we know from local theory that there exists a neighbourhood $U(a_j)$ on which

$$Y_j(z) = G_j(z)(z - a_j)^{D_j}(z - a_j)^{L_j}.$$

Suppose A_j has eigenvalues $\lambda_1^{(j)}, \cdots \lambda_m^{(j)}$ and $\lambda_k^{(j)} = d_k^{(j)} + \rho_k^{(j)}$. Similar for $z = \infty$: in a neighbourhood of $t = 0$,

$$Y_\infty(z) = Y(t) = \tilde{G}_\infty(t)t^{D_\infty}t^{L_\infty} = G_\infty(z)z^{-D_\infty}z^{-L_\infty}$$

where $G_\infty(z)$ is analytic at $z = \infty$.

Remark. To talk about the local solutions we must select a sheet of the universal cover. It is given as follow. Choose parallel branch cuts L_j which are rays from a_j's in direction η, such that no a_k is contained L_j for $j \neq k$. Then we stipulate that either $\eta - 2\pi < \arg(z - a_j) < \eta$ for $|z - a_j|$ small, or $\eta - 2\pi < \arg z < \eta$ for $|z|$ large.

Remark. Note that from definition $A_\infty + \sum_{j=1}^{m} A_j = 0$. Taking the trace, we get

$$\sum_{k=1}^{n}(\lambda_k^{(1)} + \lambda_k^{(2)} + \cdots + \lambda_k^{(m)} + \lambda_k^{(\infty)}) = 0.$$

This interdependence of eigenvalues is called *Fuchs identity.*

Since each one of $Y_j(z)$ and $Y_\infty(z)$ is a fundamental solution, we can relate them by $Y_\infty(z) = Y_j(z)C_j$ where C_j nonsingular is called the *connection matrix.* For computation, $G_j(z), G_\infty(z), D_j, L_j, D_\infty, L_\infty$ are algebraic functions of $A_1, \ldots, A_m$ so we can compute them. However C_j's are *transcendental functions* of A_i's.

Note $C_j = Y_j(z_0)^{-1}Y_\infty(z_0)$ for $z_0 \notin \{a_1, \ldots, a_m, \infty\}$ and we know each $Y_j(z)$ only locally so the study of the global Fuchsian system is reduced to asking how to compute $Y_j(z_0), Y_\infty(z_0)$ when $z_0 \notin U(a_1)$ or $z_0 \notin U(\infty)$. This is called the *connection problem.*

In several cases ($n = 2$) this can be solved (using linear special function theory such as Bessel, Airy, hypergeometric functions). The general strategy is as follow: for the (a, b) entry we write

$$(Y_j(z))_{ab} = \int_\gamma \phi_{ab}(z, s) ds$$

called *integral representation.* Then

$$(Y_j(z_0))_{ab} = \int_\gamma \phi_{ab}(z_0, s) ds$$

which is computable in terms of classical special functions.

Monodromy of global Fuchsian system We know the monodromy of the local solutions: for $|z - a_j|$ small, if we take the loop γ_j that goes around a_j counterclockwise once we have

$$Y_j(z) \mapsto Y_j(z) e^{2\pi i L_j}.$$

For Y_∞, we take big $|z|$ and loop around ∞ *clockwise* to get

$$Y_\infty(z) \mapsto Y_\infty(z) e^{2\pi i L_\infty}.$$

Combining the local monodromy with connection matrices,

$$Y_\infty(z) = Y_j(z) C_j \mapsto_{\gamma_j} Y_j(z) e^{2\pi i L_j} C_j = Y_\infty(z) \underbrace{C_j^{-1} e^{2\pi i L_j} C_j}_{M_j}$$

so the monodromy (anti)representation is given by

$$\gamma_j \mapsto M_j = C_j^{-1} e^{2\pi i L_j} C_j$$

Note that $\gamma_1 \cdots \gamma_2 \cdots \gamma_m = \gamma_\infty^{-1}$ so $M_\infty = (M_m \cdots M_1)^{-1}$. Again we stress that the difficulty lies in finding C_j.

6 Linear equations of order n

Consider the scalar ODE of order n

$$u^{(n)} + a_1(z)u^{(n-1)} + \cdots + a_n(z)u = 0$$

where $z = 0$ is an isolated singularity of $a_j(z)$. $a_j(z)$ are analytic the we can express the solutions in Taylor series. For the simplest case $a_j \in \mathbb{C}$, we can solve the indicial equation

$$f_0(\lambda) = \lambda^n + a_1\lambda^{n-1} + \cdots + a_n = \prod_{j=1}^{s}(\lambda - \lambda_j)^{m_j}$$

and $e^{\lambda_j z}, ze^{\lambda_j z}, \ldots, z^{m_j-1}e^{\lambda_j z}$ for $j = 1, \ldots, s$ is a fundamental system.

As another example

$$u^{(n)} + \frac{a_1}{z}u^{(n-1)} + \cdots + \frac{a_n}{z^n}u = 0$$

where $a_j \in \mathbb{C}$. This is called the *Euler equation*. For $z \neq 0$ multiply by z^n to get

$$E(u) = 0$$

where $E = z^n \frac{d^n}{dz^n} + \cdots + a_n$. Define the *Euler operator* $\delta = z\frac{d}{dz}$ and define

$$[\delta]_k = \delta(\delta - 1)\cdots(\delta - k + 1)$$

and a quick calculation shows $[\delta]_k = z^k \frac{d^k}{dz^k}$ so

$$E = [\delta]_n + a_1[\delta]_{n-1} + \cdots + a_2\delta + a_1$$

and we have another indicial equation

$$f_0(\lambda) = [\lambda]_n + a_1[\lambda]_{n-1} + \cdots + a_n = \prod_{j=1}^{s}(\lambda - \lambda_j)^{m_j}$$

where λ_i's are distinct. They are called *indices* or *characteristic exponents*.

Frobenius method A computation shows

$$[\delta]_k z^\lambda = [\lambda]_k z^\lambda$$

so z^λ is an eigenfunction so

$$E(z^\lambda) = f_0(\lambda)z^\lambda.$$

Thus if λ_i's are the solutions of the indicial equation then z^{λ_i}'s are solutions to the Euler equation.

Since

$$\frac{\partial^\ell}{\partial\lambda^\ell}z^\lambda = (\log z)^\ell z^\lambda,$$

it follows that

$$E(\frac{\partial^\ell z^\lambda}{\partial\lambda^\ell}) = \frac{\partial^\ell}{\partial\lambda^\ell}E(z^\ell) = \frac{\partial^\ell}{\partial\lambda^\ell}(f_0(\lambda)z^\lambda) = \sum_{k=0}^{\ell}\binom{\ell}{k}\frac{d^k f_0(\lambda)}{d\lambda^k}(\log z)^{\ell-k}z^\lambda.$$

If λ_j is a root of f_0 with multiplicity m_j then $E(\frac{\partial^\ell z^\ell}{\partial \lambda^\ell}) = 0$ for $\ell < m_j$ so

$$z^{\lambda_j}, \ln z \cdot z^{\lambda_1}, \dots, z^{\lambda_j} (\ln z)^{m_j - 1}$$

form a fundamental system.

In this example the companion system is

$$\frac{dY}{dz} = \begin{pmatrix} 0 & 1 & \\ & 0 & 1 \\ \vdots & & \\ -\frac{a_m}{z^m} & & -\frac{a_1}{z} \end{pmatrix} y$$

for which $z = 0$ is a singularity of second kind if $m \geq 2$. In addition we know this is a regular singularity from the fundamental system above. It is not clear, however, that this system is a priori regular.

We can make this more lucid by making the substituion

$$y_1 = u, y_2 = zu', \dots, y_n = z^{n-1} u^{(n-1)}$$

so

$$y_j' = \begin{cases} \frac{1}{z}((j-1)y_j + y_{j+1}) & j \leq n-1 \\ \frac{1}{z}((n-1)y_n - (a_1 y_n + \cdots + a_n y_1)) & j = n \end{cases}$$

so the system is

$$\frac{dY}{dz} = \frac{1}{z} \begin{pmatrix} 0 & 1 & & & \\ & 1 & 1 & & \\ & & 2 & 1 & \\ \vdots & & & & \\ -a_n & -a_{n-1} & \cdots & -a_2 & n-1-a \end{pmatrix} y$$

and it is obvious that $z = 0$ is a Fuchsian singularity.

Call the constant matrix A_0 so from general theory of Fuchsian singularity

$$Y(z) = z^{A_0} = G_0 z^J G_0^{-1}.$$

One can also check $\det(\lambda - A_0) = f_0(\lambda)$. Note that the original system has Poincaré rank $n-1$, while after sustituion it has Poincaré rank 0. The solutions are related by a meromorphic tranformation

$$\begin{pmatrix} 1 & & & & \\ & \frac{1}{z} & & & \\ & & \frac{1}{z^2} & & \\ & & & \ddots & \\ & & & & \frac{1}{z^{n-1}} \end{pmatrix}$$

This is called a *shearing transformation.*

6.1 Series solution

Consider

$$u^{(n)} + \frac{\hat{a}_1(z)}{z}u^{(n-1)} + \cdots + \frac{\hat{a}_n(z)}{z^n}(u) = 0$$

where $\hat{a}_j(z)$ analytic at $z = 0$. The only difference between this and the Euler equation is that the coefficients are not constant. Write $E(u) = 0$. Use the Frobenius method with a pertubation by substituting

$$u(z) = z^\lambda \sum_{\nu=0}^{\infty} c_\nu z^\nu$$

$$E(\sum_{\nu=0}^{\infty} c_\nu z^{\nu+\lambda}) = \sum_{\nu=0}^{\infty} c_\nu E(z^{\nu+\lambda}) = \sum_{\nu=0}^{\infty} c_\nu \underbrace{([\lambda+\nu]_n + \hat{a}_1(z)[\nu+\lambda]_{n-1} + \cdots + \hat{a}_n(z))}_{\bullet} z^\nu$$

Substitute $\hat{a}_j(z) = \sum_{\ell=0}^{\infty} a_\ell^{(j)} z^\ell$, we can write ($\bullet$) as

$$\sum_{\ell=0}^{\infty} f_\ell(\lambda+\nu) z^\ell$$

for some f_ℓ. Thus

$$E(\sum_{\nu} c_\nu z^{\nu+\lambda}) = \sum_{\nu,\ell} c_\nu f_\ell(\lambda+\nu) z^{\nu+\ell} = \sum_{m=0} (\sum_{\nu=0}^{m} c_\nu f_{m-\nu}(\lambda+\nu)) z^m \quad m = \nu+\ell$$

so for $E = 0$ we reuire all coefficients of z^m to vanish, so we get recurrence relations

$$\begin{aligned} c_0 f_0(\lambda) &= 0 \\ c_1 f_0(\lambda+1) + c_0 f_1(\lambda) &= 0 \\ c_m f_0(\lambda+m) + \cdots + c_0 f_m(\lambda) &= c \end{aligned}$$

Write $f_0(\lambda) = \prod(\lambda - \lambda_j)^{m_j}$. If $\lambda = \lambda_j$ then c_0 is arbitrary. If in addition $\lambda_j + m$ is not a root then we can determine all c_m in terms of c_0 and λ_j.

$$u(z) = z^{\lambda_j}(c_0 + \sum_{\nu=1}^{\infty} c_\nu(c_0, \lambda_j) z^\nu).$$

In summary, if $f_0(\lambda)$ has n distinct roots $\lambda_1, \ldots, \lambda_n$ not differing by integers then we have a fundamental system

$$u(z)_j = z^{\lambda_j}(c_0^{(j)} + \sum_{\nu=1}^{\infty} c_\nu^{(j)} z^\nu).$$

If some $m_j \geq 2$, i.e. multiple roots, or if $\lambda_j + m$ is also a root, we can implement Frobenius method to get $z^{\lambda_j}(\ln z)^k$. For reference see Ince.

