University of
Cambridge

Mathematics Tripos

Part III

3 Manifolds

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Contents

0 Why 3?

0.1 Motivation

Poincare conjecture (1904) Question: how can we distinguish S^3 fom other 3-manifolds? The strategy is to find an invariant that distinguishes S^3. The frst guess is homology but

> **Theorem 0.1** (Poincare)**.** *There exists a closed oriented 3-manifold P with $H_*(P) \simeq H_*(S^3)$ but with $P \ncong S^3$.*

Notation. We use $\cong$ to denote homeomorphism and $\simeq$ to denote isomorphism.

This is proven in the following way: first invent the fundamental group π_1, then construct P, which is now known as (-1)-Dehn surgery on left-handed trefoil knot $K_T \subseteq S^3$. Finally show that $|\pi_1(P)| = 120, |\pi_1(S^3)| = 1$ and $H_*(P) \simeq H_*(S^3)$.

0.2 Homotopy

Review of homotopy theory homotopy, fundamental groups and higher homotopy groups, homotopy equivalence, weak homotopy equivalence

Homotopy vs. homology Let X and Y be path-connected topological spaces.

> **Theorem 0.2** (Hurewicz)**.**
>
> 1. $H_1(X, \mathbb{Z}) \simeq \pi_1(X)/[\pi_1(X), \pi_1(X)]$.
> 2. *If* $\pi_i(X) = 1$ *for* $i = \{1, \dots, n\}$ *then*
>
> $$H_i(X) = 0 \text{ for } i \leq n, i \neq 0$$
> $$H_{n+1} \simeq \pi_{n+1}(X)$$

> **Theorem 0.3** (Whitehead)**.** *If X, Y are CW complexes. Then a weak homotopy equivalence of X and Y is also a homotopy equivalence.*

> **Theorem 0.4** (Whitehead-homology variant)**.** *Suppose X, Y are simply-connected CW complexes. If the induced homomorphisms $f_* : H_k(X; \mathbb{Z}) \to H_k(Y; \mathbb{Z})$ are isomorphisms for all $k \leq \dim X$ then $f : X \to Y$ is a homotopy equivalence.*

> **Theorem 0.5.** *Any homotopy equivalence $f : X \to Y$ induces isomorphisms on homology, cohomology, cohomology ring structure (for any coefficients).*

0.3 *Simplifications in higher dimension

Let $\mathcal{C}$ be the smooth category when $n \geq 5$ and topological category $n \geq 4$.

Theorem 0.6 (Whitney trick)**.** *Suppose* $\dim X = n$ *where* $n \geq 4$ *and* $P, Q \subseteq X$ *are* $\mathbb{C}$*-embedded submanifolds and* $\dim P + \dim Q = \dim X$*. Then* P, Q *can be locally* $\mathcal{C}$*-isotoped so that the geometric intersection number equal to the absolute value of algebraic intersection of* P, Q*. Note that algebraic intersection number is signed while teh geometric counterpart is not.*

Convention. When we say topological embeddings we always mean locally flat embeddings, which will be defined later in the course.

Definition (h-cobordism)**.** Let W with $\partial W = X_1 \amalg X_2$ be a cobordism from X_1 to X_2. W is an *h-cobordism* if the embeddings $X_i \hookrightarrow W$ are homotopy equivalences.

Convention. All manifolds are compact connected and oriented unless otherwise stated.

Theorem 0.7 (h-cobordism)**.** *Suppose* $\dim X_i = n, \dim W = n + 1$*,* W *is a* h*-cobordism from* X_1 *to* X_2*. If* $\pi_1(X_i) = \pi_1(W) = 1$ *and* $n \geq 4$ *then* W *is* $\mathcal{C}$*-isomorphic to* $X_1 \times [0, 1]$.

0.4 Generalised Poincare conjecture

Poincare conjecture: if S is compact oriented 3-manifold homotopy equivalent to S^n, then does $S \cong S^n$?

Generalised Poincare conjecture: if S is compact oriented n-manifold homotopy equivalent to S^n, then does $S \cong S^n$?

It turns out for $n \geq 4$, the generalised Poincare conjecture is a corollary of h-cobordism theorem. Sketch of proof for $n \geq 5$: suppose S is homotopy equivalent to S^n, Then $\pi_*(S) \simeq \pi_*(S^n), H_*(S) \simeq H_*(S^n)$. Delete two balls from S to obtain $W \cong S \setminus \mathring{B}_1^n \amalg \mathring{B}_2^n$. Claim that W is a h-cobordism: apply Mayer-Vietoris with $A = W, B = B_1^n \amalg B_2^n$. Then $A \cap B = S^{n-1} \amalg S^{n-1} =_{\text{htp}} W \amalg \{0, 1\}, A \cup B = S, A \amalg B = W$.

$$\begin{array}{l} H_n(S^{n-1} \amalg S^{n-1}) \longrightarrow H_n(W \amalg \{0,1\}) \longrightarrow H_n(S) \longrightarrow \\ \longrightarrow H_{n-1}(S^{n-1} \amalg S^{n-1}) \longrightarrow H_{n-1}(W \amalg \{0,1\}) \longrightarrow H_{n-1}(S) \end{array}$$

The first term vanishes because of dimension, the second term vanishes because W is not closed. By homotopy equivalence we get

$$0 \longrightarrow \mathbb{Z} \longrightarrow \mathbb{Z} \oplus \mathbb{Z} \longrightarrow H_{n-1}(W \amalg \{0,1\}) \longrightarrow 0$$

We can compute that $H_{n-1}(W \amalg \{0, 1\}) \simeq \mathbb{Z}$. It is an exercise to show that there is an induced isomorphism on homology $H_k(S_i^n) \to H_k(W)$ for each k. Moreover $\pi_1(W) = 1$ so $S_i^n \to W$ are homotopy equivalent.

Therefore $W \cong S^{n-1} \times [0,1]$ So $S \cong B_1^n \cup W \cup B_2^n$. By Alexander trick map on a S^{n-1} can be extended *topologically* to a map on B^n with $\partial B^n = S^n$. Extends this homeomorphism over the two balls.

Note that this only applies to topological category and smooth generalised Poincare conjecture is still open in $n \geq 4$.

0.5 Why not higher than 5?

Moral: homotopy-theoretic techniques can be used to answer most/many questions about topology or smooth structures in dimension ≥ 5.

1 Lecture 2: Why 3-manifolds? + Embeddings/Knots

Active research areas

1. An interaction with 4-dimensional manifolds (smooth/symplectic/complex structures)

   (a) Dimension reduction reduces 4-dimensional invariant to 3-dimensional ones (that are fancier "categorified") and maps induced by cobordisms.

   (b) symplectic form ω on $X^4 \implies$ *contact structure* ξ on $Y = \partial X$.

   (c) Stein structure (complex/symplectic structure) on $X \implies$ Stein-fillable contact structure.

   (d) Normal complex structure sin $(X, 0)$ is a real cone over Y =Linkm(X, 0.

2. Geometric group theory: fundamental groups, especially of 3-manifolds:

   prime, atoroidal non lens space 3 manifolds $\iff$ fundmental groups of such 3-manifolds.

3. 2-dimensional structure

   (a) contact stucture: ξ everywhere nonintegrable 2-lane field. "tight" contact structure classification

   (b) minimal genus representatives of embedded surfaces, or knot genus. This is better understood. Thurston norm. The 4-dimensional analogue is still open.

   (c) Foliations. Taut folations classification. Seifert fibered

4. 1-dimensional structure: knots and links

   (a) embedddings $\amalg_i S^1_i \hookrightarrow S^3$. Every 3-manifold can be realised as *Dehn surgery* on a link $L \hookrightarrow S^3$. Thus the theory of knot theory is richer that of 3-manifold. We study 3-manifolds via knot invariants (WIlten-Reshetikhin-Turaev invariant).

   (b) Relations to other areas

      i. Chern-Simons knot invarints: $K \subseteq S^3 \iff$ Gromov-Witten invariants on $O(-1) \underset{\mathbb{CP}^1}{\oplus} O(-1)$.

      ii. Homfly homology of $n$str braids $\iff$ DC sheaves on $\mathrm{HIlb}^n(\mathbb{C})$.

      iii. Khovanov homology of links in $S^3 \iff$ DC sheaves on other spaces.