A few more words on where $\ln z$ comes from

Similar as before we can reduce the ODE to a linear system by subsituting $y_1 = u, y_2 = zu', \dots, y_n = z^{n-1}u^{(n-1)}$, and consider the system

$$\frac{dy}{dz} = \frac{1}{z}\begin{pmatrix} 0 & 1 & & & & \\ & 1 & 1 & & & \\ & & 2 & 1 & & \\ & & & \ddots & & \\ & & & & n-2 & 1 \\ -hata_n(z) & -\hat{a}_{n-1}(z) & \cdots & -\hat{a}_1(z)+n-1 & & \end{pmatrix} y$$

and it is obvious that $z = 0$ is a Fuchsian singularity, ergo the name. Since $\hat{a}(z)$ is analytic at $z = 0$, write $\hat{a}(z) = \sum_{\ell=0}^{\infty} a_\ell^{(j)} z^\ell$. Then we find

$$A_0 = A(0) = \begin{pmatrix} 0 & 1 & & & & \\ & 1 & 1 & & & \\ & & 2 & 1 & & \\ & & & \ddots & & \\ & & & & n-2 & 1 \\ -a_0^{(n)} & -a_0^{(n-1)} & \cdots & -a_0^{(1)}+n-1 & & \end{pmatrix}$$

and we can check $\det(\lambda - A_0) = f_0(\lambda)$.

We know that exists fundamental solution $Y(z) = G(z)z^D z^{S+R}$ where G is holomorphically invertible at $z = 0$, $D + S = J$, the Jordan normal form of A_0, and R nilpotent. Suppose the first row of $Y(z)$ is $(u_1(z), \dots, u_n(z))$, which are solutions to the ODE. Then u_j containes terms $z^{\lambda_k}(\ln z)^p$ where $1 \le k < s$ and $p \le m_k - 1$.

Remark. The solutions obtained by Frobenius method converge because the companion system is Fuchsian. Thus the formal solutions are actual solutions.

A converse to proposition to Theorem 4.3:

> **Theorem 6.1.** *For an ODE, $z = 0$ is Fuchsian if and only if it is a regular singularity.*

Proof. Only need to prove only if. Induction on the order of the ODE n. For $n = 1$, suppose $z = 0$ is a regular singularity of $u' + a_1(z)u = 0$ and $u(z)$ is a solution. Then it has monodromy

$$u(ze^{2\pi i}) = \mu u(z) = e^{2\pi i\lambda}u(z)$$

where $\mu \in \mathbb{C} \setminus \{0\}$ and λ is defined up to integers. Then $u(z)z^{-\lambda}$ is single valued so $z = 0$ is regular, so $z = 0$ is at most a pole. Choose λ so that $f(z) = u(z)z^{-\lambda}$ is analytic at $z = 0$ and $f(0) \neq 0$. Then

$$a_1(z) = -\frac{u'}{u} = -\frac{\lambda}{z} + \frac{f'(z)}{f(z)}$$

which has a pole of order 1.

We do the induction step for $n = 2$, consider $u'' + a_1(z)u' + a_2(z)u = 0$ where $z = 0$ is regular. Exists $u_1(z), u_2(z)$ linearly independent such that (u_1, u_2) has monodromy matrix M. By a linear transformation wlog M is upper trianglular so $u_1(ze^{2\pi i}) = \mu_1 u_1(z)$ so as before $u_1(z) = f(z)z^\lambda$ where $f(z)$ analytic and $f(0) \neq 0$. We seek another solution $u(z) = u_1(z)v(z)$. Substitute the ansatz into the equation to get the condition

$$v'' + (\frac{2u_1(z)'}{u_1(z)} + a_1(z))v' = 0,$$

a first order ODE in v'. Since $z = 0$ is regular, $v'(z)$ has a regular singularity in $z = 0$ so the first step applies to v' and conclude that

$$\frac{2u_1'(z)}{u_1(z)} + a_1(z) = \frac{2\lambda_1}{z} + \frac{2f'(z)}{f(z)} + a_1(z)$$

has at most a simple pole at $z = 0$. It remains to compute $a_2(z)$:

$$a_2(z) = -(\frac{u_1''}{u_1} + a_1\frac{u_1'}{u_1}) = -(\frac{\lambda(\lambda-1)}{z^2} + \frac{\lambda}{z}(\frac{2f'}{f} + a_1(z)) + \frac{f''}{f} + a_1(z)\frac{f'}{f})$$

so $a_z(z)$ has at most a pole of order 2 at $z = 0$. □

We can also define Fuchsian singularity at ∞ by settting $t = \frac{1}{z}$ and consider the behaviour at $t = 0$.

Proposition 6.2. *$z = \infty$ is Fuchsian if and only if $a_j(z) = \frac{\tilde{a}_j}{z^j}$ where $\tilde{a}_j(z) = \tilde{a}_0^{(j)} + \sum_{\ell=1}^\infty \frac{\tilde{a}_\ell^{(j)}}{z^\ell}$ is analytic at $z = \infty$.*

Proof. $u^{(n)} + a_1(z)u^{(n-1)} + \cdots + a_n(z)u = 0$. Put $a_j(z) = \frac{\tilde{a}_j(z)}{z^j}$ (we do not know the analytic property of $\tilde{a}_j$). Multiply by z^n and substitute $z = \frac{1}{t}$, note

$$\delta = z\frac{z}{dz} = -t\frac{d}{dt} = -\delta_t$$

so $[\delta]_k = (-1)^k(\delta_t)_k$ where

$$[\delta_t]_k = \delta_t(\delta_t + 1)\cdots(\delta_t + n - 1).$$

Then the system is

$$[\delta_t]_n - \tilde{a}_1(\frac{1}{t})[\delta_t]_{n-1} + \cdots + (-1)^n\tilde{a}_n(\frac{1}{t})$$

so $t = 0$ is Fuchsian if and only if $\tilde{a}_j(\frac{1}{t})$ is analytic at $t = 0$. □

6.2 Completely Fuchsian ODE of order n

Suppose $z = \alpha_1, \ldots, \alpha_m, \infty$ are poles of the $a_j(z)$'s, which have the correct order.

Proposition 6.3. *The system is completely Fuchsian if and only if*

$$a_j(z) = \frac{p_j(z)}{(z-\alpha_1)^j \cdots (z-\alpha_m)^j}$$

where P_j is a polynomial of degree $\leq j \cdot m - j$.

Proof. At $z = \alpha_k$,

$$a_j(z) \sim \frac{\text{reg}(z-\alpha_k)}{(z-\alpha_k)^j}$$

and at $z = \infty$,

$$z^j a_j(z) \sim z^{j+\deg P_j - j\cdot m}.$$

□

By partial fractions we can find the explicit forms of all completely Fuchsian system. As an example calculation, take $n = 2$ and consider the equation

$$u'' + a_1 u' + a_2 u = 0$$

Then

$$a_1(z) = \frac{A_1}{z-\alpha_1} + \cdots + \frac{A_m}{z-\alpha_m}$$
$$a_2(z) = \sum_{j=1}^{m} \left(\frac{B_j}{(z-\alpha_j)^2} + \frac{C_j}{z-\alpha_j}\right)$$

where $A_j, B_j, C_J \in \mathbb{C}$ and we can show $\sum_{j=1}^{m} C_j = 0$.

- For $m = 1$ there is only one equation $u'' = 0$.
- For $m = 2$,
$$u'' + \frac{A_1}{z-\alpha_1}u' + \frac{B_1}{(z-\alpha_1)^2}u = 0,$$
the Euler equation.
- For $m = 3$,
$$u''+\left(\frac{A_1}{z-\alpha_1}+\frac{A_3}{z-\alpha_3}\right)u'+\left(\frac{B_1}{(z-\alpha_1)^2}+\frac{B_2}{(z-\alpha_2)^2}+C_1\left(\frac{1}{z-\alpha_1}+\frac{1}{z-\alpha_2}\right)\right)u = 0,$$
the *Riemann hypergeometric equation.* As an exercise, show $f_0(\lambda) = \lambda(\lambda - 1) + a_0^{(1)}\lambda + a_0^{(2)}$.

7 Hypergeometric equation

7.1 Riemann hypergeometric equation

A Riemann hypergeometric equation

$$u''+(\frac{A_1}{z-\alpha_1}+\frac{A_3}{z-\alpha_3})u'+(\frac{B_1}{(z-\alpha_1)^2}+\frac{B_2}{(z-\alpha_2)^2}+C_1(\frac{1}{z-\alpha_1}+\frac{1}{z-\alpha_2}))u=0,$$

has indicial equations are

$$\begin{aligned} z=\alpha_1 &: \lambda^2+(A_1-1)\lambda+B_1=0 \\ z=\alpha_2 &: \lambda^2+(A_2-1)\lambda+B_2=0 \\ z=\infty &: \lambda^2+(1-A_1-A_2)\lambda+B_1+B_2+(\alpha_1-\alpha_2)C_1=c \end{aligned}$$

A_1, A_2, B_2, B_2, C_1 determine the exponents $\lambda_\pm^j$. Conversely,

$$\begin{aligned} A_j &= 1-\lambda_+^{(j)}-\lambda_-^{(j)} \\ B_j &= \lambda_+^{(j)}\lambda_-^{(j)} \\ C_j &= \frac{\lambda_+^{(\infty)}\lambda_-^{(\infty)}-(B_1+B_2)}{\alpha_1-\alpha_2} \end{aligned}$$

Theorem 7.1. *The Riemannian equation is completely determined by the seven parameters α_1, α_2 and the exponents (note that the exponents sum up to 1), which is typically presented as*

$$\begin{pmatrix} \alpha_1 & \alpha_2 & \infty \\ \lambda_+^{(1)} & \lambda_+^{(2)} & \lambda_+^{(\infty)} \\ \lambda_-^{(1)} & \lambda_-^{(2)} & \lambda_-^{(\infty)} \end{pmatrix}$$

and is called a Riemann scheme*. The collection of all solutions to the Riemann equations is called the* Riemann symbol*, denoted by*

$$P\begin{pmatrix} \alpha_1 & \alpha_2 & \infty & \\ \lambda_+^{(1)} & \lambda_+^{(2)} & \lambda_+^{(\infty)} & ;z \\ \lambda_-^{(1)} & \lambda_-^{(2)} & \lambda_-^{(\infty)} & \end{pmatrix}$$

7.1.1 Möbius transformation

Recall that a Möbius transformation is an automorphism of $\overline{\mathbb{C}}$ of the form

$$z \mapsto \frac{az+b}{cz+d}$$

where $a, b, c, d \in \mathbb{C}, ad-bc \neq 0$. This is an action by $\mathrm{GL}(n, \mathbb{C})$ with kernel the scalar matrices so is an action by $\mathrm{PSL}(2, \mathbb{C})$.