1.1 Course themes

1. Decompositions/Constructions of 3-manifolds.

   (a) surface decompositions/constructures

      i. prime decomposition — cut along essential S^2

      ii. JSJ decomposition — cut along essential T.

      iii. Mapping tori $\iff$ surface fibrations.
   (b) quotient spaces
      i. Hyperbolic quotients
      ii. quotients of S^7. Seifert fibration
      iii. Morse theoretic
         A. handle decomposition
         B. Heegaard splittings/diagrams
      iv. Dehn surgery on links

2. Structure + Invariants for 3-manifolds
   (a) Knots & links
      i. complement $S^3 \setminus K$
      ii. $\pi_1(S^3 \setminus K)$
      iii. Alexander polynomials + Turaev torsion
   (b) Essential/incompressible embedded surfaces, Thurston norm
   (c) Foliations

2 Embeddings

Definition (link)**.** A *link* is an embedding $L = \amalg_i S^1_i \hookrightarrow S^3$ considered up to isotopy. This embedding is either smooth or topoogical and locally flat. These two notions are equivalent.

Let X and Y be topological manifolds.

Definition (topological embedding)**.** A *topological embedding* $X \hookrightarrow Y$ is a map $X \hookrightarrow Y$ which is a homeomorphism onto its image.

Definition (immersion)**.** If X and Y are also smooth then a map $f : X \to Y$ is an *immersion* if $d_x f : T_x X \to T_{f(x)} Y$ is injective for all $x \in X$.

As a consequence of inverse function theorem, any immersion is locally an embedding.

Definition (smooth embedding)**.** A *smooth embedding* is a topological embedding that is also an immersion.

Corollary 2.1. *If X, Y are smooth compact then any bijective immersion is an embedding.*

Theorem 2.2 (Moise)**.** *There is a canonical correpondence between topological structures and smooth structures on 3-manifolds.*

Thus 3-manifolds up to homeomorphism bijects to 3-manifolds up to diffeomorphism.

Definition (local flatness)**.** A topologically embedded submanifold $X \subseteq Y$ is *locally flat* at $x \in X$ if x has a neighbourhood $x \in U \subseteq Y$ with homeomorphisms $(U \cap X, U) \cong (\mathbb{R}^{\dim X}, \mathbb{R}^{\dim Y})$.

A *locally flat embedding* is locally flat everywhere.

Convention. From now on any embedding is smooth or locally flat.

Definition (regular neighbourhood)**.** A *regular neighbourhood* of an embedded submanifold $X \subseteq Y$ is a tubular/collar neighbourhood if the embedding is smooth/topologically flat.

In 3-dimensions normal bundles are trivial so a regular neighbourhood $\nu(X)$ is just $D^2 \times X \hookrightarrow Y$ if $\dim X = 1$ and $D^1 \times X \hookrightarrow Y$ if $\dim Y = 2$.

In particular, neighbourhood of a not $K \hookrightarrow S^3$ is just a solid torus $D^2 \times S^1 \hookrightarrow S^3$.

3 Lecture 3: Link diagrams & Alexander Skein relations

Example. Wild knot: not locally flat embedding

Definition (isotopy). An *isotopy* in category $\mathcal{C}$ from f_1 to $f_2 : X \to Y$ is a homotopy through maps of type $\mathcal{C}$.

The point is, all knots (including wild knot) are isotopic through non-locally flat embeddings to an unknot, and all knots are homotopic to an unknot so we want to exclude the "bad" homotopies where a knot can cross itself.

3.1 Knot and link diagrams

Definition (link). A *link* is an (oriented) embedding $\iota : \coprod_i S^1_i \hookrightarrow S^3$ of (oriented circles), considered up to isotopy.

Definition (link projection). A *link projection* is an immersion $L \looparrowright \Gamma \hookrightarrow \mathbb{R}^2$, induced by

$$\begin{array}{ccccccc} L & \hookrightarrow & S^3 \setminus \{x_0\} & \xrightarrow{\cong} & \mathbb{R}^3 & \xrightarrow{\cong} & \mathbb{R}^2 \times \mathbb{R} \\ \downarrow{\scriptstyle p|_L} & & & & & & \downarrow{\scriptstyle p} \\ \Gamma & & & \longrightarrow & & & \mathbb{R}^2 \end{array}$$

such that $x_0 \notin L$ and $p|_L$ is an embedding except at double point singularities.

This aweful looking definition is just a formalisation of a familiar concept that facilitates the study of knots:

Definition (link diagram). A *link diagram* $D = (\Gamma, \text{crossing}(D))$ of a link $L \subseteq S^3$ is an embedded graph $\Gamma \hookrightarrow \mathbb{R}^2$ from a link projection of D, together with decorations at double points to label crossings. We draw a gap in the lower strand.