Theorem 7.2. *Characteristic exponents are invariant under Möbius tran-*

formation, i.e.

$$P\begin{pmatrix} \alpha_1 & \alpha_2 & \infty & \\ \lambda_+^{(1)} & \lambda_+^{(2)} & \lambda_+^{(\infty)} & ;z \\ \lambda_-^{(1)} & \lambda_-^{(2)} & \lambda_-^{(\infty)} & \end{pmatrix} = P\begin{pmatrix} h(\alpha_1) & h(\alpha_2) & h(\infty) & \\ \lambda_+^{(1)} & \lambda_+^{(2)} & \lambda_+^{(\infty)} & ;t=h(z) \\ \lambda_-^{(1)} & \lambda_-^{(2)} & \lambda_-^{(\infty)} & \end{pmatrix}.$$

Proof. For simplicity we prove the result for $t = h(z) = \frac{z-\alpha_1}{\alpha_2-\alpha_1}$ so $z = \alpha_1, \alpha_2, \infty$ corresponds to $t = 0, 1, \infty$.

$$\frac{d}{dz} = \frac{1}{\alpha_2 - \alpha_1}\frac{d}{dt}$$

so

$$\frac{d^2u}{dt^2} + (\frac{A_1}{t} + \frac{A_2}{t-1})\frac{du}{dt} + (\frac{B_1}{t^2} + \frac{B_2}{(t-1)^2} + (\alpha_2 - \alpha_1)C_1(\frac{1}{t} - \frac{1}{t-1}))u = 0.$$

The indicial equations at $t = 0, 1$ are the same so $\lambda_\pm^{(1)}, \lambda_\pm^{(2)}$ are the same. For $t = \infty$, note that

$$(0-1)(\alpha_2 - \alpha_1) = \alpha_1 - \alpha_2$$

so again it is the same. □

Corollary 7.3. *We can always reduce a Riemann hypergeometric equation to the form*

$$P\begin{pmatrix} 0 & 1 & \infty & \\ \lambda_+^{(0)} & \lambda_+^{(1)} & \lambda_+^{(\infty)} & ;z \\ \lambda_-^{(0)} & \lambda_-^{(1)} & \lambda_-^{(\infty)} & \end{pmatrix}.$$

7.1.2 Gauge transformation

Consder the gauge transformation $u(z) = z^p(1-z)^q v(z), p, q \in \mathbb{C}$ that changes the exponent:

$$u(z) = z^{\lambda_\pm^{(0)}}\operatorname{reg}(z) \mapsto v(z) = z^{\lambda_\pm^{(0)}-p}\operatorname{reg}(z)$$
$$u(z) = (1-z)^{\lambda_\pm^{(1)}}\operatorname{reg}(z-1) \mapsto v(z) = (1-z)^{\lambda_\pm^{(1)}-q}\operatorname{reg}(z-1)$$
$$u(z) = z^{-\lambda_\pm^{(\infty)}}\operatorname{reg}(\frac{1}{z}) \mapsto v(z) = z^{-\lambda_\pm^{(\infty)}-p-q}\operatorname{reg}(\frac{1}{z})$$

so

$$P\begin{pmatrix} 0 & 1 & \infty & \\ \lambda_+^{(0)} & \lambda_+^{(1)} & \lambda_+^{(\infty)} & ;z \\ \lambda_-^{(0)} & \lambda_-^{(1)} & \lambda_-^{(\infty)} & \end{pmatrix} = z^p(1-z)^q P\begin{pmatrix} 0 & 1 & \infty & \\ \lambda_+^{(0)}-p & \lambda_+^{(1)}-q & \lambda_+^{(\infty)}+p+q & ;z \\ \lambda_-^{(0)}-p & \lambda_-^{(1)}-q & \lambda_-^{(\infty)}+p+q & \end{pmatrix}.$$

Thus we can always reduce to the form

$$P\begin{pmatrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & ;z \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{pmatrix}$$

whose correspondig equation is

$$u'' + (\frac{\gamma}{z} + \frac{\alpha+\beta-\beta+1}{z-1})u' + \alpha\beta(\frac{1}{z-1} - \frac{1}{z})u = 0$$

i.e.

$$z(1-z)u'' + (\gamma - (\alpha+\beta+1)z)u' - \alpha\beta u = 0.$$

This is called the *Gauss hypergeometric equation.*

7.2 Gauss hypergeometric equation

7.2.1 Local representation of solutions

We look for a series solution first at $z = 0$. $\lambda_+^{(0)} = 0$ so we expect a Taylor series $u(z) = \sum_{n=0}^{\infty} c_n z^n$. Substitute to get the recurrence relation

$$c_{n+1} = \frac{(m+\alpha)(m+\beta)}{(n+1)(n+\gamma)} c_n$$

where we require $\gamma \neq 0, -1, -2, \ldots$ so that Frobenius method works. Then

$$c_n = \frac{(\alpha)_n (\beta)_n}{n! (\gamma)_n} c_0$$

where the subscript n is the *Pochhammer symbol* and is defined as

$$(\alpha)_0 = 1, (\alpha)_n = \alpha(\alpha+1)\cdots(\alpha+n-1).$$

Note that $(\alpha)_n = 0$ for $\alpha = 0, -1, \ldots, -n+1$. Set $c_0 = 1$, we get

$$u_1^{(0)}(z) = {}_2F_1(\alpha, \beta, \gamma; z) = \sum_{n=0}^{\infty} \frac{(\alpha)_n (\beta)_n}{(\gamma)_n n!} z^n,$$

convergent for $|z| < 1$, called the *hypergeometric series.*

To get the second solution: use gauge $u = z^{1-\gamma} v$ so

$$P\left(\begin{smallmatrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & ;z \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{smallmatrix} \right) = z^{1-\gamma} P\left(\begin{smallmatrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha-\gamma+1 & ;z \\ \gamma-1 & \gamma-\alpha-\beta & \beta-\gamma+1 & \end{smallmatrix} \right)$$

so the other solution is

$$u_2^{(0)}(z) = z^{1-\gamma} \, {}_2F_1(\alpha-\gamma+1, \beta-\gamma+1, 2-\gamma; z)$$

where $\gamma \neq 2, 3, \ldots$.

To get the solution at 1, apply the Möbius transformation $t = 1 - z$ so

$$P\left(\begin{smallmatrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & ;z \\ 1-\gamma & \gamma-\alpha-\beta & \beta & \end{smallmatrix} \right) = P\left(\begin{smallmatrix} 0 & 1 & \infty & \\ 0 & 0 & \alpha & ;1-z \\ \gamma-\alpha-\beta & 1-\gamma & \beta & \end{smallmatrix} \right)$$

so by a change of variable $\gamma - \alpha - \beta = 1 - \gamma_1, \alpha_1 = \alpha, \beta_1 = \beta$, from the solutions at $z = 0$ we get

$$u_1^{(1)}(z) = F(\alpha, \beta, \alpha+\beta-\gamma+1; 1-z) \quad \gamma-\alpha-\beta \neq 1, 2, \ldots, |1-z| < 1$$
$$u_1^{(2)}(z) = (1-z)^{\gamma-\alpha-\beta};_2 F_1(\gamma-\beta, \gamma-\alpha, 1+\gamma-\alpha-\beta; 1-z) \quad a\gamma-\alpha-\beta \neq -1, -2, \ldots$$

To get the solution at ∞ we use $z = \frac{1}{t}$ and after another computation

$$u_1^{(\infty)}(z) = z^{-\alpha} \, {}_2F_1(\alpha, 1+\alpha-\gamma, 1+\alpha-\beta; \frac{1}{z}) \quad \alpha-\beta \neq -1, -2, \ldots, |z| > 1$$
$$u_1^{(\infty)}(z) = z^{-\beta} \, {}_2F_1(\beta, 1+\beta-\gamma, 1-\alpha+\beta; \frac{1}{z}) \quad \alpha-\beta \neq 1, 2, \ldots$$

7.2.2 Integral representation

Recall the gamma function is defined by the Euler integral

$$\Gamma(p) = \int_0^\infty t^{p-1} e^{-t} dt$$

for $\operatorname{Re} p > 0$. Also Beta function

$$B(p,q) = \int_0^1 t^{p-1}(1-t)^{q-1} dt$$

for $\operatorname{Re} p > 0, \operatorname{Re} q > 0, \arg t = 0, \arg(1-t) = 0$ which satisfies

$$B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}.$$

Proposition 7.4. *Suppose* $\operatorname{Re}\gamma > \operatorname{Re}\beta > 0$, $|z| < 1$, *we have the following integral representation of hypergeometric function:*

$${}_2F_1(\alpha,\beta,\gamma;z) = \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1}(1-zt)^{-\alpha} dt$$

where $\arg t = 0, \arg(1-t) = 0, |\arg(1-zt)| < \frac{\pi}{2}$ *for* $0 < t < 1$. *This is the* Euler representation *of hypergeometric functions.*

Proof.

$$\begin{aligned}
{}_2F_1(\alpha,\beta,\gamma;z) &= \sum_{n=0}^\infty \frac{(\beta)_n}{(\gamma)_n} \frac{(\alpha)_n z^n}{n!} \\
&= \sum_n \underbrace{\frac{\Gamma(\gamma)\Gamma(\beta+n)}{\Gamma(\beta)\Gamma(\gamma+n)}}_{\frac{\Gamma(\gamma)}{\Gamma(\beta)}\frac{B(n+\beta,\gamma-\beta)}{\Gamma(\gamma-\beta)}} \frac{(\alpha)_n z^n}{n!} \\
&= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \sum_n \int_0^1 t^{\beta+m-1}(1-t)^{\gamma-\beta-1} \binom{-\alpha}{n} (-z)^n dt \\
&= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)} \int_0^1 t^{\beta-1}(1-t)^{\gamma-\beta-1} \underbrace{\sum_n \binom{-\alpha}{n} (-tz)^n}_{=(1-tz)^{-\alpha}} dt \quad \text{as } |z| < 1
\end{aligned}$$

where the expansion for $(1-tz)^{-\alpha}$ is valid for $0 < zt < 1$ and z real. ... □

In the Euler representation of ${}_1F_2$, the integrand has a branch point at $z = \frac{1}{t}$ for $0 \le t \le 1$. In the z-plane, as t varies, the locus of $\frac{1}{t}$ is the real axis greater than 1. Thus we can analytically continue ${}_2F_1$ to $\mathbb{C} \setminus [1,\infty)$.

7.2.3 Solution to connection problem

Suppose $\mathrm{Re}(\gamma-\alpha-\beta)>0, \mathrm{Re}\,\gamma>\mathrm{Re}\,\beta>0$. Then

$$\begin{aligned}
{}_2F_1(\alpha,\beta,\gamma;1) &= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\int_0^1 t^{\beta-1}(1-t)^{\gamma-\alpha-\beta-1}dt \\
&= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}B(\beta,\gamma-\alpha-\beta) \\
&= \frac{\Gamma(\gamma)}{\Gamma(\beta)\Gamma(\gamma-\beta)}\frac{\Gamma(\beta)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\beta+\gamma-\alpha-\beta)} \\
&= \frac{\Gamma(\gamma)\Gamma(\gamma-\alpha-\beta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}
\end{aligned}$$

This is called the *Gauss-Kummer formula*. This formula holds by analytic continuation with $\gamma\neq 0,-1,-2,\dots$ and $\gamma-\alpha-\beta\neq 0,-1,\dots$.

Recall that we have the fundamental solution to the Gauss hypergeometric equation. We write them in row matrices $[u_1^{(0)},u_2^{(0)}]=Y^{(0)}$ etc. The main problem is to compute the connection matrices C_{01} and $C_{0\infty}$ where $Y^{(0)}=Y^{(1)}C_{01}, Y^{(0)}=Y^{(\infty)}C_{0\infty}$. We choose branch cuts so $|\arg z|<\pi, |\arg(1-z)|<\pi$. As an example computation, suppose $C_{01}=\left(\begin{smallmatrix}a & c\\ b & d\end{smallmatrix}\right)$. Then

$$u_1^{(0)}(z)=au_1^{(1)}(z)+bu_2^{(1)}(z).$$

Substitute $z=1$ into the equation and we get

$${}_2F_1(\alpha,\beta,\gamma;1)=a\cdot\underbrace{{}_2F_1(\alpha,\beta,\alpha+\beta-\gamma+1;0)}_{=1}+b\cdot 0$$

where for the last term to be 0 we require $\mathrm{Re}(\gamma-\alpha-\beta)\geq 0$. Apply Gauss-Kummer we can find a. By substituting $z=0$ we get

$$1=a\cdot\,{}_2F_1(\cdots;1)+b(1-0)^{\gamma-\alpha-\beta}\,{}_2F_1(\cdots;1).$$

Note as $z\to 1$, $1-z\to 0$ so $\arg(1-z)\to 0$ so the exponential term goes to 1. In this way we can solve for b.

This illustrates the general strategy of solving the connection problem: we find an integral representation for the solutions, analytically continue them and evaluate them in some "common point" z_0.

7.2.4 Monodromy

Since the connection matrices are known, we can compute monodromy using from local representations: under $z\mapsto ze^{2\pi 1}$, recall that $u_1^{(0)}$ has no monodromy and $u_2^{(0)}$ has the exponential term $z^{1-\gamma}$, we have

$$Y^{(0)}\mapsto Y^{(0)}\begin{pmatrix}1 & \\ & e^{-2\pi i\gamma}\end{pmatrix}$$

Similarly

$$\begin{aligned}
Y^{(1)}&\mapsto Y^{(1)}\begin{pmatrix}1 & \\ & e^{2\pi i(\gamma-\beta-a)}\end{pmatrix}\\
Y^{(\infty)}&\mapsto Y^{(\infty)}\begin{pmatrix}e^{2\pi i\alpha} & \\ & e^{2\pi i\beta}\end{pmatrix}
\end{aligned}$$

Denote the local monodromy matrices by $\tilde{M}_0$ etc, we have that for $Y^{(0)}$ the monodromy matrices are

$$M_0 = \tilde{M}_0, M_1 = C_{01}^{-1}\tilde{M}_1 C_{01}, M_\infty = C_{0\infty}^{-1}\tilde{M}_\infty C_{0\infty}.$$

We present here another approach that works for Gauss hypergeometric equations. Choose as fundamental system $u_1^{(0)}, u_1^{(1)}$ (they are linearly independent). Then immediately we know the monodromy matrices have the form

$$M_0 = \begin{pmatrix} 1 & * \\ 0 & * \end{pmatrix}, M_1 = \begin{pmatrix} * & 0 \\ * & 1 \end{pmatrix}$$

Recall that the exponents are $1-\gamma$ at $z=0$ and $\gamma-\alpha-\beta$ at $z=1$, we have

$$M_0 = \begin{pmatrix} 1 & x \\ 0 & e^{2\pi i(1-\gamma)} \end{pmatrix}, M_1 = \begin{pmatrix} e^{2\pi i(\gamma-\alpha-\beta)} & 0 \\ y & 1 \end{pmatrix}$$

for some x and y. As $M_1 M_0 = M_\infty^{-1}$ and M_∞^{-1} has eigenvalues $e^{-2\pi i\alpha}, e^{-2\pi i\beta}$, taking trace we can find xy (in terms of α, β and γ). Note that this is all we can extract: if we change the fundamental system by a linear transformation to $[ru_0^{(0)}, su^{(u)}]$ then the monodromy matrix M_j transforms to

$$\begin{pmatrix} r & \\ & s \end{pmatrix}^{-1} M_j \begin{pmatrix} r & \\ & s \end{pmatrix} = \begin{pmatrix} * & *\frac{r}{s} \\ *\frac{s}{r} & * \end{pmatrix}$$

Thus the invariant is xy and x, y individually have freedom (by diagonal conjugation). We conclude by the following “rigidity” property of the Gauss hypergeometric equation: the equation is equivalent to the characteristic exponents, and the characteristic exponent determines monodromy (up to diagonal conjugation) and vice versa. Thus in turn the equation and the monodoromy determine each other.

$$\text{equation} \leftrightarrow \text{characteristic exponent} \leftrightarrow \text{monodromy}$$

7.3 Some further results

Companion system Let $y = (y_1, y_2)^T$ where $y_1 = u, y_2 = (z-1)u'$. Then y satisfies the following system

$$\frac{dy}{dz} = (\frac{1}{z}\underbrace{\begin{pmatrix} 0 & 0 \\ -\alpha\beta & -\gamma+ \end{pmatrix}}_{A_0} + \frac{1}{z-1}\underbrace{\begin{pmatrix} 0 & 1 \\ 0 & \gamma-\alpha-\beta \end{pmatrix}}_{A_1})y$$

The eigenvalues of A_0 are $0, -\gamma$, the eigenvalues of A_1 are $0, \gamma-\alpha-\gamma$ and the eigenvalues of $A_\infty = -A_0 - A_1$ are α, β

Monodromy

Theorem 7.5. *Any irreducible representation*

$$\pi_1(\overline{\mathbb{C}} \setminus \{0, 1, \infty\}) \to \mathrm{GL}(2, \mathbb{C})$$

is realised by the Guass equation.

Theorem 7.6. *Any irreducible represdentation is realised by the monodromy of*

$$\frac{dy}{dz} = (\frac{1}{z}\begin{pmatrix} * & * \\ 0 & * \end{pmatrix} + \frac{1}{z-1}\begin{pmatrix} * & * \\ 0 & * \end{pmatrix})y.$$

Theorem 7.7. *Any 2×2 Fuchsian system*

$$\frac{dy}{dz} = (\frac{A_0}{z} + \frac{A_1}{z-1})y$$

with an irreducible monodromy is meromophically equivalent to the Gauss equation.

8 Poincaré asymptotics

As a motivating example, consider the system, which is confusingly also called the Euler equation,

$$y' + \frac{1}{z^2} y = \frac{1}{z}.$$

Note that $z = 0$ is non-Fuchsian. It has homogeneous solution $y_h(z) = ce^{1/z}$. By variation of parameters, subsitute $y_p(z) = c(z)e^{1/z}$ into the equation and solve to get

$$c(z) = \int_{z_0}^{z} \frac{e^{-1/s}}{s} ds$$

so by a change of variable

$$y_p(z) = e^{1/z} \int_{1/z}^{1/z_0} \frac{e^{-t}}{t} dt.$$

We can take $z > z_0 > 0$. If we take the limit $z_0 \to 0^+$, we get

$$y(z) = e^{1/z} \int_{1/z}^{\infty} \frac{e^{-t}}{t} dt$$

which is real for $z > 0$ and can be analytically continued to $\operatorname{Re} z > 0$. We call the integral

$$\operatorname{Ei}(z) = \int_{z}^{\infty} \frac{e^{-t}}{t} dt$$

the *exponential integral*. The solution $y(z) = e^{1/2} \operatorname{Ei}(z)$ is called *Euler function* and is denote by $E(z)$.

We look for a formal solution $y_f(z) = \sum_{m=0}^{\infty} c_m z^m$. Substitute into the equation and solve to get

$$y_f(z) = \sum_m (-1)^m m! z^{m+1}$$

which is divergent. But $E(z) \sim y_f(z)$ when $z \to 0$ in $-\frac{3\pi}{2} < \arg z < \frac{3\pi}{2}$, meaning that $|\sum_{m=0}^{N} c_m z^m - E(z)|$ is

- small for $|z|$ small and N fixed, and
- small for large N and $|z|$ fixed.

Poincaré posed the following problem. Take as example two series

$$\sum_{n=0}^{\infty} \frac{1000^n}{n!}, \sum_{n=0}^{\infty} \frac{n!}{1000^n}.$$

The first is convergence while the second is convergent. The question is: suppose we want to compute the partial sum of the first, say, 1000 terms. However terms do we need to sum so that the error is within 10^{-6}? In other words for what $N \leq 1000$ we have

$$\sum_{n=0}^{1000} s_n - \sum_{n=0}^{N} s_n < 10^{-6}?$$

With the help of Stirling formula

$$m! \sim m^m \sqrt{2\pi m} e^{-m},$$

for the first series we need $N = 1000$ but for the second we need $N = 3$. Thus Poincaré said that the second series, although divergent for mathematicians, is "convergent" for astronomers.