Theorem 3.1 (Reidemeister moves). *Let D_1 and D_2 be link diagrams for respective links $L_1, L_2 \subseteq S^3$. Then L_1 and L_2 are isotopic if and only if D_1 and D_2 are related by some combination of the fuollowing moves:*

It is more important to know that such moves exist than what they actually are.

3.2 Alexander Skein relation

To compute the alexander polynomial, you first choose an orientation for the link $L \subseteq S^3$. However, the resulting polynomial is independent of choice of orientation for knots.

Theorem 3.2 (Alexander)**.** *The* Alexander polynomial

$$\Delta : \{\textit{link diagram}\} \to \mathbb{Z}[t^{-1/2}, t^{1/2}]$$

is specified by 2 conditions:

1. *normalisation:* $\Delta(u) = 1$ *where* u *is the unknot.*
2. *Skein relation:* $\Delta(negativecrossing) - \Delta(positivecrossing) = \Delta(orientedresolution)(t^{-1/2} - t^{1/2})$ *for all* $c \in crossing(D)$.

$\Delta(D_1) = \Delta(D_2)$ *if* D_1 *and* D_2 *are diagrams for isotopic links.*

Theorem 3.3 (equivalence of Alexander polynomial)**.** *Later we will define an Alexander polynomial for 3-manifolds with* $b_1 > 0$. *With respect to this definition,*

$$\Delta_{link}(L) = \Delta_{\textit{3-manifold}}(S^3 \setminus L)$$

for any link $L \subseteq S^3$.

4 Handle decompositions from Morse Singularities

Handles: index k-handles are tubular neighbourhood of k-cell CW complex, also are neighbourhoods of Morse critical points.

4.1 Morse functions

Let $X \to \mathbb{R}$ be a smooth function on a smooth manifold X.

> **Definition** (Hessian, critial point)**.** $\mathrm{Hess}_p(x)$ is the Hessian of f at p, which is local coordintes is
> $$\left(\frac{\partial^2 f}{\partial x_i \partial x_J}|_{x=p}\right)_{ij}.$$
> crit$f$ is the set of critical points of f, i.e. $\{p \in X : \frac{\partial f}{\partial x_i} = 0 \text{ for all } i\}$, or more invariantly, $df = 0$.

> **Definition** (Morse function)**.** A smooth function $f : X \to \mathbb{R}$ on an n-manifold X is *Morse* if
>
> 1. every critical point of f is isolated. (If X is compact then this implies that critical points are finite)
> 2. $\mathrm{Hess}_p f$ is nongenerate at each $p \in \mathrm{crit} f$, if and only if $\det \neq 0$, if and only if has all nonzero eigenvalues.

4.2 Morse singularities

A list of descriptions of Morse functions:

1. If $f : X \to \mathbb{R}$ is Morse, then a Taylor series expansion around a critical point $p \in \mathrm{crit} f$ looks like
$$f(x) = f(p) + \frac{1}{2} \sum x_i x_j \frac{\partial^2 f}{\partial x_i x_j}\bigg|_p + \text{ higher order terms}$$
2. $\mathrm{Hess}_p f$ is nondegenerate means that we can rescale coordinates so that all eigenvalues are ± 1.
3. Since partial derivatives commute, $\mathrm{Hess}_p f$ is symmetric. Thus by linear algebra it is diagonalisable and we can write
$$f(x) = f(p) - \sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2 + \text{ higher order terms.}$$

Lemma 4.1 (Morse lemma)**.** *Let X be a smooth manifold and $f : X \to \mathbb{R}$ Morse. One can choose coordinates x centred at $p \in critf$ such that*

$$f(x) = f(p) - sum_{i=1}^{k} x_i^2 + \sum_{i=k+1}^{n} x_i^2.$$

Proof. Use implicit function theorem. □

Definition (index)**.** The *index* $\mathrm{ind}_p f$ of a Morse function $f : X \to \mathbb{R}$ at a critical point p is

$$\mathrm{ind}_p f = \# \text{ negative eigenvalues of } \mathrm{Hess}_p,$$

which is the k above.

Thus Morse lemma says that index is the (only?) invariant of Morse functions. Moral: there is a standard local model for each index k Morse critical point. See printed notes

Definition (k-handle)**.** An index k-handle, or just k-handle, or n-dimensional k-handle is the closure of a tubular neighbourhood of an index k critical point. $H_k^m \cong \nu(x) \cong D^k \times D^{n-k} \supseteq \mathring{B}^k$.