Let $s(\alpha, \beta) = \{z \in \widetilde{\mathbb{C} \setminus \{0\}} : \alpha < \arg z < \beta\}$ be an open sector (with vertex $z = 0$ or $z = \infty$). We can also define analogously a closed sector $\overline{s}(\alpha, \beta)$ by using nonstrict inequalities.

Definition (asymptotic series)**.** Let S be a sector with vertex at $z = 0$. Let $f : S \to \mathbb{C}$ be a function and $\sum_{k=0}^{\infty} a_k z^k$ be a formal series. We say $f(z)$ is *asymptotic* to the formal series, written $f(z) \sim \sum_{k=0}^{\infty} a_k z^k$, for $z \to 0$ in S if for every closed subsector $S' \subseteq S$ and for every $m \in \mathbb{N}$, exists $C(m, S') > 0$ such that

$$|f(z) - \sum_{k=0}^{m} a_k z^k| \leq C(m, S')|z|^{m+1}$$

for all $z \in S'$.

We say the asymptotic series is *uniform* in S if $C(m, S')$ does not depend on S'.

Of course if S is closed then we only need to check $S' = S$. The key point is that if the sector is open then $C(m, S')$ may depend on S' as S' "approaches" S.

Exercise. Verify that the definition above is equivalent to saying that for every closed $S' \subseteq S$,

$$\frac{1}{z^m}(f(z) - \sum_{k=0}^{m} a_k z^k) \to 0$$

as $z \to 0$ in S'.

At $z = \infty$, we say $f(z) \sim \sum_{k=0}^{\infty} a_k z^{-k}$ as $z \to \infty$ in S if for every closed $S' \subseteq S$ and every $m \in \mathbb{N}$, exists $C(m, S') > 0$ such that

$$|f(z) - \sum_{k=0}^{m} \frac{a_k}{z^k}| \leq \frac{C(m, S')}{|z|^{m+1}}$$

for $z \in S'$.

Some properties of asymptotic series:

1. if f has asymptotic expansion then it is unique. This is guaranteed by the uniqueness of limit as $a_0 = \lim_{z \to 0, z \in S'} = a_0$ and inductively all a_m's are uniquely determined.

2. not all functions have asymptotic expansion in a given sector. For example consider $f(z) = e^{1/z}$ and consider for example the sector $S(-\frac{\pi}{4}, \frac{\pi}{4})$. Then $f(z) \to \infty$ as $z \to 0$.

3. $\sum a_k z^k$ can be (is) asymptotic expansion of infinitely many functions. For example the zero formal series is asymptotic to $f(z) = e^{-\frac{1}{z^\sigma}} \sim 0$ in $-\frac{\pi}{2\sigma} < \arg z < \frac{\pi}{2\sigma}$ for all $\sigma > 0$, as $\lim \frac{f(z)}{z^m} = 0$ for all m in the sector.

Proposition 8.1. *Suppose $f(z) \sim \sum_{k=0}^{\infty} a_k z^k$ in S. If the opening angle is $> 2\pi$ and f is single-valued at $z = 0$ then the series converges.*

Proof. We know $a_0 = \lim_{z \to 0, z \in S} f(z)$. The limit is along any direction in $\mathbb{C} \setminus \{0\}$, so $z = 0$ is a removable singularity. Thus f is holomorphic at $z = 0$ so has a convergent Taylor expnansion which necessarily coincide with the given series. □

Wasow: Asymptotic expansions for ODE
Algebraic properties

1. Let $f(z) \sim \sum_{k=0}^{\infty} a_k z^k, g(z) \sim \sum_{k=0}^{\infty} b_k z^k$ as $z \to 0$ in S. Let $\alpha, \beta \in \mathbb{C}$, then

$$\alpha f(z) + \beta g(z) \sim \sum_{k=0}^{\infty} (\alpha a_k + \beta b_k) z^k$$

$$f(z) \cdot g(z) \sim \sum_{\ell=0}^{\infty} (\sum_{k=0}^{\ell} a_k b_{\ell-k}) z^\ell$$

$$\frac{1}{f(z)} \sim \sum_{k=0}^{\infty} c_k z^k \text{ if } a_0 \neq 0, \text{ where } \sum a_k z^k \cdot \sum c_k z^k = 1$$

2. Suppose $f \sim \sum a_k z^k$ in S_f, $g(z) = \sum b_k z^k$ in S_g. Let $\tilde{f}(z) = f(z) - a_0 \sim \sum_{k=1}^{\infty} a_k z^k$. Assume $\tilde{f}(S_f) \subseteq S_g$. Then

$$g(\tilde{f}(z)) \sim \sum_{h=0}^{\infty} b_h (\sum_{k=1}^{\infty} a_k z^k)^h$$

in S_f.

Analytic properties:

1. Suppose f is holomorphic in S with centre $z = 0$ and $f(z) \sim \sum a_k z^k$ in S. Then

$$\int_0^z f(\zeta) d\zeta \sim \sum a_k \int_0^z \zeta^k d\zeta = \sum \frac{a_k}{k+1} z^{k+1}$$

in S.

If $f(z) \sim \sum \frac{a_k}{z^k}$ as $z \to \infty$ then

$$\int_\infty^z (f(\zeta) - a_0 - \frac{a_1}{\zeta} d\zeta) \sim \sum_{k=2}^{\infty} a_k \int_\infty^z \frac{d\zeta}{\zeta^k} = \sum_{k=2}^{\infty} \frac{a_k}{1-k} z^{1-k}$$

as $z \to \infty$ in S.

2. Suppose $f(z)$ is holomorphic on S at $z = 0$ and $f(z) \sim \sum a_k z^k$ in S, a sector with interior (i.e. not a ray). Then

$$\frac{df(z)}{dz} \sim k a_k z^{k-1}$$

in every closed subsector $S' \subseteq S$.

3. Note that if we add the condition that $f(z)$ has a uniform asymptotic series, it is *not* true that we get a uniform asymptotic for $\frac{df(z)}{dz}$.
4. Suppose $f(z) \sim \sum a_k z^k$ as $z \to 0$ in S, a sector with interior. Then the limit $\lim_{z\to 0} f^{(k)}(z)$ exists and equals to $k!a_k$ in every $S' \subseteq S$.
5. Suppose $f(z)$ is holomorphic in S and assume $\lim_{z\to 0} f^{(k)}(z) = f_k$. Then
$$f(z) \sim \sum \frac{f_k}{k!} z^k$$
as $z \in 0$ in S.

Theorem 8.2 (Borel-Ritt)**.** *For every formal series $\sum_{k=0}^{\infty} a_k z^k$ and every sector S (open or closed) at $z = 0$, there exists $f(z)$ holomorphic in S such that $f(z) \sim \sum a_k z^k$ in S.*

example of computation of asymptotic expansion Recall the exponential integral
$$\mathrm{Ei}(z) = \int_z^{\infty} \frac{e^{-t}}{t} dt, z > 0.$$

Proposition 8.3. *The exponential integral defines an analytic function in $\{z \in \mathbb{C} \setminus \{0\} : -\pi < \arg z < \pi\}$. It is real for $z > 0$ and*
$$\mathrm{Ei}(z) = -\ln z - \gamma - \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n \cdot n!} z^n$$
where $\ln z$ is the principal branch, i.e. $\ln z \in \mathbb{R}$ for $z > 0$, γ is the Euler-Mascheroni constant, and the series converges uniformly in every compact subset of $\mathbb{C}$.

Proof.
$$\begin{aligned}
\mathrm{Ei}(z) &= \int_1^z \frac{e^{-t}}{t} dt + \int_z^1 \frac{e^{-t}}{t} dt \\
&= c_1 + \int_z^1 \sum_0^{\infty} \frac{(-1)^n t^{n-1}}{n!} dt \\
&= c_1 + \int_z^1 \frac{dt}{t} + \sum_{n=1}^{\infty} \frac{(-1)^n}{n!} \int_z^1 t^{n-1} dt \\
&= c_2 - \ln z + \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n!} \frac{z^n}{n}
\end{aligned}$$
so $\mathrm{Ei}(z) + \ln z$ is analytic at $z = 0$. As $\mathrm{Ei}(z) \in \mathbb{R}$ for $z > 0$, $\ln z$ takes the principal branch.

To find c_2, integrate by parts
$$\mathrm{Ei}(z) = -e^{-z} \ln z + \int_z^{\infty} e^{-t} \ln dt$$

so

$$c_2 = \lim_{z\to 0}(\mathrm{Ei}(z) + \ln z) = \lim_{z\to 0}[(1 - e^{-z}\ln z) + \int_z^\infty e^{-t}\ln t dt] = \int_0^\infty e^{-t}\ln t$$

which we call $-\gamma$. □

Exercise. Show

$$-\gamma = \frac{d\ln\Gamma(z)}{dz}|_{z=1}, -\gamma = \int_0^\infty e^{-t}\ln t dt.$$

From the expression we also know that the analytic continuation of $\mathrm{Ei}(z)$ on $\widetilde{\mathbb{C}\setminus\{0\}}$ has monodromy

$$\mathrm{Ei}(ze^{2\pi i}) = \mathrm{Ei}(z) - 2\pi i.$$

Proposition 8.4. $\mathrm{Ei}(z)$ *defined on* $\widetilde{\mathbb{C}\setminus\{0\}}$ *has the following asymptotic representation:*

$$ze^z\,\mathrm{Ei}(z) \sim \sum_{n=0}^\infty (-1)^n n! z^{-n}$$

as $z\to\infty$ *in* $S = \{z \in \widetilde{\mathbb{C}\setminus\{0\}} : -\frac{3\pi}{2} < \arg z < \frac{3\pi}{2}$.

Proof.

$$\begin{aligned}\int_z^\infty \frac{e^{-t}}{t}dt &= \frac{e^{-z}}{z} - \int_z^\infty \frac{e^{-t}}{t^2}dt \\ &= \frac{e^{-z}}{z} - \frac{e^{-z}}{z^2} + 2\int_z^\infty \frac{e^{-t}}{t^3}dt \\ &= \cdots \\ &= \frac{e^{-z}}{z}[\sum_{n=0}^m \frac{(-1)^n n!}{z^n} + \underbrace{(-1)^{m+1}(m+1)!z\int_z^\infty \frac{e^{z-t}}{t^{m+2}}dt]}_{R_m(z)}\end{aligned}$$

so

$$ze^z\,\mathrm{Ei}(z) = \sum_{n=0}^m \frac{(-1)^n n!}{z^n} + R_m(z).$$

Check $|R_m(z)| \leq C(m)|z|^{m+1}$:

$$\begin{aligned}|R_m(z)| &= (m+1)!|z|\left|\int_z^\infty \frac{e^{z-t}}{t^{m+2}}dt\right| \\ &= (m+1)!|z|\left|\frac{1}{z^{m+2}} - (m+2)\int_z^\infty \frac{e^{z-t}}{t^{m+3}}dt\right| \\ &\leq (m+1)!|z|^{-m-1} + (m+2)!\left|\int_z^\infty \frac{e^{z-t}}{t^{m+3}}dt\right|\end{aligned}$$

For $z > 0$, we have estimate

$$\left| \int_z^\infty \frac{e^{z-t}}{t^{m+3}} dt \right| \leq \int_z^\infty \frac{dt}{t^{m+3}} = \frac{z^{-m-2}}{m+2}$$

so

$$|R_m(z)| \leq 2(m+1)!|z|^{-m-1}$$

as desired. □

Exercise. Show the asymptotic series is valid on $-\pi \leq \arg g \leq \pi$, by deriving

$$|R_m(z)| \leq (2(m+1)! + (m+2)\pi)|z|^{-m-1}.$$

extension of asymptotics outside $-\pi \leq \arg z \leq \pi$ We have asymptotic series for

$$\begin{aligned} e^z \operatorname{Ei}(z), &-\pi \leq \arg z \leq \pi \\ e^z \operatorname{Ei}(ze^{2\pi i}), &-3\pi \leq \arg z \leq -\pi \\ e^z \operatorname{Ei}(ze^{-2\pi i}), &\pi \leq \arg z \leq 3\pi \end{aligned}$$

Note that the domains overlap. Recall

$$e^z \operatorname{Ei}(z) = e^z \operatorname{Ei}(ze^{\pm 2\pi i}) \pm 2\pi i e^2$$

Finally note that $e^z \sim 0$ as $z \to \infty$ in $\frac{\pi}{2} + 2k\pi < \arg z < \frac{3\pi}{2} + 2k\pi$. Thus

$$e^z \operatorname{Ei}(z) - e^z \operatorname{Ei}(ze^{\pm 2\pi i}) \sim 0$$

in the sector. Set $k = 0$ and -1 and apply to the three expansions, we get the sector $-\frac{3\pi}{2} < \arg z < \frac{3\pi}{2}$.

Note. Note the error term

$$|R_m(z)| \leq \underbrace{(2(m+1)! + (m+2)\pi)}_{C_n} |z|^{-m-1}.$$

is typical of asymptotic expansions:

1. ...

9 Singularities of the second kind

We will assume the singularity is at $z = \infty$ so we study the system

$$\frac{dy}{dz} = z^{r-1} A(z) y$$

where $A(z)$ is holomorphic for $|z| \geq R$ and $A(z) = \sum \frac{A_i}{z^i}$ for $|z| \geq R$. If $r = 0$ then it is a Fuchsian singularity, if $r \geq 1$ then it is a singularity of the seoncd kind. We assume $A(z) \sim \sum_{i=0}^{\infty} \frac{A_i}{z^i}$ in $S = S(\alpha, \beta)$ and $|z| > R$. The object we are dealing with in tis chapter is the solution

$$y(z) = G(z) z^B \exp(\Lambda(z))$$

where $G(z)$ is analytic in a sector at ∞ and has an asymptotic expansion in z^{-1}, $\Lambda(z)$ is a polynomial in z.

9.1 Formal simplification

Suppose $A(z) = \sum_{i=0}^{\infty} \frac{A_i}{z^i}$ where A_0 has at least two distinct eigenvalues. Divide the eigenvalues of A_0 in two sequences $(\lambda_1, \ldots, \lambda_p), (\lambda_{p+1}, \ldots, \lambda_m)$ which are disjoint. We can find G_0 invertible such that $G_0^{-1} A_0 G^0$ is block diagonal with each block having the respective sequence of eigenvalues. We look for a gauge transformation $Y(z) = G(z)\tilde{Y}(z)$, $G(z) = I + \sum_{j=1}^{\infty} G_j z^{-j}$ such that

$$\frac{d\tilde{Y}}{dz} = z^{r-1} B(z) \tilde{Y}$$

which is “simpler”.

Substitute into the differential equatation, we get

$$G'\tilde{y} + G\tilde{Y}' = z^{r-1} AG\tilde{Y},$$

i.e.

$$z^{1-r} G' + GB = AG.$$

Substitute $G = \sum G_i z^{-i}, A = \sum A_i z^{-i}, B = \sum B_i z^{-i}$,

- $\ell = 0$: set $G_0 = I, B_0 = A_0$,
- $1 \leq \ell \leq r$: $A - G_\ell - G_\ell A_0 = (\sum_{j=1}^{\ell-1} (G_j B_{\ell-j} - A_{\ell-j} G_j) - A_\ell) + B_\ell$
- $r+1 \leq \ell$: $A - G_\ell - G_\ell A_0 = (\sum_{j=1}^{\ell-1} (G_j B_{\ell-j} - A_{\ell-j} G_j) - (\ell - r) G_{\ell-r}) + B_\ell$

Thus the recurrence relation for G_ℓ, B_ℓ is

$$A_0 G_\ell - G_\ell A_0 = K_\ell + B_\ell$$

where K_ℓ is known from previous steps.

Recall

$$A_0 = \begin{pmatrix} * & 0 \\ 0 & * \end{pmatrix}, B_\ell = \begin{pmatrix} B_{11}^{(\ell)} & B_{12}^{(\ell)} \\ B_{21}^{(\ell)} & B_{22}^{(\ell)} \end{pmatrix}$$

etc so we get *Sylvester equation*

$$A_{ii}^{(0)}G_{ij}^{(\ell)} - G_{ij}^{(\ell)}A_{jj}^{(0)} = K_{ij}^{(\ell)} + B_{ij}^{(\ell)}.$$

For $i = j$, take $B_{ii}^{(\ell)} = -K_{ii}^{(\ell)}$. Then $G_{ii}^{(\ell)}$ is the solution of a homogeneous equation so has a nontrivial solution (we can choose $G_{ii}^{(\ell)} = 0$). For $i \neq j$, take $B_{ij}^{(\ell)} = 0$. As $A_{ii}^{(0)}$ and $A_{jj}^{(0)}$ do not have common eigenvalues, there is a solution $G_{ij}^{(\ell)}$.

In conclusion, we can find gauge transformation $G(z)$ such that $Y(z) = G_0G(z)\tilde{Y}(z)$ satisfies

$$\frac{d\tilde{Y}}{dz} = z^{r-1}B(z)\tilde{Y}(z)$$

where $B(z)$ is block diagonal.

The block diagonal system admits fundamental matrix solution

$$\tilde{Y}(z) = \begin{pmatrix} \tilde{Y}_1(z) & 0 \\ 0 & \tilde{Y}_2(z) \end{pmatrix}$$

and

$$\frac{d\tilde{Y}_i}{dz} = z^{r-1}B_{ii}(z)\tilde{Y}_i.$$

We can repeat the above computation for $\tilde{Y}_i$ if $B_{ii}^{(0)}$ has at least two distinct eigenvalues. By repeated applications we conclude that the following holds:

Theorem 9.1. *If A_0 has $s \leq n$ distinct eigenvalues $\lambda_1, \dots, \lambda_s$, we can formally determine $G(z)$ such that $Y = G_0G(z)\tilde{Y}$ gives*

$$\frac{d\tilde{Y}}{dz} = z^{r-1}B(z)\tilde{Y}$$

where $B(z)$ is block diagonal with s blocks.

Analytic issue of the solution is addressed by Malmquist (1944) and Sibuya (1962) (see Wasow). We present an easier result.

Theorem 9.2.

1. *Let $A(z)$ be analytic for $|z| > R$ with Taylor series $\sum_{i=0}^{\infty} A_i z^{-i}$, absolutely convergent on $|z| \geq \rho > R$.*
2. *It suffices to suppose that $A(z)$ is analytic in a sector S for opening angle $< \frac{\pi}{r}, |z| > R$ with $A(z) \sim \sum A_i z^{-i}$ in S as $z \to \infty$.*

Suppose A_0 has eigenvalues divided into two disjoint sequences. Then exists $R_1 \geq R$ and actual solutions $G(z), B(z)$ to

$$z^{1-r}G' + GB = A(z)G$$

analytic for $|z| \geq R_1$ in S with opening $< \frac{\pi}{r}$.

Remark. Sibuya proved that the result holds for S with opening $\frac{\pi}{r} + \varepsilon$ for $\varepsilon > 0$.

Now suppose A_0 has pairwise distinct eigenvalues $\lambda_1, \dots, \lambda_n$. Then $G_0^{-1} A_0 G_0 = \Lambda = \operatorname{diag}(\lambda_1, \dots, \lambda_n)$, $B(z) = \operatorname{diag}(b_1(z), \dots, b_n(z))$ is diagonal, $b_k(z) \sim \sum b_\ell^{(k)} z^{-\ell}$ with $b_0^{(k)} = \lambda_k$ in some S. Then the sysmte

$$\frac{d\tilde{Y}}{dz} = z^{r-1} B(z) \tilde{Y}$$

so exists a fundamental matrix solution $\tilde{Y}(z) = \operatorname{diag}(y_1(z), \dots, y_m(z))$: by separation of variables

$$y(z) = \exp(\int_a^z t^{r-1} b(t) dt)$$

where the integration should be taken on a path that lies in S.

$$\int_a^z t^{r-1} b(t) dt = \int_a^z t^{r-1}(b_0 + \frac{b_1}{t} + \dots + \frac{b_n}{t^n}) dt + \underbrace{\int_a^z t^{r-1}(b(t) - \sum_{j=0}^r \frac{b_j}{t^j})\, dt}_{\bullet}$$

$$= q(z) + b_n(z) + \text{const} + \int_a^z t^{r-1}(b(t) - \sum_{j=0}^r \frac{b_j}{t^j}) dt$$

The term $(\bullet)$ vanishes in S as $\frac{1}{t^2}$. Thus the limit of the integral as $z \to \infty$ exists, so

$$\int_a^z t^{r-1}(b(t) - \sum_{j=0}^r \frac{b_j}{t^j}) dt = C + \underbrace{\int_\infty^z t^{r-1}(b(t) - \sum_{j=0}^r \frac{b_j}{t^j})\, dt}_{h(z)}$$

and

$$h(z) \sim \int_\infty^z \sum_{\ell=1}^\infty \frac{b_{\ell+r}}{t^{\ell+1}} dt = \sum_{\ell=1}^\infty \int_\infty^z \frac{b_{\ell+r}}{t^{\ell+1}} dt = \sum_{\ell=1}^\infty \frac{b_{\ell+r}}{-\ell z^\ell}$$

Thus in conclusion,

$$y(z) = \text{const} \cdot z^{b_r} e^{q(z)} e^{h(z)} = f(z) z^{b_r} e^{q(z)}$$

wher $f(z)$ is analytic in S. We can choose $f(z) \sim 1 + \sum_{k=1}^\infty f_k z^{-k}$ and

$$Y(z) = \underbrace{G_0\, G(z) \begin{pmatrix} f_1(z) & & \\ & \ddots & \\ & & f_n(z) \end{pmatrix}}_{\mathcal{G}(z)} z^{B_r} e^{Q(z)}$$

where

$$Q(z) = \frac{B_0}{r} z^r + \frac{B_1}{r-1} + \dots + B_{r-1} z.$$

$\mathcal{G}(z) \sim F(z) = I + \sum_{k=1}^\infty \frac{F_k}{z^k}$ as $z \to 0$.