Note that the corners in D^k and D^{n-k} are different

5 Lecture 5: Handles from cells, Heegard diagrams

Cell ecomplex interpretation

Definition (handle, core, cocore)**.** An *n-dimension k-handle* H_k^n or *index k-handle* is a product decomposition

$$H_k^n \cong D^k \times D^{n-k} \cong B^n$$

of the closed n-ball into a k-dimensional k-cell *core* D^k and *cocore* D^{n-k}.

If you choose a metric, the core D^k is fat and the cocore D^{n-k} is thin.

Definition (attaching region, belt region)**.** The boundary ∂H_k^n of an n-dimensional k-handle decomposes as

$$\begin{aligned}\partial H_k^n &\cong \partial(\text{core} \times \text{cocore}) \\ &\cong \partial(\text{core}) \times \text{cocore} \cup \text{core} \times \partial(\text{cocore}) \\ &\cong \underbrace{\partial D^k \times D^{n-k}}_{\text{attaching region}} \cup \underbrace{D^k \times \partial D^{n-k}}_{\text{belt region}}\end{aligned}$$

In a cell complex, we attach a k-cell D^k by gluing its boundary $\partial D^k \cong S^{k-1}$ to the cell-complex we have built so far.

$$\text{attaching region}(H_k^n) \cong \overline{\nu(\partial D^k)} \cong \overline{\nu S^{k-1}} \cong \partial D^k \times D^{n-k}$$

Definition (handle attachment)**.** The attachment of a k-handle H_k^n to an n-manifold X to product an n-manifold X' is induced by a *k-handle attachment* cobordism Z from $-\partial X$ to $\partial X'$.

$$Z = (\partial X \times I) \cup_{\text{a.r.}} H_r^n.$$

We have

$$\partial X' \cong (\partial X \setminus \text{a.r.}(H_k^n)) \cup (\text{b.r.}(H_k^n)).$$

As $\partial X \times I$ deformation retracts to ∂X, we have

$$X' \cong X \cup Z \cong X \cup H_k^n.$$

Convention. We usually say that we attach a handle along the *core* of the attaching region.

Definition (attaching/belt sphere)**.**

$$\begin{aligned}\text{attaching region}(H_k^n) &\cong \text{core}(\text{attaching region}(H_k^n)) \\ &\cong \text{core}(\partial D^k \times D^{n-k}) \\ &\cong \partial D^k\end{aligned}$$

$$\begin{aligned}\text{belt region}(H^n_k) &\cong \text{core}(\text{belt region}(H^n_k))\\ &\cong \partial D^{n-k}\end{aligned}$$

Convention reexpressed: to attach a k-handle, we specify where the *attaching sphere* will be glued.

Morse interpretation, revisited

Definition (gradient)**.** Choose a Riemannian metric g on a smooth n-manifold X. Let $f : X \to \mathbb{R}$ be a smooth function. the gradient $\operatorname{grad} f \in \Gamma(TX)$ of f is the vector field satisfying

$$g(\operatorname{grad} f, V) = df(V)$$

for $V \in \text{Vect}X = \Gamma(TX)$. Locally,

$$g_x((\operatorname{grad} f)_x, V_x) = df_x(V_x).$$

In local coordinates,

$$\operatorname{grad} f = \sum g^{ik} \frac{\partial f}{\partial x^k} e_i$$

where $e_i = \frac{\partial}{\partial x^i}$.

Idea: invariant object from partials of f, $df = \sum \frac{\partial f}{\partial x^i} dx^i$. To get a vector field, need a bilinear form to dualise df. $\operatorname{grad} f$ comes from bilinear form (metric), and Hamiltonian vector field f comes from symplectic form.

Moral (Morse theorey intepretation)

Q: In what sense $H^n_k \cong \overline{\nu(x)}$, $x \in \text{crit} f, \text{ind}_x f = k$?

A: Gradient flow at the boundary of H^n_k: $\operatorname{grad} f$ flows into attaching region H^n_k, into $x \in \text{crit} f$ in k directions. $\operatorname{grad} f$ flows out of belt region H^n_k, out of x in $n-k$ directions.

For example, for

$$f = -\sum_{i=1}^{k} x_i^2 + \sum_{j=k+1}^{n} x_j^2$$

Index