Definition (formal fundamental solution)**.** $Y_F(z) = G_0 F(z) z^{B_r} e^{Q(z)}$ is called the *formal fundamental solution*.

Remark. $\mathcal{G}(z) \sim F(z)$ asymptotically so $Y(z) z^{-B_r} e^{-Q(z)} \sim G_0 F(z)$. By abuse of notation we will write $Y(z) \sim Y_F(z)$.

Proposition 9.3.

- *Suppose* Λ *has been fixed. Then* $B_0 = \Lambda, B_1, \ldots, B_r$ *are uniquely determined.*
- *Suppose* Λ *and* G_0 *have been fixed. Then* $Y_F(z)$ *is uniquely determined.*

Proof. Any formal solution has the form $Y_F(z) \cdot C$. Then

$$\begin{aligned} Y_F(z)C &= G_0(I + \sum_{k=1}^{\infty} F_k z^{-k}) z^{B_r} e^{Q(z)} C \\ &= G_0(I + \sum_{k=1}^{\infty} F_k z^{-k}) C z^{C^{-1} B_r C} e^{C^{-1} Q(z) C} \\ &= G_0 C(I + \sum_{k=1}^{\infty} C^{-1} F_k C z^{-k}) z^{C^{-1} B_r C} e^{C^{-1} Q(z) C} \end{aligned}$$

We are requiring that $C^{-1} \Lambda C = \Lambda$. Since Λ has distinct eigenvalues, this is equivalent to C being diagonal so $C^{-1} B_j C = B_j$ for $j = 1, \ldots, r$. This proves the first statement.

If in addition G_0 is fixed, $G_0 C = G_0$ so $C = I$. □

Now we adpot a different notation. Let $B_1 = \Lambda_1, \ldots, B_r = \Lambda_r$. Then

$$z^{B_r} e^{Q(z)} = \exp(\frac{\Lambda_r}{r} z^r + \cdots + \Lambda_{r-1} z + \Lambda_r \ln z) =: e^{\Lambda(z)}.$$

Remark. Though Y_F is unique, this is not the case for $Y(z)$. In Wasow, the sector opening $< \frac{\pi}{r}$. If we increase the opening we can prove uniqueness.

Proposition 9.4. *Two systems*

$$\frac{dY}{dz} = z^{r_1 - 1} A_1(z) Y, \frac{dX}{dz} = z^{r_2 - 1} A_2(z) X$$

are formally holomorphically equivalent if and only if $\pi_1 = \pi_2$ *and they have the same* $\Lambda, \Lambda_1, \ldots, \Lambda_r$ *where* Λ *is a digonal form of both* $A_0^{(1)}$ *and* $A_0^{(2)}$*. These are called* formal invariants*.*

Proof. Suppose they have the same formal invariants. Then

$$\begin{aligned} Y_F(z) &= G_0^{(1)} F_1(z) \exp(\frac{\Lambda}{r_1} z^{r_1} + \cdots + \Lambda_{r_1} \ln z) \\ X_F(z) &= G_0^{(2)} F_2(z) \exp(\frac{\Lambda}{r_1} z^{r_1} + \cdots + \Lambda_{r_1} \ln z) \end{aligned}$$

Then $X_F(z) = H(z)Y_F(z)$ where $H(z) = G_0^{(2)}F_2(z)(G_0^{(1)}F_1(z))^{-1}$.

Conversely suppose $X(z) = H(z)Y(z)$ wher $\det H_0 \neq 0$. Let Λ be a diagonal form of $A_0^{(1)}$. Then exists

$$Y_F(z) = G_0^{(1)}F_1(z)\exp(\frac{\Lambda}{r_1}z^{r_1} + \cdots + \Lambda_{r_1}\ln z)$$

so by assumption

$$X_F(z) = \underbrace{H_z(z)G_0^{(1)}F_1(z)}_{\mathcal{F}(z)}\exp(\frac{\Lambda}{r_1}z^{r_1} + \cdots + \Lambda_{r_1}\ln z).$$

The equation for X_F is

$$\frac{dX_F}{dz} = (\frac{d\mathcal{F}}{dz}\mathcal{F}^{-1} + \mathcal{F}(\Lambda z^{r_1-1} + \cdots + \frac{\Lambda_{r_1}}{z})\mathcal{F}^{-1})X_F = z^{r_1-1}(\underbrace{\mathcal{F}_0\Lambda\mathcal{F}_0^{-1}}_{A_0^{(2)}} + \sum\frac{A_i^{(2)}}{z^i})X_F$$

so $A_0^{(2)}$ has diagonal form Λ, $r_2 = r_1$ and also $\Lambda_1, \ldots, \Lambda_r$ are the same. □

9.2 Case of non-distinct eigenvalues

Theorem 9.5. *If $A(z)$ is holomorphic in S, $A(z) \sim \sum A_i z^i$ then for every "sufficiently small" subsector of S there is fundamental matrix solution*

$$Y(z) = G(z)z^L e^{Q(z)}$$

where $Q(z)$ is diagonal polynomial in $z^{1/N}$ for some natural number N, L is inte general not diagonal, $G(z) \sim z^{k_0/N}\sum_{j=0}^{\infty}\frac{F_j}{z^{j/N}}$ in some small sector for $k_0 \in \mathbb{Z}$. In general $\det F_0 = 0$. *$z = \infty$ is called a ramified singularity.*

Example: Airy equation The Airy equation is the equation

$$u'' = zu.$$

By a substitution

$$\frac{dy}{dz} = \begin{pmatrix} 0 & 1 \\ z & 0 \end{pmatrix} y = z(\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \frac{1}{z}\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix})y.$$

By the gauge transformation $y = (\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix})y_1$, we write

$$\frac{dy_1}{dz} = z(\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \frac{1}{z}\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix})y_1$$

from which we see that there are two identical eigenvalues $\lambda_1 = \lambda_2 = 0$.

If we apply the shearing transformation $y_1 = (\begin{smallmatrix} 1 & 0 \\ 0 & z^{-1/2} \end{smallmatrix})y_2$ so

$$\frac{dy_2}{dz} = z^{1/2}(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2z^{3/2}}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})y_2$$

so at the cost of a fractional power we now have two distinct eigenvalues. Let $t = cz^{1/2}$ so $\frac{d}{dz} = \frac{c^2}{2}\frac{1}{t}\frac{d}{dt}$. Set $c = 2^{1/3}$,

$$\frac{dY_2}{dt} = t^2(\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} + \frac{1}{t^3}\begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix})y_2$$

This has Poincaré rank 3. Eigenvalues of the term in the parethesis is

$$\pm 1 + \frac{1}{2t^3} + O(\frac{1}{t^6})$$

so

$$Y_2(z) = G_0 G(t) t^{\Lambda_3} \exp(\frac{\Lambda}{3}t^3 + \frac{\Lambda_1}{2t})$$

so

$$Y(z) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & z^{-1/2} \end{pmatrix}\begin{pmatrix} -\gamma_1 & \gamma_2 \\ \gamma_1 & \gamma_2 \end{pmatrix} G(z^{1/2}) z^{\frac{1}{4}I} \exp(-\frac{2}{3}z^{3/2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}).$$

9.3 Stokes phenomenon

Now back to the case A_0 with distinct eigenvalues. Suppose $Y(z) \sim Y_F(z)$ in S. We can extend asympototics to $S \supseteq S'$ but in general we meet "separating rays" beyond which the asymptotics no longer holds. This is called *Stokes phenomenon*. The ray is called *Stokes ray*.

Exercise. As $z \to \infty$,

$$f(z) = 1 + e^z = \begin{cases} \sim 1 & \operatorname{Re} z < 0 \\ \text{oscilates} & \operatorname{Re} z = 0 \\ \infty & \operatorname{Re} z > 0 \end{cases}$$

Consider the λ-plane. Choose $\eta \in \mathbb{R}$ such that $\arg(\lambda_j - \lambda_k) \neq \eta \pmod{\pi}$. This is called an *admissible direction* in the λ-plane. Take any determination of $\arg(\lambda_j - \lambda_k)$, for example $\eta - 2\pi < \widehat{\arg}(\lambda_j - \lambda_k) < \eta$ for $j \neq k$.

Definition (Stokes ray). The *Stokes rays* associated with $(\lambda_j, \lambda_k)_{j \neq k}$ are infinitely many rays in $\widetilde{\mathbb{C} \setminus \{0\}}$ such that

$$\operatorname{Re}((\lambda_j - \lambda_k)z^n) = 0, \operatorname{im}((\lambda_j - \lambda_k)z^n) < 0.$$

Proposition 9.6. *To* (λ_j, λ_k) *the associated rays are*

$$\arg z = \theta_{jk} + 2\pi\frac{N}{r}$$

for all $N \in \mathbb{Z}$, $\theta_{jk} = \frac{1}{r}(\frac{3\pi}{2} - \widehat{\arg}(\lambda_j - \lambda_k))$.

Proof. Exercise. □

There is an even number 2μ of possible values of $\hat{\arg}(\lambda_j - \lambda_k)$. Label $\hat{\arg}(\lambda_j - \lambda_k)$ with $\eta_{\tilde{\nu}}$ for $0 \le \tilde{\nu} \le 2\mu - 1$. If η is an admissible direction in λ-plane,

$$\eta > \eta_0 > \eta_1 > \dots \eta - \pi > \eta_\mu > \dots > \cdot_{2\pi-1} > \eta - 2\pi.$$

Let $\pi = \frac{3\pi}{2} - \eta$. Stokes rays have directions $\tau_{\tilde{\nu}} = (\frac{3\pi}{2} - \eta_{\tilde{\nu}})\frac{1}{r}$ and they satisfy

$$\tau < \tau_0 < \tau_1 < \dots < \tau_{\mu-1} < \tau + \frac{\pi}{2} < \tau_\mu < \dots < \tau_{2\pi-1} < \tau + \frac{2\pi}{r}.$$

τ is called *admissible direction in the z-plane.*

From the proposition all Stokes rays are $\tau_\nu = \tau_{\tilde{\nu}} + \frac{2N\pi}{r}$. In fact, we can express all Stokes rays by

$$\tau_{\tilde{\nu}+N\mu} := \tau_{\tilde{\nu}} + N\frac{\pi}{r}$$

for $0 \le \tilde{\nu} \le \mu - 1$.

One can also conclude that a section not containing a Stokes ray has opening $< \frac{\pi}{r}$.

Lemma 9.7. *Suppose $Y(z)$ is a fundamental matrix solution such that $Y(z)\tilde{Y}_F(z)$ as $z \to \infty$ in S. Assume exists $\tilde{S}$ that does not contain Stokes rays and $S \cap \tilde{S} \ne \emptyset$, then $Y(z) \sim Y_F(z)$ in $S \cup \tilde{S}$.*

Proof. $\tilde{S}$ has central opening $< \frac{\pi}{r}$. By Wasow exists $\tilde{(}z) \sim Y_F(z)$ in $\tilde{S}$. Then $Y(z) = \tilde{Y}(z)C$ for some C. As $Y(z) = G_0\mathcal{G}(z)e^{\Lambda(z)}, \tilde{Y}(z) = G_0\tilde{\mathcal{G}}(z)e^{\Lambda(z)}$ so

$$e^{\Lambda(z)}Ce^{-\Lambda(z)} = \tilde{\mathcal{G}}(z)^{-1}\mathcal{G}(z)$$

but $\mathcal{G}$ and $\tilde{\mathcal{G}}$ has exactly the same same asymptotic expansion in $S \cap \tilde{S}$ so RHS $\sim I$ as $z \to \infty$. Then

$$e^{\Lambda_i(z)-\Lambda_j(z)}C_{ij} = \exp\left(\frac{\lambda_i - \lambda_j}{r}z^r(1 + O(\frac{1}{z}))\right)C_{ij} \sim \delta_{ij}. \qquad (*)$$

By hypothesis in $S \cap \tilde{S}$ there are no Stokes rays so the sign of $\operatorname{Re}(\lambda_i - \lambda_j)z^r$ is fixed in $S \cap \tilde{S}$, and also in $\tilde{S}$. Thus $(*)$ holds in $\tilde{S}$ and

$$\mathcal{G}(z) = \tilde{\mathcal{G}}(z)\underbrace{e^{\Lambda(z)}Ce^{-\Lambda(z)}}_{\sim I}$$

so the asymptotic expansion holds in $S \cup \tilde{S}$. ☐

As a corollary we have

Theorem 9.8 (extension). *Let $Y(z)$ be a fundamental matrix solution which by Wasow's result satisfies $Y(z) \sim Y_F(z)$ in an S containing a set of basic Stokes rays. Then $Y(z) \sim Y_F(z)$ in an open sector $\mathcal{S}$ containing S. $\mathcal{S}$ extending up to the nearest Stokes rays outside S.*

Theorem 9.9 (uniqueness)**.** *Let $Y(z)$ be a fundamental matrix solution such that $Y(z) \sim Y_F(z)$ in a sector containing a set of basic Stokes rays. Then $Y(z)$ is unique.*

Proof. Suppose exists $\tilde{Y}(z) \sim Y_F(z)$ in the sector. Then as usual $Y(z) = \tilde{Y}(z)C$ and

$$e^{\Lambda(z)}Ce^{-\Lambda(z)} = \tilde{\mathcal{G}}(z)^{-1}\mathcal{G}(z) \sim I$$

so

$$e^{\Lambda_i(z)-\Lambda_j(z)}C_{ij} \sim \delta_{ij}. \tag{$*$}$$

Since the sector contains basic rays, $Re(\lambda_i - \lambda_j)z^n$ changes signas z varies in the sector. Thus there exists a subsector where $e^{(\lambda_i-\lambda_j)z^r} \to \infty$ as $z \to \infty$, so $(*)$ holds if and only if $C_{ij} = \delta_{ij}$ so $C = I$. □

Remark. We can take the sector to be $\mathcal{S}$ in the extension theorem.

9.4 Stokes matrix

Let $Y_1(z), Y_2(z)$ are fundamental solutions such that $Y_j(z) \sim Y_F(z)$ in S_j such that the opening of $S_j > \frac{\pi}{r}$. Suppose $S_1 \cap S_2 \neq \emptyset$ and does not contain Stokes rays.

Definition (Stokes matrix)**.** The *Stokes matrix* is the connection matrix S defined by

$$Y_2(z) = Y_1(z)S$$

for $z \in S_1 \cap S_2$ (and analytically exnteded in z).

Exercise. Consider Stokes rays associated with (λ_j, λ_k). Prove that for $\arg z = \theta_{jk} + \frac{2\pi N}{r} + \delta$, $\delta \in \mathbb{R}$ (taken modulo $\frac{2\pi}{r}$),

$$\mathrm{Re}(\lambda_j - \lambda_k)z^n \begin{cases} < 0 & -\frac{\pi}{r} < \delta < 0 \\ = 0 & \delta = 0, \pm\frac{\pi}{r} \\ > 0 & 0 < \delta < \frac{\pi}{r} \end{cases}$$

Definition. If in an open sector $\mathrm{Re}(\lambda_j - \lambda_k)z^n > 0$ we write $\lambda_j \succ \lambda_k$ and say λ_j is *dominant*. If $\mathrm{Re}(\lambda_j - \lambda_k)z^n < 0$ then write $\lambda_j \prec \lambda_k$.

If S does not contain Stokes rays then $\prec$ defines an ordering in $\{\lambda_1, \dots, \lambda_n\}$.

Proposition 9.10. *Suppose $Y_2 = Y_1 S$ for some Stokes matrix S in $S_1 \cap S_2$. Then S has the following form: $S_{jj} = 1$ for all j, $S_{jk} = 0$ for $\lambda_j \succ \lambda_k$ in $S_1 \cap S_2$,*

Proof. As usual write $Y_a(z) = G_0\mathcal{G}_a(z)e^{\Lambda(z)}$. Then

$$e^{\Lambda_j(z)-\Lambda_k(z)}S_{jk} \sim \delta_{jk}$$

This holds if and only if $S_{jj} = 1$ and $S_{jk} = 0$ for $\mathrm{Re}(\lambda_j - \lambda_k)z^n \geq 0$, which is precisely the definition of $\lambda_j \succ \lambda_k$ (S_{jk} can be any number for $\mathrm{Re}(\lambda_j - \lambda_k)z^n < 0$). □

Let $\mathcal{S}_\nu = S(\tau_{\nu-\mu}, \tau_{\nu+1})$ be the open sector containing basic rays $\tau_{\nu-\mu+1}, \ldots, \tau_\nu$. Let $\mathcal{S}_{\nu+\mu} = S(\tau_\nu, \tau_{\nu+\mu+1})$ and so on. On each sector $\mathcal{S}_{\nu+h\mu}$, $h \in \mathbb{Z}$, exists a unique solution $Y_{\nu+h\mu(z)} \sim Y_F(z)$. There are Stokes matrices relating overlapping sector, which we label using

$$Y_{\nu+(h+1)\mu}(z) = Y_{\nu+h\mu} S_{\mu+h\mu}.$$

Exercise. Show that $(S_\nu)_{jk} = 0$ if and only if $(S_{\nu+\mu})_{jk} = 0$, if and only if $(S_{\nu-\mu})_{kj} = 0$ (so they are upper/lower triangular alternating in ν).

9.5 Monodromy

Consider the formal solution $Y_F(z) = G_0 F(z) e^{\Lambda(z)}$. As $\Lambda(z)$ can be written as a polynomial plus $\Lambda_r \log z$,

$$Y_F(ze^{2\pi i}) = G_0 F(z) e^{\Lambda(z)} e^{2\pi i \Lambda_r}.$$

We call $e^{2\pi i \Lambda_r}$ *formal monodromy.*

Given $z \in \mathcal{S}_\nu$, $z_\nu \mapsto z_{\nu+2r\mu} = z_\nu \cdot e^{2\pi i} \in \mathcal{S}_{\nu+2r\mu}$.

Theorem 9.11.

1. $Y_{\nu+2r\mu}(z_{\nu+2\pi\mu}) = Y_\nu(z_\nu) \cdot e^{2\pi i \Lambda_r}$.
2. $Y_{\nu+2r\mu}(z) = Y_\nu(z) S_\nu S_{\nu+\mu} \cdots S_{\nu+(2r-1)\mu}$.
3. $Y_\nu(ze^{2\pi i}) = Y_\nu(z) e^{2\pi i \Lambda_r} (S_\nu \cdots S_{\nu+(2r-1)\mu})^{-1} = Y_\nu(z)[M_\infty^{(\nu)}]^{-1}$.

Proof. 1 follows from formal monodromy and uniqueness of asymptotics. 2 is repeated application of Stokes matrix. Combining them gives 3. □

Proposition 9.12. *For every $\nu \in \mathbb{Z}$, we have the relation*

$$S_{\nu+2r\mu} = e^{-2\pi i \Lambda_r} S_\nu e^{2\pi i \Lambda_r}.$$

Proof.

$$\begin{aligned} Y_{\nu+(2r+1)\mu}(z_{\nu+(2r+1)\mu}) &= Y_{\nu+\mu}(z_{\nu+\mu}) e^{2\pi i \Lambda_r} \\ &= Y_\nu(z_\nu) S_\nu \cdot e^{2\pi i \Lambda_r} \end{aligned}$$

LHS can also be expressed as

$$Y_{\nu+2r\mu}(z_{\nu+(2r+1)\mu}) S_{\nu+2r\mu} = Y_\nu(z_\nu) e^{2\pi i \Lambda_r} S_{\nu+2r\mu}.$$

□

Theorem 9.13.

1. $\Lambda_r, S_\nu, S_{\nu+\mu}, \ldots, S_{n+(2r-1)\mu}$ *generated* $S_{\nu+h\mu}$ *for all* $h \in \mathbb{Z}$.
2. *They are sufficient to compute the monodromy at* ∞ *of every* $Y_{\nu+h\mu}$.

Definition (complete set of Stokes matrices). $S_\nu, \ldots, S_{\nu+(2r-1)\mu}$ is called a *complete set of Stokes matrices.* $S_\nu, \ldots, S_{n+(2r-1)\mu}, \Lambda_r$ are called *monodromy data at* $z = \infty$.

Theorem 9.14. $\frac{dY}{dz} = z^{r-1}A(z)Y, \frac{d\tilde{Y}}{dz} = z^{\tilde{r}-1}\tilde{A}(z)\tilde{Y}$ *are holomorphically equivalent if and only if* $= \tilde{r}$*, have the same* $\Lambda, \Lambda_1, \ldots, \Lambda_r$*, and for some* ν_0 *they have the same* $S_{\nu_0}, \ldots, S_{\nu_0+(2r-1)\mu}$ *(this is true for all* ν*).*

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